Guided activities for precalculus

Aligned to Precalculus by Stitz and Zeager

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Notes to the teacher

These activities are designed as a guide to help your students navigate standard precalculus content. The purpose is to shift the focus from you, the instructor, to the student(s) working in small groups. When I teach precalculus a typical class period begins with me **briefly** introducing the daily topic, and setting the student expectations for that class period. Then students will work in small groups through the guided activities while I move around the room helping them. If I find there is a lot of struggle surrounding a particular topic I typically bring everyone together to discuss. Ideally there will be a student who can share their progress but if no students are able to find success I will go to the board and work through some examples.

As you read through the materials you will notice that there are places where students are directed to come back together for an in class discussion. These questions often serve as reminders to me to bring the class back together and see what everyone has discovered, even if everyone is doing well. The hope is to have students discuss the content, and I will ask individual students to share what they have found. This gives me a little more control over which discoveries are emphasized, and I feel it brings a more uniform understanding amongst the students.

It is common in guided activities to have space for students to write their answers directly on the guides, however I chose to intentionally leave very little space for students to write on the sheets. My intention is to get students to keep a notebook as a companion to working through these materials. I have observed students often get a lot from rephrasing the questions and solution in their own words and this is my effort to help them do that.

Each section in these materials begins with a book reference. This is the section in "Precalculus" by Stitz and Zeager that students should reference if they want additional material. I chose to align my materials with the book by Stitz and Zeager for a few reasons: it is freely available online to all students, it has hundreds of questions on each topic along with their answers, and it covers everything I want and much more. I often direct students to the questions at the end of each section for additional practice.

For many of the activities I have created Desmos graphs to complement the student work. If you have enough time I feel it is valuable to have students create many of these graphs, but if you are in a time crunch they are available here for you to use.

To go along with each section I have created a video that serves as an additional resource for students. I have found that many students find comfort in having a lecture, or video, or some other explanation to listen to and these are my way of filling that need. Using these videos one could run their classroom in a typical flipped classroom style with the guided activities in class, and videos outside of class. You can access the **Video Playlist** by clicking on this link. Similar links are located throughout the book.

Each chapter has at least one modeling activity for students to work through using the functions from that chapter. In my experience being able to read scenarios, parse them, and write the equations that describe the situation is the most difficult thing for students to do in Calculus. Many students also struggle with algebra, but with advances in technology gaps in those skills are becoming easier to overcome. To demonstrate this many of the modeling activities chosen are standard calculus problems that can be solved graphically via a technology such as Desmos.

I have included many of the traditional approaches to solving equations but some of them come with the title "Extension." I typically use these as extra credit opportunities for students to work through, or as supplemental work for advanced students. If your audience is primarily math majors, future teachers, or students who want to excel at standardized testing these would be a valuable inclusion.

I hope you enjoy!

Contents

Chapter 1

Functions

Throughout this course we are going to be exploring different types of functions. As such we are going to begin with some explorations surrounding functions in general with an eye toward answering the following questions:

- What is a function? How can I determine if I have one or not?
- What type of notation do we typically use for functions?
- How can we represent functions? How are they related to graphs?
- Can we combine functions? What are some ways to do that?
- If a function changes my input is there a way for me to undo that change? Is there always an undo button?
- How can we measure how fast a function is changing?

1.1 What is a function, and how can I tell?

Book Reference: Section 1.3

Throughout this course we are going to explore many different types of **functions**. In order to study functions we should know what a function is:

Functions

A **function** is a relationship in which every input is matched with exactly one output.

1. Consider the following tables. Which ones are functions? Justify your answers.

1.

Input	Output
3	2
2	5
8	8
7	11
4	20

2.

Input	Output
2	5
4	7
6	7
8	5
2	3

3.

Input	Output
7	5
3	9
5	15
7	5
1	25

2. Go to Desmos.com and click on Start Graphing. Desmos is a graphing calculator (like a TI-84) that is free for everyone. We will use it frequently in this class so it may be helpful for you to download the Desmos App on your phone or tablet. The app looks like this:

3. In the upper left of the screen you should see a big + sign. Click on it and choose to add a table. Enter the data from the 3 tables above. What letter does Desmos use for the input? Which letter is used for the output? What is different about the tables that are not functions?

4. Enter the following into the left sidebar on Desmos. Determine which ones are functions and which ones are not functions.

   (a) $x^2 - y = x + 1$

   (b) $y^2 = x^3 + 7$

   (c) $7y - 2x + 5 = 0$

   (d) $\frac{(x-2)^2}{2} + \frac{(y-3)^2}{3} = 3$

5. Can you come up with a test that will allow us to check if a given graph is a function? What would you call this test? Write down your test in your notebook and prepare to share it with the class.

6. We are going to do a class activity around functions on Desmos. Look for the class code to join Card Sort: Functions.

1.2 Function Notation

Book Reference: Section 1.4

Now that we know what a function is our next task is to understand the notation that we use to describe functions. For example consider the description of the line below:

$$y = 2x + 5.$$

Plug this into Desmos to verify that this is a line then type in the notation below:

$$f(x) = 2x + 5.$$

Desmos recognizes these as the same because it understand the notation that we use for functions. Instead of using the letter y for our output we use the expression $f(x)$ as this gives us a better description. The letter f is the name of our function. We could use anything we want here to help us describe our function, or differentiate from other functions. The letter x is telling us what letter we are using for our variable. See the chart below:

Name of function	Variable for input	=	Formula for output	
f	(x)	=	$2x + 5$	Multiply by 2 then add 5

1. Given then function $f(x) = 2x + 5$ we can use Desmos to find different outputs quickly. If I want to know that output of the function when I have an input of 2, I would find $f(2)$. Type this into Desmos and it should tell you that $f(2) = 9$. Try using Desmos to find the outputs at $x = 7$, $x = 292$, and $x = e^5$.

2. Describe a scenario that could be modeled by the function $f(x) = 2x + 5$. That is, describe some inputs and outputs that would fit this function.

3. For each of the functions below identify the name of the function, the variable for the input, and the formula for the output. Then describe the formula in words and give a real world scenario that could be described by the function.

   (a) $g(x) = \frac{5}{2} - 3x$

   (b) $A(x) = (x + 5)(x)$

   (c) $T(h) = 60h$

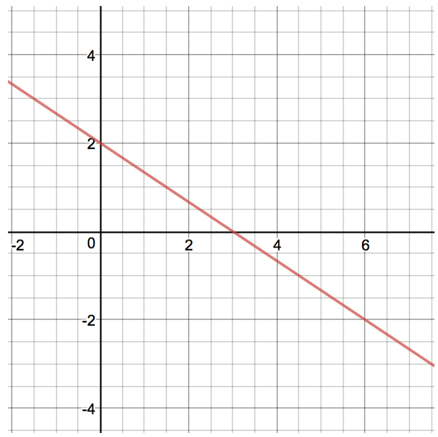
4. Use Desmos to find the out puts of the functions g, A, and T, at $x = -1, 7, \frac{7}{3}$ and give an interpretation of these outputs based on the description of your functions above.

5. We are going to do another class activity around functions on Desmos. Look for the class code to join Function Carnival which will help us connect graphs of functions to real world scenarios.

1.3 Functions and Graphs

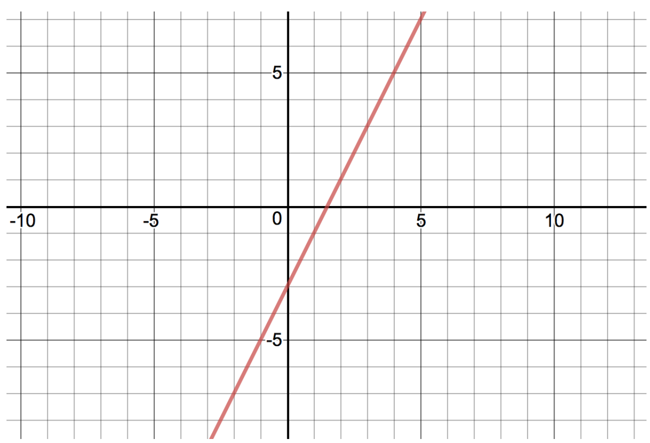
Book Reference: Section 1.6

When we are looking at the graph of a function we are seeing all of the possible input-output pairs. We traditionally use the x-axis to represent our inputs and the y-axis to represent our outputs. For example, the graph below is from the function $f(x) = -\frac{2}{3}x + 2$. Observe that the point $(3, 0)$ is on the graph of this function. This tells us that if we plug in $x = 3$ we will get an output of 0.



$f(x) = -\frac{2}{3}x + 2$

1. Using the graph above find $f(6)$, the output corresponding to an input of 6.
2. Using the graph above solve the equation $f(x) = 2$. That is, find the input that gives you an output of 2.



3. Use the graph above to answer the following questions
   (a) Find $f(1)$
   (b) Find $f(3)$
   (c) Solve $f(x) = 5$
   (d) Solve $f(x) = -1$

1.4 Function Composition

Book Reference: Section 5.1

When we are using function notation we often use the letter x to represent our variable. For example we can consider the function defined by

$$f(x) = x^2 - 5$$

We see that our function is named f, and whatever we have as an input we are going to square it and then subtract 5. We usually replace x with a number but we can also plug in many other expressions into our function. For example:

- $f(2) = (2)^2 + 5$
- $f(h) = (h)^2 + 5$
- $f(2+3) = (2+3)^2 + 5$
- $f(x+3) = (x+3)^2 + 5$

Any time we see an x that represents our input, so we always replace **ALL** of the variables with our chosen input.

1. Let $g(x) = x^2 + 2x - 1$. Determine the following:

   (a) $g(2)$
   (b) $g(h)$
   (c) $g(x^2)$
   (d) $g(x+3)$

   Now we are ready to see a new operation that we call **function composition**. Informally students often think about this as plugging one function into another.

   > **Function Composition**
   >
   > We use the following notation for function composition:
   >
   > $$(g \circ f)(x) = g(f(x))$$

   Example: Let $g(x) = x^3$ and $f(x) = 2x + 5$. Then

   $$(g \circ f)(x) = g(f(x)) = g(2x+5) = (2x+5)^3$$

2. Let $g(x) = x^3$, and $f(x) = 4x$. Determine the following:

   (a) $g(2x)$
   (b) $f(x+5)$
   (c) $(f \circ g)(x)$
   (d) $(g \circ f)(x)$

3. Let $g(x) = x^2 - 2x$, and $f(x) = 3 - 2x$. Determine the following:

   (a) $(g \circ f)(x)$
   (b) $(g \circ g)(x)$
   (c) $(f \circ g)(x)$
   (d) $(f \circ f)(x)$

1.5 Inverse Functions

Book Reference: Section 5.2

So far we have seen that functions take an input and change it through a series of operations. What we want to know now is: Can these changes be undone? Is there some way to undo whatever a function does?

Example:

- Pick a Function:
- f(x) = 2x + 5
- Multiply by 2
- Then add 5

- Undo the function:
- Subtract 5
- Divide by 2
- $g(x) = \frac{x-5}{2}$

The two functions f and g above are called **inverse functions** because they "undo" each other.

Inverse functions

We say that two functions $f(x)$ and $g(x)$ are inverses if:

- $(f \circ g)(x) = x$ and
- $(g \circ f)(x) = x$.

In this case we write $g(x) = f^{-1}(x)$ and say that f and g are **invertible**.

1. Use function composition to verify that $f(x) = 2x + 5$ and $g(x) = \frac{x-5}{2}$ are inverses.
2. Consider the function $f(x) = mx + b$ where we take an input, multiply by m, and then add b. Determine the inverse for this function.
3. Use Desmos to graph $f(x) = mx + b$ and its inverse that you found in 2. Type them into Desmos with the letters m and b and add the sliders when it prompts you to add them. Move around the values of m and b using the sliders. What interesting properties do you notice? Record your observations in your notebook to prepare for an in class discussion.
4. As we see in our daily lives not all actions can be undone. For example, completely burning a piece of firewood is something that cannot be undone. Similarly with functions we cannot always undo them. The list of functions below are all examples of functions that do not have inverses. Use Desmos to graph them and look for some patterns that may indicate why they are not invertible.

   (a) $f(x) = x^2$

   (b) $g(x) = \frac{x+5}{5x^2}$

   (c) $h(x) = x^3 - 2x + 1$

   (d) $p(x) = -|x| + 4$

   Hint 1: Using our observations about the graphs of inverses, what would the inverse of these functions look like? Would they be functions?

   Hint 2: The property that we are looking for is called One to One, and we can use something called the Horizontal line test to determine it. Prepare your observations for an in class discussion.

1.6 Average rate of change

Book Reference: Section 2.1

Throughout the calculus sequence we are often studying how fast quantities are changing. One of the ways we can measure how fast a function is changing is called the average rate of change.

Average rate of change

The average rate of change of a function f on an interval $[a, b]$ is defined to be

$$\frac{f(b) - f(a)}{b - a}$$

1. When I walk my dogs in the evening we typically walk $\frac{3}{4}$ of a mile and the entire walk takes about 15 minutes. What is my average walking speed during this trip? Describe how the average rate of change formula applies to this situation. What would the function f be measuring? What are a and b measuring?

2. The two tables below give input and output values for functions f and g.

$f(x)$:

Input	Output
1	2
2	5
3	8
4	11
5	14

$g(x)$:

Input	Output
1	3
2	6
3	11
4	18
5	27

   Determine the average rate of change of f and g on the intervals

   (a) $[1, 2]$
   (b) $[2, 4]$
   (c) $[3, 5]$
   (d) $[1, 5]$

3. Find the average rate of change of the following functions on the interval $[-2, 2]$.

   (a) $f(x) = 2x + 5$
   (b) $g(x) = x^2 - 1$
   (c) $h(x) = \frac{1}{x+1}$
   (d) $p(x) = 14$

4. I am drinking a 12 ounce cup of coffee and notice that 2 minutes after getting my cofee I have 10 ounces of coffee remaining. Then after 5 minutes I have 4 ounces of coffee remaining. How fast am I drinking my coffee, on average, between 2 and 5 minutes after receiving it? How fast am I drinking, on average, in the first 5 minutes? Include appropriate units on your answer.

Chapter 2

Linear Functions

The first type of function we are going to study is a linear function. Our goal is to answer the following questions:

- What makes a function linear?
- What are the main parts of a linear function and what information can we get from them?
- What does the slope of a line tell me and how do I find it?
- How many different ways can we represent a linear function?
- What information do I need to determine an equation of a line?
- How can I find the equation of a line from a graph?
- Which real world situations can be modeled by linear functions?

2.1 Introduction to Linear Functions

Book Reference: Section 2.1

Linear Function

A **linear function** is a function of the form

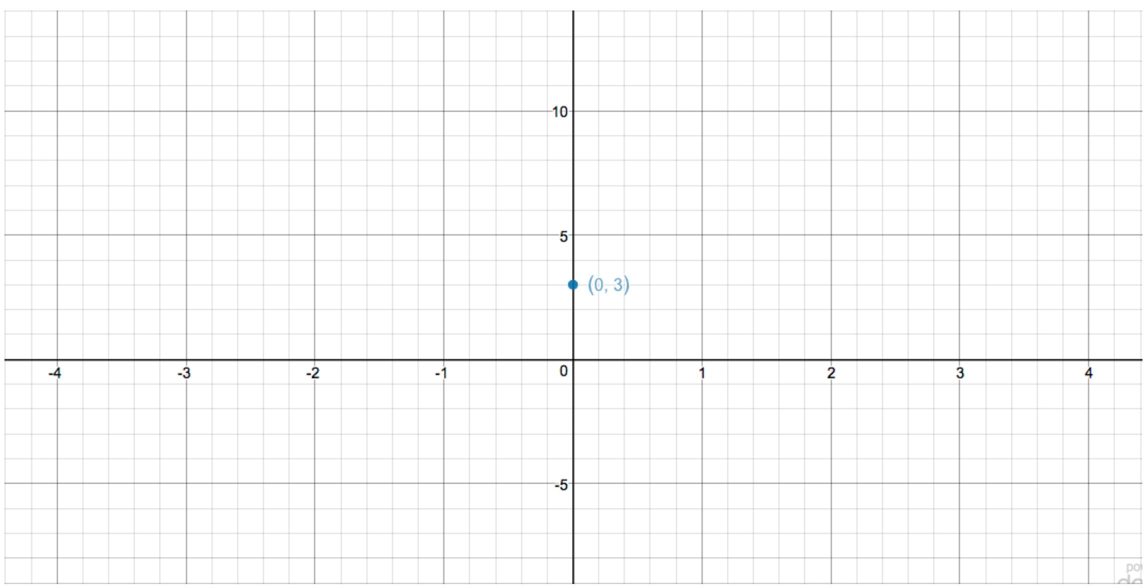
$$f(x) = mx + b.$$

This form is called the **slope-intercept form** where m is the **slope** of the function and b is the **y-intercept**.

Use the function $f(n) = 2n + 3$ for the table below

n	$f(n)$
-1	
0	3
1	
2	
3	
4	11

1. Fill out the missing outputs in the table above.
2. Plot the points from the table above onto the axes below, the pair $(0, 3)$ is already listed. Write down some observations you have about these points in your notebook. Do you think these points lie on the graph of a linear function? Why?



3. Solve the equation defined by $f(x) = -1$. Describe this equation in words.
4. What is $f(-4)$? Describe this quantity in words.
5. Determine the average rate of change of the function f on the intervals $[0, 4]$, $[2, 4]$, and $[2, 3]$. What do you notice? Prepare your observations for an in class discussion.

2.2 Modeling Activity: Pond border

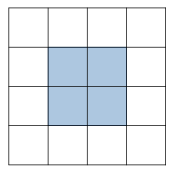
Book Reference: Section 2.1

Materials: pen, paper, square pattern blocks, Desmos.

Goal: Model an area scenario using linear functions. Observe and note some of the properties we have associated with linear functions.

Scenario:

Zoey is a tile specialist at a local pool company Westfield Pool and Tile. When an inground pool is installed she comes in and creates the border surrounding the edge of the pool. She uses 1 foot long square tiles and completely surrounds the pool, an example is shown below:



Square pools

1. The pool in the picture above is a 2×2 pool and requires 12 tiles to create the border. Determine how many tiles are needed to create the border for a 3×3 pool, 4×4 pool, up to a 7×7 pool.
2. Using the data you gathered above do you notice any patterns? Can you quickly determine how many tiles it would take to create the border for an 8×8 pool, 9×9 pool? Describe in words how you would do that.
3. How many tiles are needed for a 15×15 pool? Describe in words how to determine the number of tiles. How many tiles are needed for a 50×50 pool?
4. Write down an equation for the number of tiles needed to create the border for an $n \times n$ pool.
5. What is the size of the biggest pool Zoey can tile around using 100 tiles?

Rectangular pools

The new trend in pools is to have a lane pool in order to swim laps. The lane pool is 6 feet wide and can be as long as you would like it to be. Use the Desmos graph **Pool Border** for the following questions.

6. How many tiles are needed to border a 10 foot long lane pool?
7. How many tiles are added when the length of the lane pool increases by 1 foot?
8. Write down an equation for the number of tiles needed to create the border for a lane pool that is n feet long.
9. What is the size of the biggest lane pool Zoey can tile around using 100 tiles?

2.3 Linear functions from tables

Book Reference: Section 2.1

Consider the table below giving a value n and the value of the function f evaluated at n, which we denote $f(n)$.

n	$f(n)$
1	2
2	5
3	8
4	11
7	20

x	$g(x)$
2	16
4	12
6	8
8	4
11	-2

t	$h(t)$
1	5
3	9
6	15
8	19
11	25

1. Describe the data that you see in the table for $f(n)$ above. What do you notice about the average rate of change? What specifically makes you think this is a linear function?
2. Make a graph of the values in the table for $f(n)$.
3. Write an equation for $f(n)$ and describe in detail how you figured that out.
4. Calculate $f(52)$.

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5. Describe the data that you see in the table for $g(x)$ above. How does it compare to data in the table for $f(n)$? What specifically makes you think this is a linear function?
6. Make a graph of the values in the table for $g(x)$.
7. Write an equation for $g(x)$ and describe in detail how you figured that out.
8. Solve the equation $g(x) = 34$.

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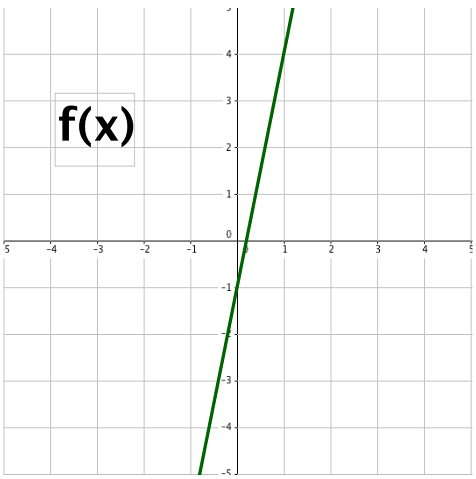
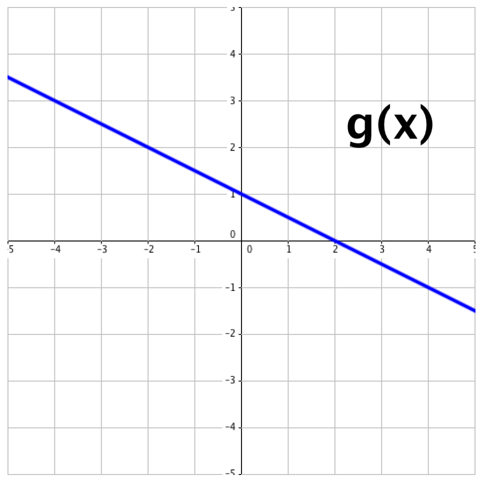
9. Describe the data that you see in the table for $h(t)$ above. How does it compare to data in the table for $f(n)$? $g(x)$? What specifically makes you think this is a linear function?
10. Write an equation for $h(t)$ and describe in detail how you figured that out.
11. Sketch a graph of the function $h(t)$.
12. For which values of x does $f(x) = h(x)$?

2.4 Linear functions from graphs

Book Reference: Section 2.1





Recall: A linear function can be described as $f(x) = mx+b$ where m is the slope and b is the y-intercept.

1. Write down the equation for the function $f(x)$. Describe in detail how you figured this out.
2. Describe specifically how you found the slope of $f(x)$, and how you found the y-intercept of $f(x)$.
3. In your own words describe what information the **slope** of the graph gives you?

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4. Determine the slope of the graph of $g(x)$. Describe in detail how you figured this out.
5. Determine the y-intercept of $g(x)$. Describe in detail how you figured this out.
6. Write down the equation of $g(x)$.

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7. On a new set of axes plot the points $(2, 5)$, and $(4, 9)$.
8. Draw in the linear function that connects the two points and label it $h(x)$.
9. Find the equation of $h(x)$. Describe in detail how you figured this out.

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10. Find the equation of the line that connects the points $(3, 7)$ and $(1, 9)$.

2.5 Point slope form

Book Reference: Section 2.1

Point-Slope form

The **point-slope form** of a linear function is

$$y - y_0 = m(x - x_0).$$

The quantity m is the **slope** of the function and (x_0, y_0) is any point on the function.

Using the point-slope form of a linear function often results in faster and more accurate computations. When using the slope-intercept form of a linear function we have to compute the y-intercept which can often be challenging when the data we have is not "nice." For the following questions it is recommended that you try to compute some of them using both the point-slope form, and the slope-intercept form to see which method is preferable to you.

1. Find the equation of the line with the following information:

   (a) Slope: 5, Point: $(1, 1)$

   (b) Slope: 2, Point: $(-3, 5)$

   (c) Slope: $-\frac{1}{2}$, Point: $(9, 0)$

   (d) Slope: -3, Point: $(\frac{3}{4}, \pi)$

   (e) Slope: 0, Point: $(0, 5)$

   (f) Slope: π, Point: $(-1, 2)$

2. Find the equation of the line passing through the following two points:

   (a) $(0, 5)$ and $(-1, 8)$

   (b) $(-1, -4)$ and $(6, 12)$

   (c) $(\frac{1}{2}, 7)$ and $(\frac{3}{2}, 10)$

   (d) $(0, \pi)$ and $(-4, 2\pi)$

   (e) $(3, 5)$ and $(2, 5)$

   (f) $(0, 6)$ and $(0, 8)$

Chapter 3

Quadratic Functions

The next type of functions we are going to study are called quadratic functions and fall into a larger class of functions that we call polynomials. Polynomials will be studied in more detail later. For now we are interested in answering the following questions:

- What are the different forms of a quadratic function?
- How do I evaluate quadratic functions?
- How can we solve quadratic equations? Specifically how do we
  - Factor;
  - Use the vertex of the function;
  - Use the quadratic formula (what is that?).
- What information do I need to find the equation of a quadratic function from data?
- Extension: Where does the quadratic formula come from?

3.1 General form of quadratic functions and factoring

Book Reference: Section 2.3

Quadratic functions

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers with $a \neq 0$. This is called the **general form** of a quadratic function.

For the functions below try evaluating them at the given points in two ways: Analytically with pen and paper, then verify your answers in Desmos.

1. Evaluate analytically and graphically:

   (a) $f(x) = 2x^2 - 5x + 7$. Find $f(2)$.

   (b) $g(x) = x^2 + x + 2$. Find $g\left(\frac{3}{2}\right)$.

   (c) $h(x) = 7x^2 - 3x - 10$. Find $h(0)$.

   (d) $p(x) = 15x^2 + 12x - 9$. Find $p(1)$.

   Now we want to focus on solving quadratic equations, that is finding an answer the question: When does $f(x) = c$ where $f(x)$ is a quadratic function, and c is a number. There are several methods to do this and the first one we are going to focus on is factoring. Work through the guided questions below to make progress toward this goal.

2. Suppose that you have to numbers A and B. If you know that $A \cdot B = 0$ what can you say about either A or B?

3. What are the solutions to the following equation. How do you know?

$$(x - 7)(x + 3) = 0$$

   The property that we see in the previous questions is called the **zero product property**. It is extraordinarily useful and the main reason why we are often interested in factoring, or writing an equation as the product of several terms. If we want to solve an equation of the form $f(x) = 0$ our main strategy is often to factor and then set each part of our product equal to zero.

4. If we expand the product in the above equation we get the following

$$(x - 7)(x + 3) = x^2 - 4x - 21$$

   How do you think the coefficients in the above expressions are related to one another? That is, how are -7 and 3 related to -4 and -21?

5. Expand the following products and think about how the factored form and general form of a quadratic function are related.

   (a) $f(x) = (x + 3)(x + 2)$

   (b) $g(x) = (x - 1)(x + 4)$

Zero-product property

Given any two real numbers A and B, if

$$A \cdot B = 0$$

then either $A = 0$ or $B = 0$.

6. Observe the following
$$(x + a)(x + b) = x^2 + (a + b)x + ab$$
Does this help you find a relationship between the factored form and general form? What do you think the relationship is?

7. When we are factoring we are looking for two numbers and we want the product of those two numbers to be our constant term, and the sum of those two numbers to be our middle coefficient. Consider the following
$$x^2 + x - 6 = \left(x + \qquad\right)\left(x + \qquad\right)$$
Can you find two numbers that multiply to -6 and add to 1?

8. Solve the following quadratic equations by factoring:

   (a) $x^2 - 3x - 10 = 0$

   (b) $x^2 - 6x - 7 = 0$

   (c) $x^2 + 7x + 12 = 0$

   (d) $x^2 - 3x + 2 = 0$

9. Try solving the following quadratic equations by factoring. What is different about them? What do you have to do differently? After you solve the equations analytically use Desmos to solve them graphically.

   (a) $2x^2 - 6x + 4 = 0$

   (b) $2x^2 - 5x - 3 = 0$

   (c) $x^2 + x + 1 = 3$

   (d) $2x^2 - 8x + 1 = -5$

   Observe that if we have a quadratic equation that is not equal to zero, like two of the examples above, it is to our advantage to rearrange the terms so that one side is equal to zero. Then we can factor and use the zero product property.

   When we are setting an equation equal to zero we often say that we are looking for the **zeros** or the **roots** of the equation. Asking someone to determine these is equivalent to asking them to solve the equation $f(x) = 0$.

10. Find the roots of the following equations. First try to find them analytically, then find them graphically.

   (a) $f(x) = x^2 - 4$

   (b) $g(x) = x^2 - 10x + 21$

   (c) $h(x) = x^2 + 19x + 84$

   (d) $t(x) = x^2 + 13x - 30$

   (e) $r(x) = 3x^2 - 13x + 12$

   (f) $p(x) = 2x^2 - 5x + 3$

3.2 Modeling activity: maximizing area

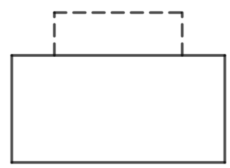
Book Reference: Section 2.3

Materials: pen, paper, scissors, string, tape, Desmos.

Goal: Model an area scenario with a quadratic function. Observe the properties of the vertex of the associated parabola and identify the information that it gives us.

Scenario:

I have two dogs named Cayley and Barley who are very interested in the neighbor dogs. When I let them out into my backyard they always want to run next door to find them, even in the middle of the night. Due to this I want to create a fenced in area in my backyard so that they cannot go next door. I went to the hardware store and bought 35 yards of fencing to create a pen in my backyard next to my house. See the diagram below as an example.



Objective: I want my dogs to be happy so I would like to create a pen that has the largest possible area for them to run around in. Help me find the dimensions of the pen that will have the largest possible area.

1. Begin by creating a pen on your tables using the string and tape. Record the length and width of your pen and find the area. (Remember that the area of a rectangle is length $\times$ width). Prepare to share your answer with the class so we can collect some data.
2. If you know the length of your pen can you find the width? How do you do that? Write down an equation that gives you the width of the pen from the length:

   Width =

3. Plot the data we have received in class in Desmos. Use the length of each rectangle as the input and the area as the output.
4. Based on your observations from the data create the pen that you think has the largest area.
5. Find a function that fits the data you have plugged into Desmos. You should use the formula for width that you found above.
6. What are the dimensions that produce the maximum area? What is the maximum area?

Notes:

- The function that you found above should be a quadratic function, do you see it?
- The highest (or lowest) point on the quadratic function is a point that we call the **vertex**. This is a point that we are often interested in as it represents an optimal point.

3.3 Vertex form of quadratic functions

Book Reference: Section 2.3

Vertex form of a quadratic function

An alternative form of a quadratic function is

$$f(x) = a(x-h)^2 + k$$

where a, k, and h are real numbers with $a \neq 0$. This is called the **vertex form** and the **vertex** of the quadratic function is the point (h, k).

1. Compare the two quadratic functions given below, what do you notice about them? What did you do to compare them?

$$f(x) = x^2 - 2x + 10 \qquad\qquad g(x) = (x-1)^2 + 9$$

2. Compare the general and vertex forms of the following functions. Do you notice any patterns?

General Form	Vertex Form
$x^2 - 4x + 10$	$(x-2)^2 + 6$
$x^2 - 6x + 10$	$(x-3)^2 + 1$
$x^2 - 8x + 10$	$(x-4)^2 - 6$
$3x^2 - 6x + 6$	$3(x-1)^2 + 3$
$2x^2 - 7x + 1$	$2\left(x - \frac{7}{4}\right)^2 - \frac{41}{8}$

3. For each of the following convert from vertex form to general form:

   (a) $(x-4)^2 + 1$

   (b) $(x+1)^2 - 11$

4. For each of the following convert from general form to vertex form:

   (a) $x^2 + 4x + 7$

   (b) $x^2 - 6x + 5$

   (c) $2x^2 - 8x + 8$

   (d) $3x^2 + 5$

5. Based on all of the work we have done so far we should be seeing some patterns. The general form of a quadratic function has the form $f(x) = ax^2 + bx + c$, and the vertex form is given above. We use the letter a in both cases because they are always the same. Use your experiences to guess the formulas for h and k. Prepare for an in class discussion.

$h =$ $\qquad\qquad$ $k =$

6. Use the examples of vertex form we have on this page to explore the following questions to prepare for a group discussion. Use other representations of the functions (tables, graphs) to aid in your thinking.

   (a) Why is the vertex form of a quadratic function of interest to us?

   (b) What is the vertex and why do we call it vertex form?

   (c) Are there any computations that are easier to do when the function is in vertex form?

7. Find the zeros of the following quadratic functions

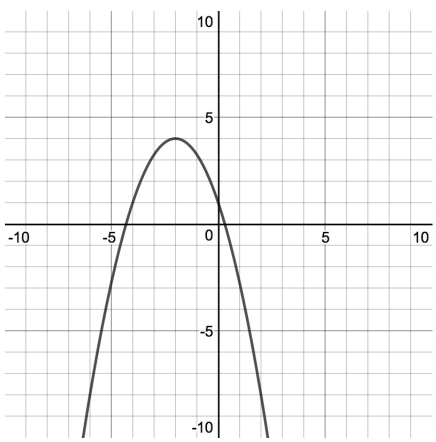
   (a) $f(x) = 2(x-3)^2 - 4$

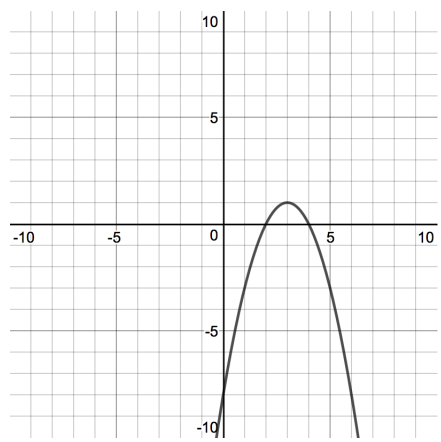
   (b) $g(x) = 3(x-7)^2 + 5$

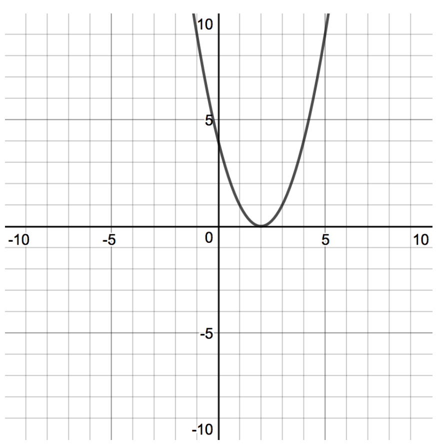
8. Is the vertex of a parabola enough information to find the vertex form of the quadratic function? If not, what other information do you need?

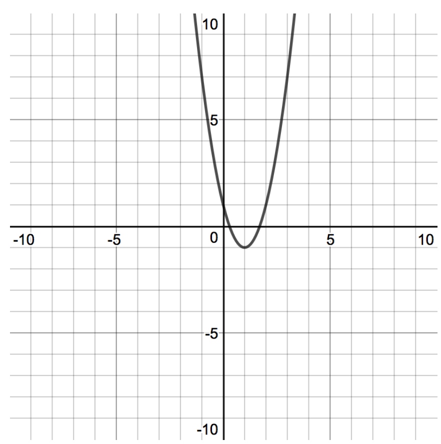
   - If you can, find two quadratic functions that have a vertex of $(0, 0)$.
   - Find the equation of a quadratic function with vertex $(0, 0)$ that also passes through $(2, 5)$.

9. Find the equation of the quadratic functions given in the graphs below:

(a) 

(c) 

(b) 

(d) 

3.4 Using the quadratic formula

Book Reference: Section 2.3

So far we have seen several ways to solve quadratic functions, factoring, graphing, and vertex form. Each of these methods gives us some insight into the behavior of quadratic functions and some understanding of the inner workings of these parabolas. In this section we will see a formula that gives us these zeros.

Quadratic formula

Given the quadratic function $f(x) = ax^2 + bx + c$ the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives the zeros of the quadratic function. This formula above is called the **quadratic formula**.

This section gives no insight into *why* this works, but that is the focus of the following extension.

Example: Use the quadratic formula to find the zeros of the function $f(x) = x^2 - 10x + 16$.

We begin by identify that $a = 1$, $b = -10$, and $c = 16$. Next we plug those values into the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(16)}}{2(1)} \\ &= \frac{10 \pm \sqrt{100 - 64}}{2} \\ &= \frac{10 \pm \sqrt{36}}{2} \\ &= \frac{10 \pm 6}{2} \\ &= \frac{10 + 6}{2} \text{ or } \frac{10 - 6}{2} \\ &= 8 \text{ or } 2 \end{aligned}$$

Use the quadratic formula to find the zeros of the following functions:

1. $p(x) = 2x^2 - 14x + 24$
2. $f(x) = 6x^2 - 7x + 2$
3. $g(x) = x^2 + 2x - 9$
4. $h(x) = 2x^2 + 3x - 4$
5. $k(x) = 2x^2 - 7x + 3$
6. $m(x) = x^2 - 2x + 2$
7. $t(x) = x^2 - 10x + 25$
8. $r(x) = 2x^2 - 8x + 9$

3.5 Extension: Proof of the quadratic formula

Book Reference: Section 2.3

For this activity we will be exploring the famous quadratic formula! You have probably seen this before and may even be able to recite it from memory

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

However, there are usually a lot of questions that remain unanswered: Where did this come from? What are a, b, and c? What information does this give me? Our goal is to answer those questions.

1. Below is an example of completing the square for reference. We will consider $3x^2 - 6x - 5$.
   - Factor out the leading coefficient: $3(x^2 - 2x) - 5$
   - Take half of our interior coefficient and square it: $\left(\frac{2}{2}\right)^2 = 1$
   - Add and subtract this inside the parentheses: $3(x^2 - 2x + 1 - 1) - 5$
   - Regroup your subtraction outside of the parentheses: $3(x^2 - 2x + 1) - 3 - 5$
   - Factor and simplify: $3(x - 1)^2 - 8$
2. Practice this procedure on the quadratics below:
   - $2x^2 + 4x - 7$
   - $2x^2 + 6x - 5$
   - $2x^2 + 3x - 7$
3. We are often interested in determining where quadratics are equal to zero. Having a quadratic in vertex form gives us a standard way to figure out where a quadratic equals zero. In part 1 we saw that
$$3x^2 - 6x - 5 = 3(x - 1)^2 - 8$$
   Our goal: Determine the values of x for which $3(x - 1)^2 - 8 = 0$.
   - Bring your constant term over:
   $$3(x - 1)^2 = 8$$
   - Divide by the leading coefficient:
   $$(x - 1)^2 = \frac{8}{3}$$
   - Take the square root:
   $$x - 1 = \pm\sqrt{\frac{8}{3}}$$
   - Add over the constant:
   $$x = 1 \pm \sqrt{\frac{8}{3}}$$
4. Practice this procedure on the vertex forms you found in question 2.

Now we are ready see where the quadratic formula comes from. Follow the notes below with your group and fill in the spots where appropriate. Our goal is to answer the following question: Find values of x for which

$$ax^2 + bx + c = 0$$

We will use completing the square just like we saw in the last activity.

1. Begin by factoring out the leading coefficient:

$$ax^2 + bx + c = a\left(\qquad\qquad\qquad\qquad \right) + c$$

2. Take half of our interior coefficient and square it:

$$\left(\frac{b}{2a}\right)^2$$

3. Add and subtract this inside the parentheses:

$$a\left(x^2 + \frac{b}{a}x + \underline{\qquad\quad} - \underline{\qquad\quad}\right) + c$$

4. Regroup your subtraction outside of the parentheses:

$$a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) - a\left(\frac{b^2}{4a^2}\right) + c$$

5. Factor and simplify:

$$a\left(x + \underline{\qquad\quad}\right)^2 - \frac{b^2 - 4ac}{4a}$$

6. What we have done so far is found the vertex form for a general quadratic. Compare this to the quadratic formula that we have seen before

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   What similarities do you notice?

Now that we have our vertex form, lets see if we can determine which values of x will make it equal to zero.

Question: For which values of x does

$$a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}=0$$

- Bring your constant term over:

$$a\left(x+\frac{b}{2a}\right)^2=$$

- Divide by the leading coefficient:

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{\qquad\qquad}$$

- Take the square root:

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

- Add over the constant:

$$x=-\frac{b}{2a}\pm$$

- Finally you combine the common denominators to get your solutions:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(a) Practice using the quadratic formula on the following functions:

- $f(x)=3x^2-7x+2$
- $g(x)=5x^2-3x+9$
- $h(x)=7x^2-14x+3$
- $p(x)=5x^2-10x+5$

Chapter 4

Unit Circle Trigonometry

In this chapter we will study circles, specifically the unit circle, and explore how we can use the unit circle to define trigonometric functions. We will try to answer the following questions:

1. What is the general equation of a circle, and what is the equation of the unit circle?
2. What are radians and how are they connected to degrees?
3. How do we define the functions $\sin(x)$ and $\cos(x)$?
4. What are some important trigonometric identities?
5. How can I use the first quadrant to determine values of $\sin(x)$ and $\cos(x)$?
6. What are the definitions of other trigonometric functions?

4.1 The unit circle, radians, and degrees

Book Reference: Section 10.1

For our next topic we are going to study Trigonometry. Specifically we are going to look at the relationship between the unit circle, angles, and right triangles. To begin we will explore the equation of circles and the unit circle.

1. Begin by opening the Desmos graph titled **Circles**.
   (a) Move the sliders for h, k, and r. With your groups discuss how changing the values moves the circle around. How would you define the point (h, k)? What is the meaning of r?
   (b) Move around your circle until you find a circle that is centered at the origin and goes through the point $(1, 0)$. What is the radius of this circle?
   (c) Write down the equation of the circle you just created:

   ### Equation of unit circle:

2. Recall that the formula for the circumference for a circle is $C = 2\pi r$ where r is the radius of the circle. Write down the circumference for the circle that you just created:

   ### Circumference of unit circle: =

3. Using the image on the back of this page mark the point $(1, 0)$ and then mark off 12 points, including $(1, 0)$, that are equally spaced around the circle.
4. Using the circumference of the circle that you found above determine the distance along the circle from $(1, 0)$ to each of the other 11 points that you found. Assume that you start at $(1, 0)$ and move in a counter-clockwise direction.

   The distances that you have just written down define the **Radian measure** of the angles that correspond to the associated points on the unit circle. Many of us are used to measuring angles in degrees so it will benefit us to talk about the relationship between degrees and radians. Explore the Desmos activity **Radians Visually**.
5. Talk to your group about how many degrees are in a circle. From our work above you should see that there are also 2π radians in a circle. Write down your guess for the relationship between these two quantities and prepare for a class discussion about what we discovered up to this point in class.

______________ radians = ______________ degrees

From our previous discussions you should have come up with the following relationship between degrees and radians:

$$\pi \text{ radians} = 180^\circ$$

6. From this relationship we can create two conversion factors. That means a factor we multiply by to convert from one to the other. Write the conversion factors in the spaces below:

   Radians to degrees:

   Degrees to radians:

7. Convert the following angle measures between radians and degrees:

   (a) π

   (b) $\frac{\pi}{3}$

   (c) $\frac{5\pi}{4}$

   (d) $\frac{11\pi}{6}$

   (e) 120°

   (f) 30°

   (g) 270°

   (h) 300°

What happens if we look at negative angle measures? Angles more than 2π radians? More than 360°?

8. Convert the following angle measures between radians and degrees. Then determine the corresponding points on the unit circle.

   (a) $-\pi$

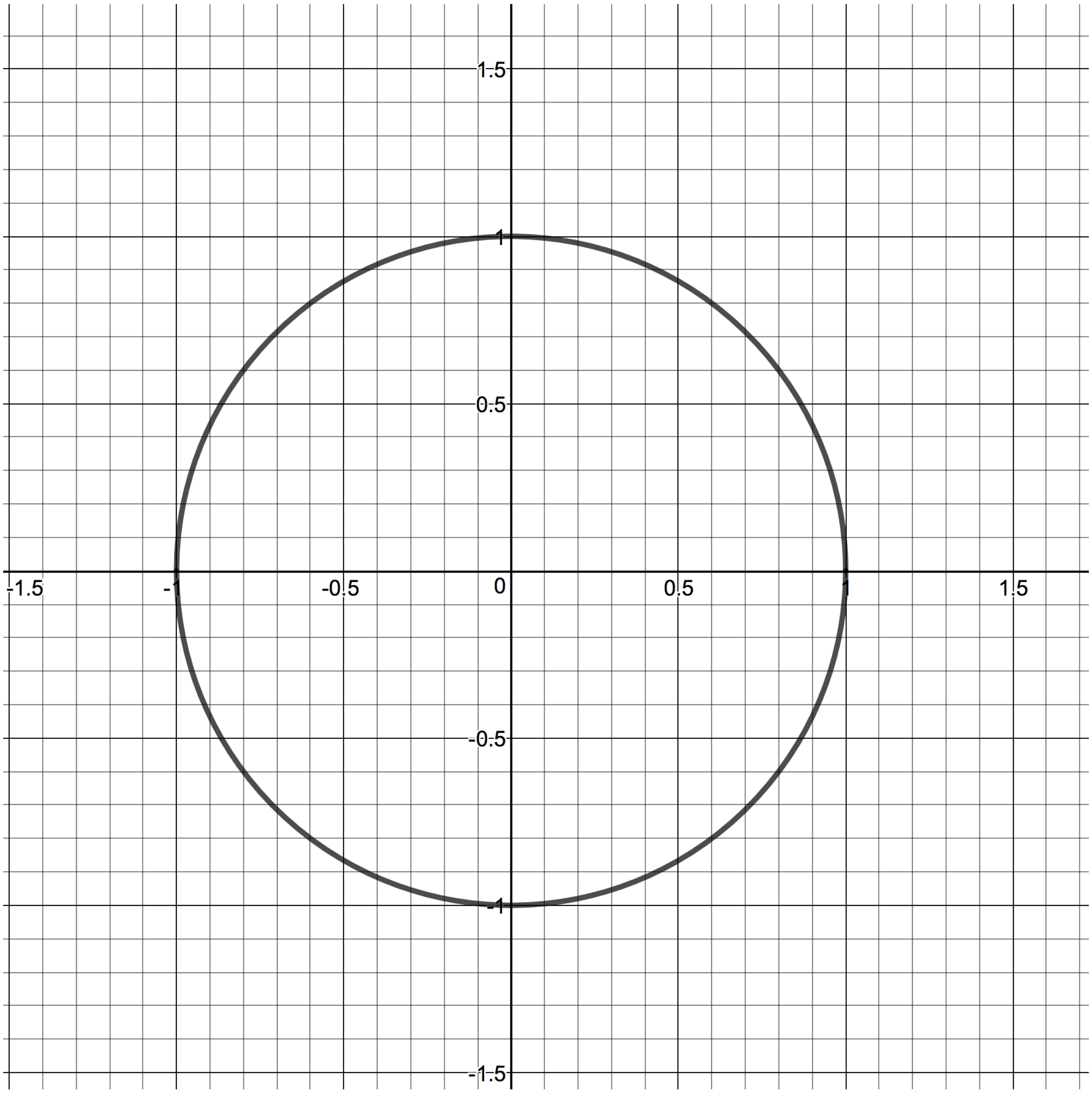
   (b) $\frac{17\pi}{3}$

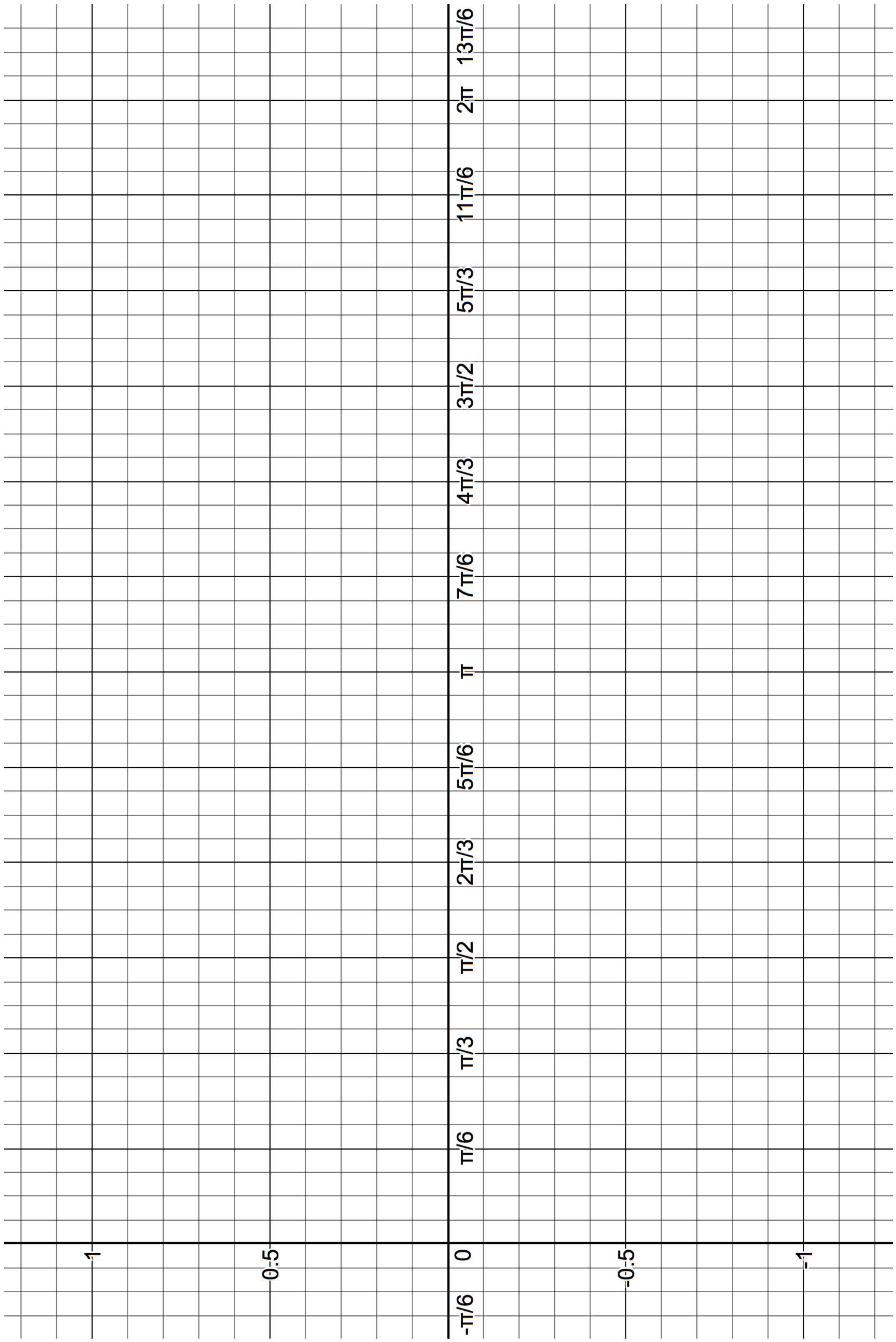
   (c) $\frac{-15\pi}{4}$

   (d) -120°

   (e) 480°

   (f) -540°


1.5
1
0.5
-1.5
-1
-0.5
0
0.5
1
1.5
-0.5
-1
-1.5


1
0.5
0
-0.5
-1
-π/6
π/6
π/3
π/2
2π/3
5π/6
π
7π/6
4π/3
3π/2
5π/3
11π/6
2π
13π/6

4.2 The functions $\sin(x)$ and $\cos(x)$

Book Reference: Section 10.2

For our next exploration we are going to define two function based on the x and y coordinates of the points on the unit circle.

1. Open the Desmos graph titled **Unit Circle**. Use the coordinates of the points to fill out the tables below. For each of the 12 points marked on the unit circle record the radian measure of their defined angle and then either the x or y coordinate of that point.

Radians	x-coordinate
0	
$\frac{\pi}{6}$	
$\frac{11\pi}{6}$	

Radians	y-coordinate
0	
$\frac{\pi}{6}$	
$\frac{11\pi}{6}$	

2. Using the tables you created above and either graph paper or Desmos sketch the graphs of the functions defined by these tables. Label the graph created by the x-coordinates as $\cos(\theta)$ and the graph created by the y-coordinates as $\sin(\theta)$.

3. Determine the following values of the $\sin(\theta)$ and $\cos(\theta)$.

   (a) $\sin(\pi)$

   (b) $\sin\left(\frac{5\pi}{2}\right)$

   (c) $\sin(-\frac{5\pi}{4})$

   (d) $\cos\left(\frac{7\pi}{3}\right)$

   (e) $\cos\left(\frac{-5\pi}{4}\right)$

   (f) $\cos\left(\frac{11\pi}{2}\right)$

4. Based on these definitions the points on the unit circle have the form $(\cos(\theta), \sin(\theta))$. Earlier we found that the equation of the unit circle was $x^2 + y^2 = 1$. Use these two facts to write down a trigonometric identity:

Fundamental Trigonometric Identitiy

4.3 The first quadrant and reference angles

Book Reference: Section 10.2

When we study trigonometry we will see that we can relate any angle to one in the first quadrant of the unit circle. The most common angles that we see are listed in the table below. Use our Desmos graph and what we have already seen to fill out the table below.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\theta)$					
$\cos(\theta)$					

1. Use your calculator to determine the decimal approximations for $\frac{\sqrt{3}}{2}$ and $\frac{\sqrt{2}}{2}$. Replace any decimals in the table above with these fractions. Do you notice any patterns in the table above? Can you come up with an easy to way to remember this table? Prepare your method for a group discussion.

2. Choose angles that fall into quadrant II and study how the values of $\sin(\theta)$ and $\cos(\theta)$ relate to the values in the first quadrant. How can you use the table above to evaluate $\sin(\theta)$ and $\cos(\theta)$ for these angles? Go through the same process for quadrants III and IV. Write down general formulas to find reference angles for angles in each quadrant. The first two are given here:

   - **Quadrant I**
     Angle: θ
     Reference angle: θ
   - **Quadrant II**
     Angle: θ
     Reference angle: $\pi - \theta$
   - **Quadrant III**
     Angle: θ
     Reference angle:
   - **Quadrant IV**
     Angle: θ
     Reference angle:

3. Consider the angle $\theta = \frac{5\pi}{3}$.
   (a) Which quadrant is θ in?
   (b) Which angle in quadrant I does θ relate to?
   (c) Is $\sin(\theta)$ positive or negative?
   (d) Find $\sin(\theta)$ and $\cos(\theta)$.

4. Use the first quadrant table above to determine the **exact** values for the following. Write each of the values and identify the quadrant that each angle lies in.
   (a) $\sin\left(\frac{5\pi}{6}\right)$
   (b) $\cos\left(\frac{7\pi}{6}\right)$
   (c) $\sin\left(\frac{3\pi}{2}\right)$
   (d) $\cos\left(\frac{3\pi}{4}\right)$
   (e) $\sin\left(\pi\right)$
   (f) $\cos\left(\frac{2\pi}{3}\right)$
   (g) $\sin\left(\frac{4\pi}{3}\right)$
   (h) $\cos\left(\frac{7\pi}{4}\right)$

4.4 Other trigonometric functions

Book Reference: Section 10.3

Trigonometric functions

There are four other trigonometric functions that we often use, these functions are called **tangent**, **cotangent**, **secant**, and **cosecant**. The functions are defined as follows:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

1. Use the definitions above to find the exact values of the following:

   (a) $\tan\left(\frac{5\pi}{6}\right)$

   (b) $\cot\left(\frac{7\pi}{6}\right)$

   (c) $\sec\left(\frac{3\pi}{2}\right)$

   (d) $\csc\left(\frac{3\pi}{4}\right)$

   (e) $\tan(\pi)$

   (f) $\cot\left(\frac{2\pi}{3}\right)$

   (g) $\sec\left(\frac{4\pi}{3}\right)$

   (h) $\csc\left(\frac{7\pi}{4}\right)$

2. Determine all of the angles where the functions $\tan(\theta)$ and $\sec(\theta)$ are undefined. Why are they undefined at these angles?

3. Determine all of the angles where the functions $\cot(\theta)$ and $\csc(\theta)$ are undefined. Why are they undefined at these angles?

4. For each of the four trigonometric functions above try to sketch their graphs without using Desmos. Use your knowledge of their domains and plot several more points to sketch your graphs.

5. Verify that your graphs are correct by plotting them in Desmos.

4.5 Inverse Trigonometry

In chapter 1 we learned about inverse functions and used the horizontal line test to determine if a function was invertible. Consider the functions $\sin(x)$ and $\tan(x)$ on Desmos or on the attached handout.

1. Are the functions $\sin(x)$ and $\tan(x)$ invertible? Why?
2. Since the functions we are considering are periodic, we want to restrict the domain of these functions so that they pass the horizontal line test. Find a restricted domain for each function so that it passes the horizontal line test.
3. Check: Is every possible output for these functions represented in your restricted domain? If not, choose a new restricted domain so that every output is represented. Prepare for an in class discussion.
4. We know that the graphs of inverse functions are symmetric about the line $y = x$. Considering the graphs of $\sin(x)$ and $\tan(x)$ on our restricted domain, sketch a graph of their inverse function on the axes provided by reflecting the across the line $y = x$.

   Each of the six trigonometric functions has an inverse defined similarly to the inverses of sine and tangent but these two will be our primary focus. Formal definitions are given below:

   Inverse trigonometric functions

   - The function **arcsin(x)** is the inverse of the sine function. It has domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We can define it as follows

   $$y = \arcsin(x) \text{ if and only if } x = \sin(y) \text{ where } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

   - The function **arctan(x)** is the inverse of the tangent function. It has domain $[-\infty, \infty]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We can define it as follows

   $$y = \arctan(x) \text{ if and only if } x = \tan(y) \text{ where } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

5. Use the definition of arcsin(x) and arctan(x) along with their graphs to find the exact values of the following quantities:

   (a) $\arcsin(0)$
   (b) $\arcsin(1)$
   (c) $\arcsin\left(-\frac{1}{2}\right)$
   (d) $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
   (e) $\arctan(0)$
   (f) $\arctan(1)$
   (g) $\arctan\left(-\sqrt{3}\right)$
   (h) $\arctan\left(\frac{\sqrt{3}}{3}\right)$

   Important note: Sometimes we use the notation $\sin^{-1}(x)$ for arcsin(x) and $\tan^{-1}(x)$ for arctan(x). It is easy to confuse what the -1 means so we will typically use arcsin(x) and arctan(x), but you may see both notations out in the wild.

Example: Solving trigonometric functions

One of the uses for inverse trigonometric functions is they give us a tool to solve equations that have trigonometric functions in them. For example, if we have the equation

$$2\sin(\theta) = 1$$

We can rewrite this equation to get

$$\sin(\theta) = \frac{1}{2}$$

and then using our knowledge of inverse trigonometry we see that

$$\theta = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

But wait! We know that there are more answers than this. Since we restricted the domain on our trig functions to define the inverse trig functions we know have to use our understanding of trigonometry to get all of the solutions. Since $\sin(x) > 0$ in Quadrant I and Quadrant II we will look for solutions in both.

- Quadrant I. We have our first solution from the arcsin function as $\frac{\pi}{6}$. We can also have any full rotation around the circle so we get solutions of the form $\frac{\pi}{6} + 2\pi k$ for any integer k.
- Quadrant II. Since one solution is $\theta = \frac{\pi}{6}$ in the first quadrant we know that a solution in Quadrant II will be $\pi - \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. Similarly we get all of the full rotations around the circle. Our complete solutions for Quadrant II have the form $\frac{5\pi}{6} + 2\pi k$ for integers k.

Thus the solutions to the equation $2\sin(\theta) = 1$ are $\theta = \frac{\pi}{6} + 2\pi k$ or $\theta = \frac{5\pi}{6} + 2\pi k$ for integers k.

6. Use the example above to solve the following trigonometric equations.

   (a) $2\sin(\theta) = \sqrt{2}$

   (b) $3\sin(\theta) - 1 = 0$

   (c) $\tan(\theta) = \sqrt{3}$

   (d) $5\tan(\theta) - 5 = 0$

7. Challenge: Solve the following trigonometric equations

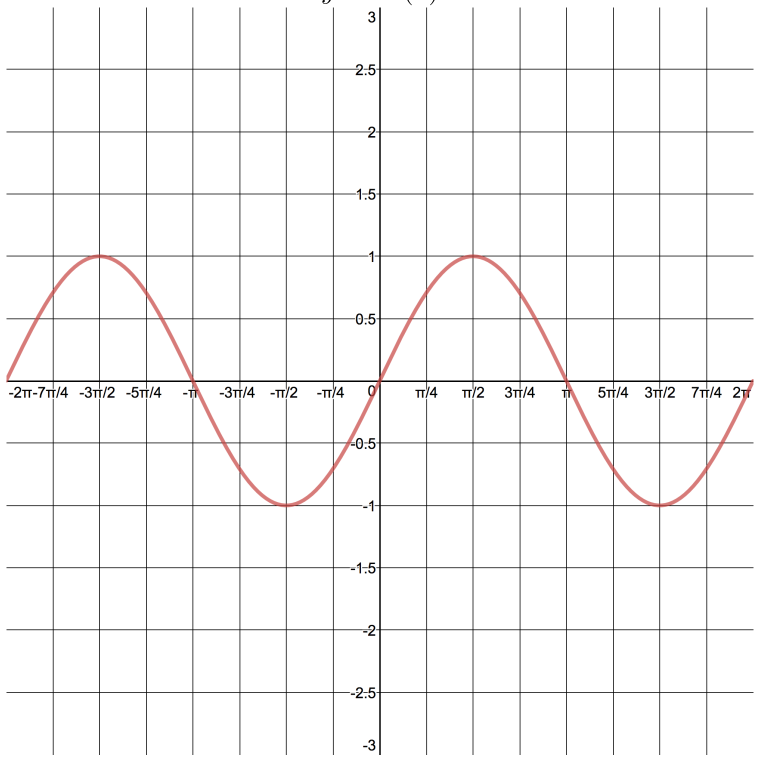
   (a) $5\sin(2\theta) - 3 = 0$

   (b) $3\tan(5\theta - 1) = \sqrt{3}$

   (c) $\sin^2(\theta) = \sin(\theta)$

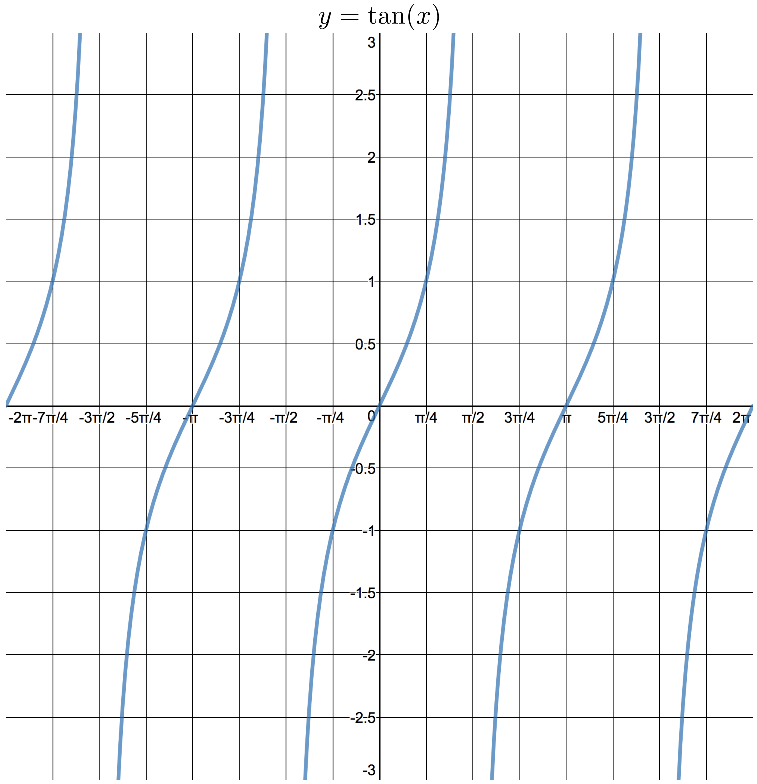
   (d) $\tan^2(\theta) + \tan(\theta) = 2$

$y = \sin(x)$



3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-2π -7π/4 -3π/2 -5π/4 -π -3π/4 -π/2 -π/4 π/4 π/2 3π/4 π 5π/4 3π/2 7π/4 2π

$y = \tan(x)$



3
2.5
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0.5
0
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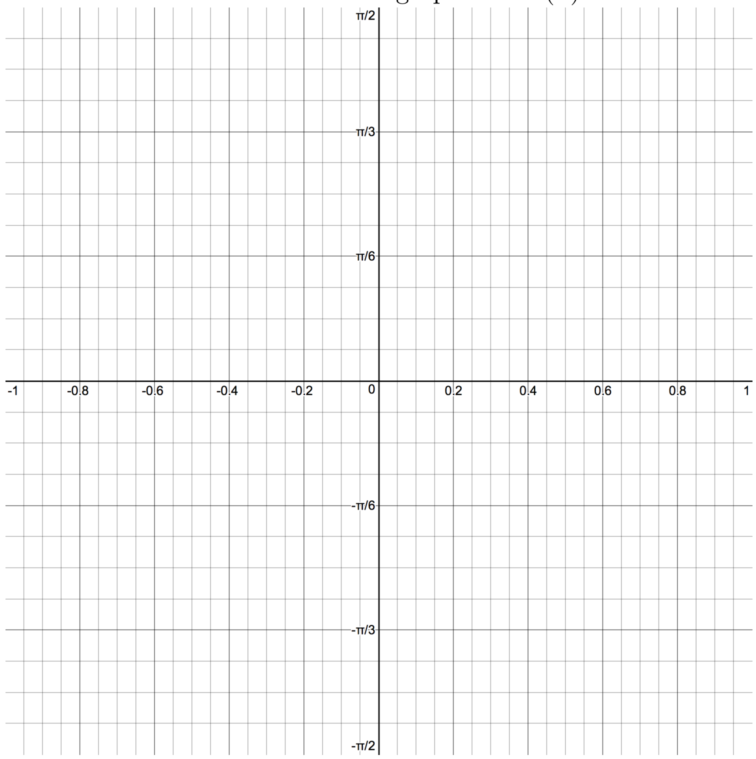
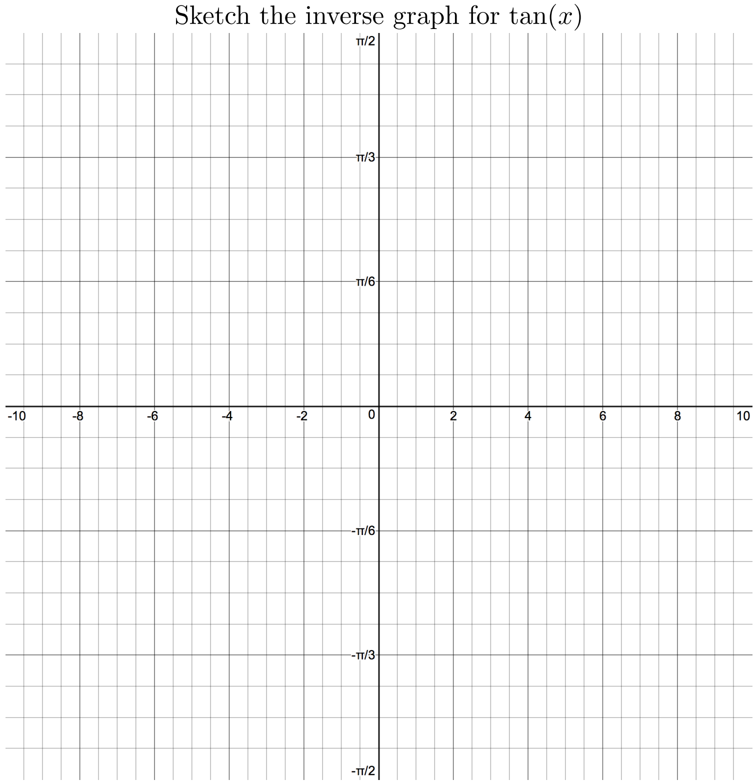
-1
-1.5
-2
-2.5
-3
-2π -7π/4 -3π/2 -5π/4 -π -3π/4 -π/2 -π/4 π/4 π/2 3π/4 π 5π/4 3π/2 7π/4 2π

Sketch the inverse graph for $\sin(x)$



Sketch the inverse graph for $\tan(x)$



Chapter 5

Exponential Functions

In this chapter we introduce the reader to exponential functions. We start with an refresh on our knowledge of exponent rules and then look to answer the following questions:

1. What is the basic form of an exponential function? What information do we get from the different parts?
2. How can we find the equation of an exponential function given two points?
3. What types of scenarios can we model with exponential functions?
4. What difficulties do we run into when we try to solve exponential functions?

5.1 Exponent rule review

Book Reference: Section 6.1

Rules for exponents:

- When mutliplying numbers with the same base we can add the exponents.

  $a^m \cdot a^n = a^{m+n}$ For example: $2^5 \cdot 2^4 = 2^{5+4} = 2^9$

- When dividing numbers with the same base we can subtract the exponents.

  $\frac{a^m}{a^n} = a^{m-n}$ For example: $\frac{2^5}{2^4} = 2^{5-4} = 2^1$

- When we have a power to a power we can multiply the exponents.

  $(a^m)^n = a^{mn}$ For example: $(2^4)^3 = 2^{4\cdot 3} = 2^{12}$

- A power of a product is the product of the powers.

  $(ab)^n = a^n b^n$ For example: $(2 \cdot 3)^5 = 2^5 \cdot 3^5$

- A power of a quotient is the quotient of the powers.

  $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ For example: $\left(\frac{2}{3}\right)^5 = \frac{2^5}{3^5}$

Definitions of negative and fractional powers:

- Any nonzero number to the 0^{th} power is 1.

  $a^0 = 1$ For example: $7^0 = 1$

- A negative exponent represents the reciprocal.

  $a^{-n} = \frac{1}{a^n}$ For example: $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

- The denominators in fractional exponents represent roots.

  $a^{1/n} = \sqrt[n]{a}$ For example: $8^{1/3} = \sqrt[3]{8} = 2$

- The numerators in fractional exponents represent powers.

  $a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$ For example: $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = \sqrt[3]{8^2} = 4$

Exponent rule practice

Use the exponent rules on the previous page to simplify the following:

1. $(3^5)(3^8)(3^{-4})$
2. -3^4
3. $(-2)^{-4}$
4. $27^{2/3}$
5. $(5^{-2})^4$
6. $5^{-2} + \left(\frac{1}{5}\right)^2$
7. $\left(\frac{12}{5}\right)^{-2}$
8. $\dfrac{-7^{-1}}{3^{-2}}$
9. $\dfrac{(9^5)^2}{9^8}$
10. $\left(5\pi^2 + 7e\right)^0$
11. $(2a^3b^2)(3ab^4)^3$
12. $\dfrac{(3x+9)^3}{9}$
13. $\left(\dfrac{x}{y}\right)^3 \left(\dfrac{y^2x}{z}\right)^4$
14. $\dfrac{6st^{-4}}{2s^{-2}t^2}$
15. $\left(\dfrac{y}{3z^2}\right)^{-2}$
16. $\left(\dfrac{x^2y^{-4}}{z^{-8}}\right)^{3/2}$
17. $\left(\dfrac{2}{5}\right)^2 + 3(4)^{-1/2}$
18. $\dfrac{27x^{-3}y^5}{9x^{-4}y^7}$
19. $\dfrac{(-3)^2a^5(bc)^2}{(-2)^3a^2b^3c^4}$
20. $\dfrac{(x+h)^3 - x^3}{h}$

5.2 The structure of exponential functions

Book Reference: Section 6.1

Exponential function

An **exponential function** is one that can be written in the form

$$f(t) = A \cdot b^t$$

for some real numbers A and b such that $A \neq 0$, $b > 0$ and $b \neq 1$.

1. Explore the exponential function on Desmos via the link provided in PLATO. What information does A give you about the exponential function? What information does b give you about the exponential function?

2. The two data points $(2, 126)$ and $(3, 756)$ are on the graph of an exponential function. Using the model $f(t) = Ab^t$ those points give us the following information:

$$126 = Ab^2 \qquad\qquad 756 = Ab^3$$

Use this information to first find the value of b and then once you have b find the value for A. Check your results on Desmos.

3. Use the same process to find an exponential function of the form $f(t) = Ab^t$ that connects the data points $(3, 4.5)$ $(5, 10.125)$.

4. The following question is from section 6.5 #39.

According to Facebook, the number of active users of Facebook has grown significantly since its initial launch from a Harvard dorm room in February 2004. The chart below has the approximate number $U(x)$ of active users, in millions, x months after February 2004. For example, the first entry $(10, 1)$ means that there were 1 million active users in December 2004 and the last entry $(77, 500)$ means that there were 500 million active users in July 2010.

Month x	10	22	34	38	44	54	59	60	62	65	67	70	72	77
Active Users in millions	1	5.5	12	20	50	100	150	175	200	250	300	350	400	500

With the help of your classmates find a model for this data. Use the Desmos graph **Facebook Data** to check for accuracy.

5. Compute the average rate of change of the following exponential functions on the given intervals. Use the graphs of each function to make sense of this information.

(a) $f(x) = 2^x$ on $[0, 1]$

(b) $g(x) = 5 \cdot 3^x$ on $[1, 2]$

(c) $h(x) = \left(\frac{1}{2}\right)^x$ on $[0, 2]$

(d) $p(x) = 8 \cdot \left(\frac{2}{3}\right)^x$ on $[1, 3]$

5.3 Modeling Activity: Chessboard Fable

Legend has it that, when chess was invented in ancient times, the emperor was so enchanted that they said to the inventor, "Name your reward." The inventor thought for a moment and pulled a single grain of rice out of her pocket to show the emperor. "If you please, emperor, match my one grain of rice today on the first square of my chessboard," said the inventor. "Then, tomorrow place two grains on the second square, four grains on the third square the day after, eight grains on the fourth square the day after, and so on." The emperor gladly agreed, thinking she was a fool for asking for a few grains of rice when she could have had gold or jewels.

Little did the emperor know they were about to look like a fool. Why?

Explore: Fill in the rest of the table to see why the emperor looks like a fool.

Square	Grains on this square	Total grain on the board up to this point	Formula for total grains
1	1	1	1 (on board) + 1 (in hand) = 2
2	2	1 + 2 = 3	
3			
4			
5			
64			

1. Complete the table above to gather some data on the amount of rice that is being given as a reward. Try to make sense of the amount of rice by comparing the quantity to something else.
2. Write down a function $f(x)$ giving the total amount of rice after rice is placed on square x.
3. How much rice does the inventor have after rice is placed on the 50^{th} square of the board?
4. When will the inventor reach $1,000,000,000,000$ grains of rice? Describe the difficulties you are having with this computation.
5. Determine how long it will take to get the following amounts of rice:

   2 grains 16 grains 152 grains 1,024 grains 20,171 grains 1,048,576 grains

6. What is the difficulty in doing these computations? Can we answer them graphically? Can we answer them analytically?

5.4 Modeling Activity: Student loan growth

This modeling activity will show us how exponential functions model something that has a big effect on most of our lives: student loans.

Materials: pen, paper, Desmos.

Goal: Understand how student loan growth can be modeled by exponential functions.

Scenario: At the end of your time at Westfield State you may find yourself with $30,000 in student loans that have an Annual Percentage Yield (APY) of 5.2%. These numbers are close to the average student loan debt and the current interest rate for 2018. This exploration will allow us to see how these loans grow over time.

1. The APY on a loan gives you a normalized way to compare interest rates that takes into account the compounding of interest. It simplifies many of our computations as we can start thinking about growth on a yearly basis, rather than a daily, or monthly basis. If you are unfamiliar with APY take a minute to do some research and discuss with your group.

2. Assuming that you make no payments, how much money will you owe on your student loans after 1 year?

3. Create a table in your notebook like the one shown below that gives the amount you will owe on your loans after 1, 2, 3, 4, ..., 10 years. For each year include the amount owed and the way you computed that amount.

Year	amount owed	Formula
0	$30,000	
1		
2		
3		

4. Write down an exponential function $L(t)$ that gives the amount owed on the loan after t years if no payments are made. Use this formula to compute how much you would owe on the loan after 15, 20, 25, and 30 years.

5. How long will it take for your amount owed to double? You can use Desmos to come up with an approximate answer. Discuss with your group how to determine an exact answer.

6. How long will it take for your amount owed to grow to $80,000? You can use Desmos to come up with an approximate answer. Discuss with your group how to determine an exact answer.

Chapter 6

Logarithmic Functions

In this chapter we are going to explore a new function called a logarithm which is the inverse of the exponential function. We are interested in answering the following questions:

1. Why do I want to use a logarithm? What types of questions does it help me answer?
2. What is the definition of a logarithm and how does it relate to the exponential function?
3. What are the various properties of logarithms?
4. How can I use logarithms to solve exponential equations?
5. How do I solve logarithmic equations?
6. What are the various scenarios that can be modeled with exponential and logarithmic equations?

6.1 The definition of logarithm

Book Reference: Section 6.1 and 6.2

We have explored the connection between logarithms and exponential functions and saw that logarithms were a tool that we could use to determine certain exponents. To explore this further we need a formal definition of a logarithm, which is given below:

Logarithmic function

The function $f(x) = \log_b(x)$ is define as

$$\log_b(x) = y \qquad \text{if and only if} \qquad b^y = x$$

The result of this is that $\log_b(x)$ is an exponent, it is the power of b that gives x as a result.

Example: Consider the quantity

$$\log_2(8)$$

If we let $\log_2(8) = y$ we can rewrite the equation as $2^y = 8$. We ask ourselves: What power of 2 makes 8? Since $2^3 = 8$ we get

$$\log_2(8) = 3.$$

Practice using the definition of the logarithm by computing the following without a calculator

1. $$\log_2(16)$$
2. $$\log_3(27)$$
3. $$\log_5\left(\frac{1}{5}\right)$$
4. $$\log_{1/2}(4)$$
5. $$\log_7\left(7^{15}\right)$$
6. $$\log_{2/3}\left(\frac{8}{27}\right)$$
7. $$\log_{122}(1)$$
8. $$4^{\log_4(17)}$$

Based on our work above we should be able to identify the logarithm as the inverse of the exponential function. Think about what it means for functions to be inverses and complete the properties below:

- For all x we have $\log_b(b^x) =$ _______ and
- For all $x > 0$ we have $b^{\log_b(x)} =$ _______.

6.2 Logarithm Rules

6.2.1 The rule for products

Book Reference: Section 6.2

Exploration 1: Rule for products

For this exploration we want to determine a way to rewrite the the logarithm of a product.

$$\log_b(uw) = ???$$

In order to determine this rule we are going to let:

$$\log_b(uw) = a \qquad \log_b(u) = c \qquad \log_b(w) = d$$

1. Rewrite a, c, and d using the definition of logarithms.
2. When you have three equations use some algebraic techniques to combine them into one equation.
3. Use exponent rules to come up with a relationship between a, c, and d.
4. Write a conclusion for the rule for products.

Rule for Products

$\log_b(uw) =$

Practice using the rule for products:

5. Evaluate: $\log_2\left(\frac{8}{3}\right) + \log_2(6)$
6. Evaluate: $\log_3\left(\frac{15}{4}\right) + \log_3\left(\frac{2}{5}\right) + \log_3(18)$
7. Write without an exponent: $\log_3(t^3)$
8. Solve for x: $\log_3(x) + \log_3(x+2) = 1$

6.2.2 The rule for quotients

Book Reference: Section 6.2

Exploration 2: Rule for quotients

For this exploration we want to determine a way to rewrite the the logarithm of a quotient.

$$\log_b\left(\frac{u}{w}\right) = ???$$

In order to determine this rule we are going to let:

$$\log_b\left(\frac{u}{w}\right) = a \qquad \log_b(u) = c \qquad \log_b(w) = d$$

1. Rewrite a, c, and d using the definition of logarithms.
2. When you have three equations use some algebraic techniques to combine them into one equation.
3. Use exponent rules to come up with a relationship between a, c, and d.
4. Write a conclusion for the rule for quotients.

Rule for Quotients

$\log_b\left(\frac{u}{w}\right) =$

Practice using the rule for quotients:

5. Evaluate: $\log_2(160) - \log_2(5)$
6. Evaluate: $\log_3(100) - \log_3(18) - \log_3(50)$
7. Expand: $\log_4\left(\frac{x}{yz}\right)$
8. Solve for x: $\log_5(x^2 - 1) - \log_5(x - 1) = 2$

6.2.3 The rule for powers

Book Reference: Section 6.2

Exploration 3: Rule for powers

For this exploration we want to determine a way to rewrite the the logarithm of a power.

$$\log_b(u^w) = ???$$

In order to determine this rule we are going to let:

$$\log_b(u^w) = a \qquad \log_b(u) = c$$

1. Rewrite a and c using the definition of logarithms.
2. When you have the two equations use some algebraic techniques to combine them into one equation.
3. Use exponent rules to come up with a relationship between a and c.
4. Write a conclusion for the rule for powers.

Rule for powers

$$\log_b\left(u^w\right) =$$

Practice using the rule for powers:

5. Evaluate:
$$\log_2(4^{23})$$
6. Evaluate:
$$\log\left(\frac{1}{\sqrt{1000}}\right)$$
7. Expand:
$$\log\left(\frac{x^3}{y^4 z^5}\right)$$
8. Expand:
$$\log_2(xy)^{10}$$

6.2.4 The change of base formula

Book Reference: Section 6.2

Exploration 4: Changing bases of logarithms

For this exploration we want to determine a relationship between logarithms with different bases. In general we would like to know how $\log_a(x)$ is related to $\log_b(x)$.

In order to determine this relationship I want you to start with the product

$$\log_a(x) \cdot \log_b(a)$$

1. Use the rule of powers that we have discovered to rewrite the product above.
2. Use the definition of logarithms to simplify the expression you have just created.
3. Write down the relationship below:

$$\log_a(x) \cdot \log_b(a) =$$

4. Divide your equation by $\log_b(a)$ to find the change of base formula.

Change of base formula

$\log_a(x) =$

Practice using the change of base formula:

5. Convert to log: $\log_7(22)$
6. Convert to log: $\log_{13}(25)$
7. Convert to ln: $\log_{11}(4)$
8. Convert to ln: $\log_6(8)$

6.2.5 Logarithm summary sheet

Book Reference: Sections 6.1 and 6.2

What is a logarithm?

Logarithms are the tool we use to answer questions like: For what values of x does $3^x = 152$? They are very powerful as they allow us to undo exponentiation in the same way that division undoes multiplication, or subtraction undoes addition.

Definition: We define logarithms in the following way

$$\log_b(x) = y \text{ if and only if } b^y = x$$

When you see $\log_b(x)$ you should be asking yourself a question: "What power do I have to raise b to in order to get x?" So a logarithm is an exponent.

Example: From our original question above we get:

$$3^x = 152 \text{ if and only if } \log_3(152) = x$$

and so the precise answer to our question is $\log_3(152)$. In other words "What power do I have to raise 3 to in order to get 152?" In this case $\log_3(152)$ is the exponent you have to put on 3 to get back 152.

Conventions: There are two special logarithms that we use a lot and are given their own notation:

$$\ln(x) = \log_e(x) \qquad \text{and} \qquad \log(x) = \log_{10}(x)$$

Through our explorations we have seen the strong connection between logarithms, exponents and exponential functions. As such the rules that we have for logarithms are very similar to those we have for exponents. The rules we discovered are summarized below.

Properties of Logarithms

- **Inverse property:** For all x we have $\log_b(b^x) = x$ and for all $x > 0$ we have $b^{\log_b(x)} = x$.
- **Rule for Products:** $\log_b(uw) = \log_b(u) + \log_b(w)$
- **Rule for Quotients:** $\log_b\left(\frac{u}{w}\right) = \log_b(u) - \log_b(w)$
- **Rule for Powers:** $\log_b(u^w) = w\log_b(u)$
- **Change of Base:** $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$

Practice using the properties of logarithms

Book Reference: Section 6.2

Expand the the following using the properties of logarithms and simplify:

1. $\log_2\left(\frac{8}{x}\right)$
2. $\log(10x^2)$
3. $\ln\left(\frac{3}{ex}\right)^2$
4. $\log\left(\sqrt[3]{\frac{100x^2}{yz^5}}\right)$
5. $\log_2\left(\frac{128}{x^2-4}\right)$
6. $\log(1000x^3y^5)$
7. $\log_6\left(\frac{216}{x^3y}\right)^4$
8. $\ln\left(\frac{\sqrt{z}}{xy}\right)$

Use the properties of logarithms to write the following as a single logarithm:

9. $4\ln(x) + 2\ln(y)$
10. $\log_2(x) + \log_2(y) - \log_2(z)$
11. $3 - \log(x)$
12. $2\ln(x) - 3\ln(y) - 4\ln(z)$
13. $\log_3(x) - 2\log_3(y)$
14. $\log_2(x) + \log_4(x-1)$
15. $\ln(x) + \frac{1}{2}$
16. $\log_7(x) + \log_7(x-3) - 2$

6.3 Solving logarithmic equations

Book Reference: Section 6.3

Now that we have studied the properties of logarithms we are going to see how we can use these properties to solve logarithmic functions. For example, we will solve the equation

$$4 - 3\log_2(x+1) = -8$$

$4 - 3\log_2(x+1) = -8$	
$-3\log_2(x+1) = -12$	Subtract 4 from both sides
$\log_2(x+1) = 4$	Divide both sides by -3
$x + 1 = 2^4$	Use the definition of logarithm (snail!)
$x = 15$	Subtract 1 from both sides

Steps for solving equations with involving logarithms

1. Isolate the logarithmic term
   - Look for terms that are of the form $\log_b(x)$ and try to get those terms by themselves.
   - Use the properties of logarithms to create a single logarithmic term.
2. Use the definition of logarithm (remember the snail!).
   - This will allow us to remove the logarithm and access the x terms we are interested in.
3. Use your known algebraic techniques to find a solution.

Example:
Solve the following equation:

$$12 + \log_3(x+6) + \log_3(x) = 15$$

$12 + \log_3(x+6) + \log_3(x) = 15$	
$\log_3(x+6) + \log_3(x) = 3$	Isolate the logarithm
$\log_3((x+6)x) = 3$	Combine logarithms using product rule
$x^2 + 6x = 27$	Use the definition of logarithm
$x^2 + 6x - 27 = 0$	Recognize a quadratic equation
$(x+9)(x-3) = 0$	Factor
$x = -9$ or $x = 3$	Identify the solutions
-9 is not in the domain so 3 is the only solution	Check that solutions make sense

Solve the following logarithmic equations.

1. $\log_2(x) = 7$
2. $\log_5(x) = 2$
3. $\log(5x) = 3$
4. $\log_5(2x - 1) = 2$
5. $\log_3(x) + \log_3(2x) = 1$
6. $\log_2(x) - \log_2(7) = 2$
7. $\ln(8) - \ln(x) + 5 = 12$
8. $\ln(7x + 15) - 9 = 46$
9. $\log_2(x) + \log_2(x + 15) = 4$
10. $\log_5(x) + \log_5(x + 20) = 3$

6.4 Using logarithms to solve exponential equations

Book Reference: Section 6.3

Now that we have our logarithm rules we are going to explore how we can use logarithms to solve exponential functions. We have seen previously an interest in solving the following equation:

$$2^x = 153$$

We could arrive at a solution directly through the definition of logarithms but lets take a different route:

$$\begin{aligned} 2^x &= 153 & & \\ \ln(2^x) &= \ln(153) & & \text{Take the natural log of both sides} \\ x \cdot \ln(2) &= \ln(153) & & \text{Rule of powers} \\ x &= \frac{\ln(153)}{\ln(2)} & & \text{Use algebra} \end{aligned}$$

Steps for solving an equation involving exponential functions

1. Isolate the exponential function
   - Look for terms that are of the form b^x and try to get those terms by themselves.
   - We do this because we are looking to use logarithms to get the variable not as an exponent.
2. Take the natural log of both sides of the equation and use the rule of powers.
   - This will allow us to rewrite the equation where the variable is not in the exponent.
3. Use your known algebraic techniques to find a solution.

Example:
Solve the following equation:

$$7 \cdot 3^x - 15 = 34$$

$$\begin{aligned} 7 \cdot 3^x - 15 &= 34 & & \\ 7 \cdot 3^x &= 49 & & \text{Isolate the exponential} \\ 3^x &= 7 & & \text{Isolate the exponential} \\ \ln(3^x) &= \ln(7) & & \text{Use natural log} \\ x \ln(3) &= \ln(7) & & \text{Rule of powers} \\ x &= \frac{\ln(7)}{\ln(3)} & & \end{aligned}$$

Use logarithms to solve the following exponential equations.

1. $4^x = 212$

2. $6^x = 501$

3. $7^{2x} = 59$

4. $9^{4x+1} = 15$

5. $100 \cdot (2.3)^x = 517$

6. $2000 = 1000 \cdot 3^{-.01t}$

7. $2^{3x} = 16^{1-x}$

8. $9 \cdot 3^x = 4^{5x}$

9. $75 = \frac{100}{1+3e^{-2t}}$

10. $80 = 50 + 8e^{-4x}$

Challenge questions: Use logarithms to solve the following equations.

Hint: Try to relate these to quadratic equations.

11. $25^x = 5^x + 6$

12. $\frac{e^x - e^{-x}}{2} = 5$

6.5 Modeling with exponential functions and logarithms

Book Reference: Section 6.3

Here we are going to see various scenarios that we can model using exponential and logarithmic functions. Use the skills we have built up over this chapter to set up the exponential and/or logarithmic equations that model the the scenarios presented in each question, then answer the proposed questions.

1. Suppose you are an ecologist studying the reintroduction of salamanders in a wooded area of New England. You are wondering how they are affected by the predators in the area and are observing their population over time. After you gather you data you find that you can model their population growth by the following formula:
$$P(t) = 50e^{.00145t}$$
where t is the number of months after the study began.
   (a) How many salamanders were originally reintroduced into the area?
   (b) When will the population of salamanders double?
2. Suppose that you have a portfolio of \$5000 in investments that have an annual yield of 6%. The growth of your portfolio can be modeled by the function
$$S(t) = 5000(1.06)^t$$
where t is measured in years. If you want to use your savings for a \$25,000 trip to Italy, how long will you have to wait until you have enough money?
3. According to Newton's Law of Cooling the temperature of coffee, in degrees Fahrenheit, t minutes after it is served is given by
$$T(t) = 70 + 90e^{-.01t}$$
   (a) Find and interpret $T(0)$.
   (b) When will the coffee be $100°$?
   (c) If you let the coffee sit for a week, approximately what temperature will it be?
4. We are going to consider the reintroduction of wolves to Yellowstone National Park. According to the National Park Service, the wolf population in Yellowstone National Park was 52 in 1996 and 118 in 1999. Using these data, find a function of the form $N(t) = N_0e^{kt}$ which models the number of wolves t years after 1996. (Use t = 0 to represent the year 1996. Also, round your value of k to four decimal places.) According to the model, how many wolves were in Yellowstone in 2002? (The recorded number is 272.)
5. You are growing bacteria in a dish that has a carrying capacity of 10,000 bacteria. You began by introducing some bacteria into the dish and found that after 1000 hours there were approximately 1200 bacteria. You are using the following equation to model their population over time where t is measured in hours after the initial 1000
$$P(t) = \frac{10,000}{1 + e^{-0.02(t-1000)}}$$
   (a) How many bacteria will there be if you wait 1000 more hours?
   (b) After how many more hours will there be 6000 bacteria?

6. The information entropy H, in bits, of a randomly generated password consisting of L characters is given by $H = L\log_2(N)$, where N is the number of possible symbols for each character in the password. In general, the higher the entropy, the stronger the password.

   (a) If a 7 character case-sensitive password is comprised of letters and numbers only, find the associated information entropy.

   (b) How many possible symbol options per character is required to produce a 7 character password with an information entropy of 50 bits?

7. The pH of a solution is a measure of its acidity or alkalinity. Specifically, $\mathrm{pH} = -\log[\mathrm{H}^+]$ where $[\mathrm{H}^+]$ is the hydrogen ion concentration in moles per liter. A solution with a pH less than 7 is an acid, one with a pH greater than 7 is a base (alkaline) and a pH of 7 is regarded as neutral.

   (a) The hydrogen ion concentration of pure water is $[\mathrm{H}^+] = 10^{-7}$. Find its pH.

   (b) Find the pH of a solution with $[\mathrm{H}^+] = 6.3 \times 10^{-13}$.

   (c) The pH of gastric acid (the acid in your stomach) is about 0.7. What is the corresponding hydrogen ion concentration?

8. Chemical systems known as buffer solutions have the ability to adjust to small changes in acidity to maintain a range of pH values. Buffer solutions have a wide variety of applications from maintaining a healthy fish tank to regulating the pH levels in blood.

   Blood is a buffer solution. When carbon dioxide is absorbed into the bloodstream it produces carbonic acid and lowers the pH. The body compensates by producing bicarbonate, a weak base to partially neutralize the acid. The equation which models blood pH in this situation

$$\mathrm{pH} = 6.1 + \log\left(\frac{800}{x}\right)$$

   where x is the partial pressure of carbon dioxide in arterial blood, measured in torr. Find the partial pressure of carbon dioxide in arterial blood if the pH is 7.4.

Chapter 7

Triangle Trigonometry

This chapter is devoted to the connection between our six trigonometric functions and right triangles. We will build these relationships, connect them to the unit circle and put them into context. Specifically we are guided by the following questions:

1. How can we introduce a triangle into the unit circle, and how do the trigonometric functions fit into that triangle?
2. What happens if we move from the unit circle and instead consider a circle of radius r?
3. How can we use the trigonometric functions to determine missing information in a right triangle?
4. What types of situations can I model with trigonometric functions?
5. Do these new relationships give us access to any trigonometric identities?

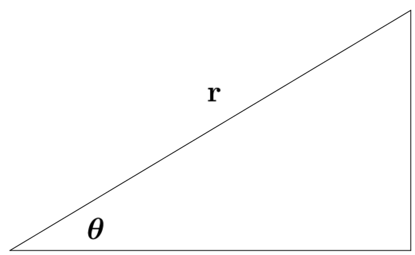
7.1 Trigonometric functions and right triangles

Book Reference: Section 10.2

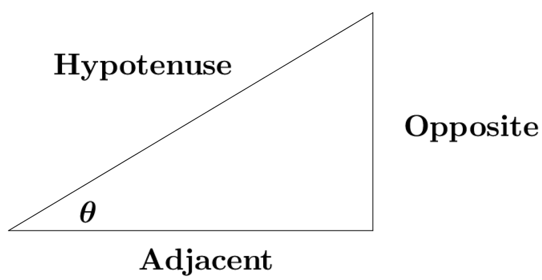
Return to our **Unit Circle** graph in Desmos. Click on the folder icon next to Triangle to make the green triangle appear on the unit circle.

1. Use the radius slider on Desmos to change the radius of the circle. How can you use the coordinates of the unit circle points to determine the lengths of the sides of the green triangle? How does the size of the triangle change as you change the radius? For example, how much bigger does the triangle get if the circle has radius 3, compared to radius 1?

2. Use your knowledge of $\sin(\theta)$ and $\cos(\theta)$ to determine the lengths of the two legs of the triangle below.



3. In relation to the angle θ we can name the sides of a right triangle as follows.



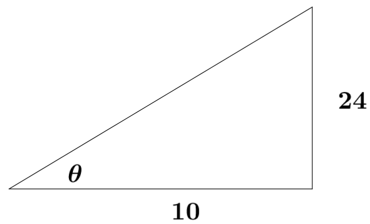
Use the two triangles on this page to represent $\sin(\theta)$ and $\cos(\theta)$ as the ratio of the sides of a right triangle. Use the letters a, h, o, for adjacent, hypotenuse, and opposite.

4. Using the fact that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and your answers from questions 3, represent $\tan(\theta)$ as the ratio of the sides of a right triangle.

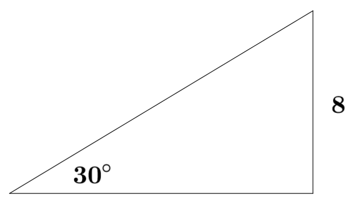
5. Write down a familiar mnemonic device to remember these ratios and give a detailed explanation describing how you determined them.

6. Use the definition of $\sec(\theta)$, $\csc(\theta)$, and $\cot(\theta)$ to represent each as the ratio of the sides of a right triangle.

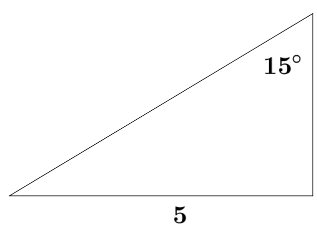
7. Use SohCahToa and the Pythagorean Theorem to determine the value of all six trigonometric functions evaluated at θ.



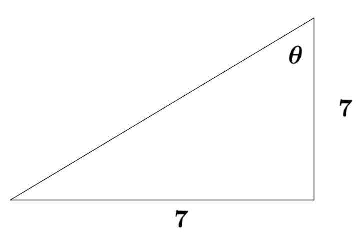
8. Use your knowledge of trigonometry and the pythagorean theorem to determine the lengths of all of the sides of the triangle. What are the measures of the other angles in this triangle?



9. Use your knowledge of trigonometry and the pythagorean theorem to determine the lengths of all of the sides of the triangle. What are the measures of the other angles in this triangle?



10. Use SohCahToa and the Pythagorean Theorem to determine the value of all six trigonometric functions evaluated at θ. What is the angle measure of θ?



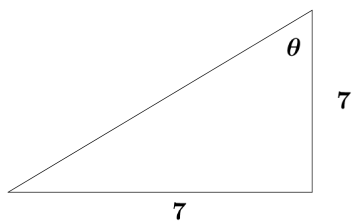
7.2 Finding angles with inverse trigonometric functions

Book Reference: Section 10.6

We will often have the side lengths of a triangle without any angle measurements. We can use inverse trigonometry to help us find the angles of such a triangle.

1. If you don't remember the two inverse trigonometry functions we have studied go back to the unit circle trigonometry chapter and do a quick review with your group.

2. In the previous section we considered the triangle below. Work through the guided questions to determine the measure of the angle θ



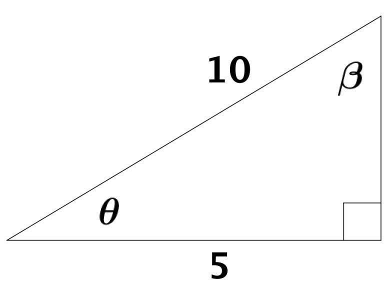
   (a) Which trigonometric function relates the sides and angle in the triangle above?

   (b) Find the value of this trigonometric function evaluated at θ.

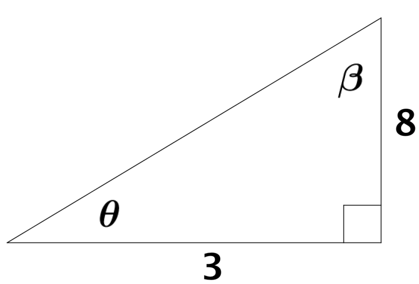
   (c) Use an appropriate inverse trig function to solve the equation above for θ.

   (d) Do we have to consider the general solution to this equation? Why? Prepare for an in class discussion.

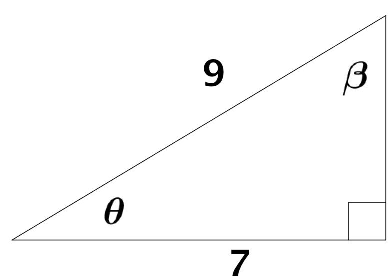
3. Find all missing angle and side measurements for the triangles below.



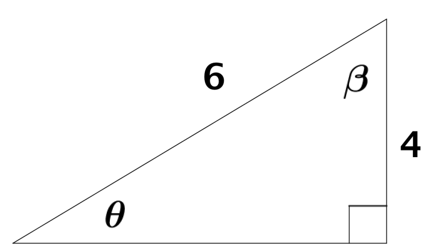
(a)



(b)



(c)



(d)

7.3 Modeling with Trigonometry

Book Reference: Chapter 11

Modeling with Trigonometry

1. A 12-foot ladder is going to be leaned against a wall. The distance between the base of the ladder and the wall is 5 feet. How high up will the ladder make contact with the wall?

2. A wire is stretched from the top of a building to the ground. The wire meets the ground 50 feet away from the building and forms an angle of 30° with the ground. Use trigonometry to find the height of the building.

3. A ladder is going to be leaned against a wall. The ladder will be sturdy if the angle it makes with the wall is between $\frac{\pi}{12}$ and $\frac{\pi}{6}$.

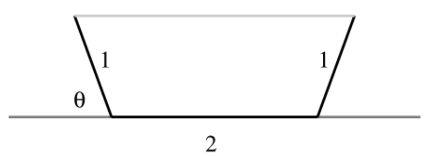
   (a) How high up can a 10 foot ladder make contact with the wall while still remaining sturdy?

   (b) A 15 foot ladder has its footing set 7 feet from the wall. Is the ladder sturdy?

4. I am building shelves in my office that will be 12 inches deep. In order to support the shelves I want to place a support that runs from the front of the shelf and meets the wall 8 inches below the back of the shelf forming a right triangle. What angle should I cut the supports so that they sit flush against both the shelf and the wall?

5. This question is related to activity 3.4.5 from Active Calculus:

   A trough is being constructed by bending a 4×24 (measured in feet) rectangular piece of sheet metal. Two symmetric folds 2 feet apart will be made parallel to the longest side of the rectangle so that the trough has cross-sections in the shape of a trapezoid, as pictured below:



   At what angle should the folds be made to produce a trough with volume 40 ft^3? Use the interactive Desmos graphic **Metal Trough** to help your understanding.

6. A bungee jumper plummets from a high bridge to the river below and then bounces back over and over again. At time t seconds after her jump, her height H (in meters) above the river is given by

$$H(t) = 100 + 75e^{-t/20}\cos\left(\frac{\pi}{4}t\right).$$

   Find her height at the times indicated below.

   (a) 0 seconds

   (b) 1 second

   (c) 2 seconds

   (d) 4 seconds

   (e) 6 seconds

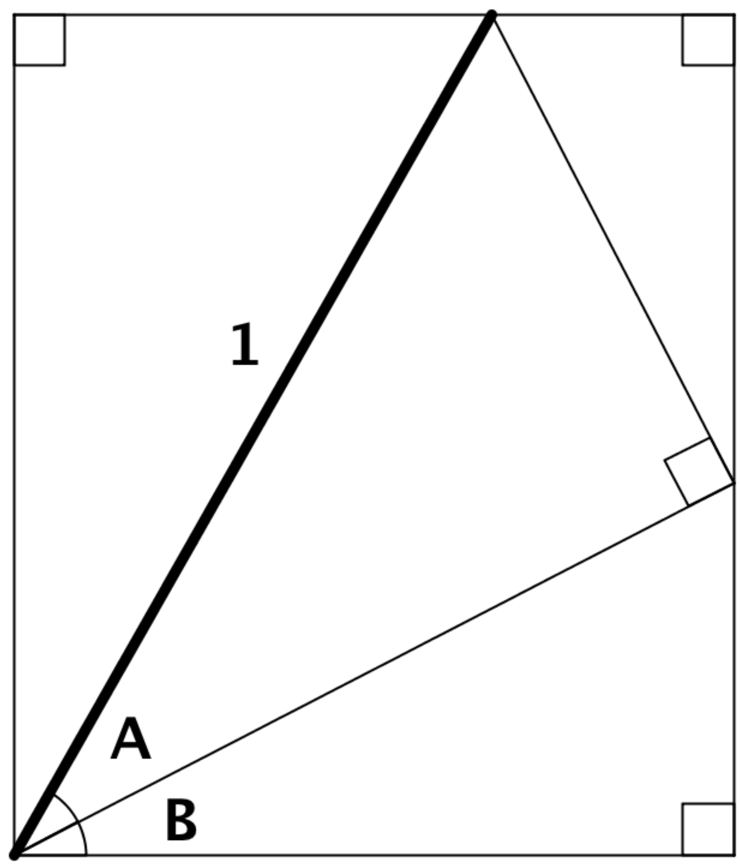
   (f) 8 seconds

7.4 Sum and Difference Identities for $\sin(x)$ and $\cos(x)$

Book Reference: Section 10.4

Sum and Difference Identities

1. Use your knowledge about the angle sum of a triangle to label all of the missing angles below. Then use your knowledge of the functions $\sin(x)$ and $\cos(x)$ to determine the length of the missing sides.



2. Now compare the lengths of the sides of the outer rectangle above. Use this comparison to write down two formulas:

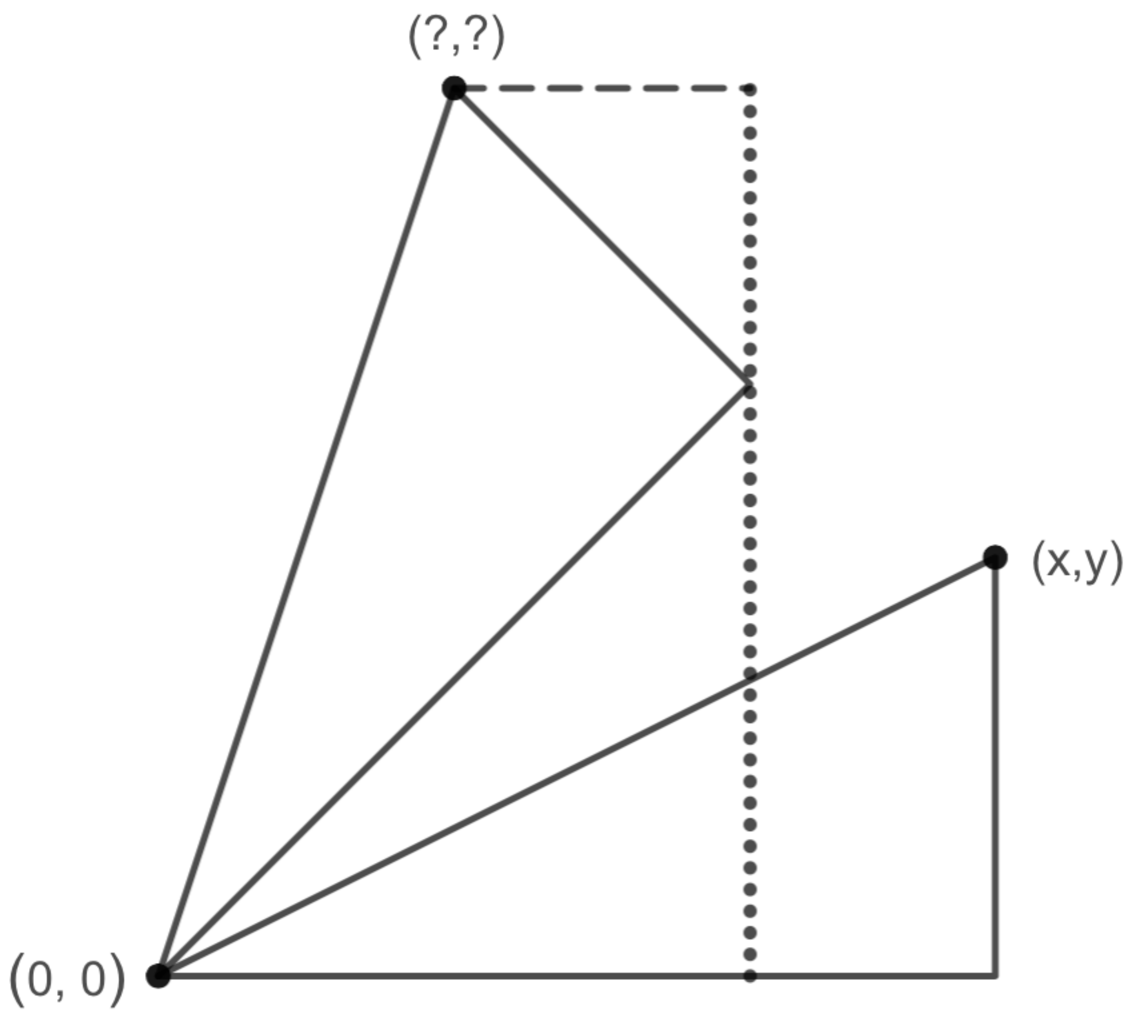
- Formula 1:

- Formula 2:

7.5 Extension: Using Trigonometry for Rotations

In the image below the point with coordinate (x, y) was rotated around the origin by an angle of θ degrees. Following that the right triangle formed by the point (x, y) and the origin was also rotated around the origin by θ degrees. Our goal is to determine the coordinates of the point labeled $(?, ?)$, giving us the equations needed to rotate images around the origin.

In order to find these coordinates, use your knowledge of triangle trigonometry to find the lengths of all of the line segments in this image. The dotted and dashed lines will be the key to our solution. Use the interactive version of **Rotations** on Desmos.



Chapter 8

Polynomial Functions

Polynomial functions are the generalization of linear and quadratic functions. In this section we will look to answer some of the following questions:

- What is a polynomial function? How can I identify them, and what are some of the important parts of a polynomial?
- How do I evaluate polynomial functions? How do I solve polynomial equations?
- How are the factors of a polynomial related to the roots?
- What types of scenarios can I model using polynomials?
- Extension: How can I use polynomial division to solve polynomial equations and how is it related to synthetic division?

8.1 Introduction to Polynomials

Book Reference: Sections 3.1, 3.2

We have seen quadratic functions, that is functions of the form $f(x) = ax^2 + bx + c$. These fall into a larger class of functions that we call **polynomials**.

Polynomial function

A polynomial is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

Here are some examples of polynomials:

a) $f(x) = x^2 + 5x - 1$

b) $g(x) = 17x^6 - 5x^3 + 2x$

c) $h(x) = 3x^7 - .75x^{199}$

d) $m(x) = x^3 - x^2 + 7x - 12$

1. Evaluate these polynomials at $x = 0$ and $x = 1$. What do you notice?

Some Definitions:

- The largest power of x that appears in a polynomial is called the **degree** of the polynomial.
- The coefficient on the largest power is called the **leading coefficient**.
- The leading coefficient together with the largest power of x is called the **leading term**.
- The number a_0 is called the **constant term**, or constant coefficient.

2. Identify the leading term, the degree, the leading coefficient, and the constant term for each of the polynomials listed above.

3. The following polynomials are given in their factored form. Expand them and then identify the leading term, the degree, the leading coefficient, and the constant term. Can you identify these without fully expanding the polynomial?

   a) $f(x) = x(x-1)(x-2)$

   b) $g(x) = (x-2)(x+3)(x+7)$

   c) $h(x) = (x-1)(x+1)(x-2)(x+2)$

   d) $m(x) = (x+1)^3(x-2)(x-1)$

   Recall that a number a is a zero of the polynomial p if $p(a) = 0$. What are the zeros of these polynomials? Make a conjecture regarding how the factors of a polynomial are related to the zeros.

4. Verify that the following values are zeros of the given polynomials.

(a) $h(x) = x^5 - x^3 + x;\ a = 0$

(b) $p(x) = x^3 - 2x^2 - 5x + 6;\ a = 3$

(c) $q(x) = x^6 + 5x^5 + 6x^4;\ a = -2$

(d) $r(x) = 2x^3 - x^2 - 2x + 1;\ a = 1$

The Factor Theorem

Suppose p is a nonzero polynomial. The real number c is a zero of p if and only if $(x - c)$ is a factor of $p(x)$.

5. Use what you know about factoring and quadratic functions to find the solutions of the polynomial equations given below:

(a) $x^3 - x = 0$

(b) $x^3 + 2x^2 - 15x = 0$

(c) $(x - 15)(x + 12)(4x - 3) = 0$

(d) $(x + 5)(x^2 - 6x - 7) = 0$

6. In the following you are given the zeros of a polynomial. Create a polynomial that has the given zeros. How many polynomials can you find with the given zeros? What additional information do you need to ensure there will only be one polynomial with the given zeros?

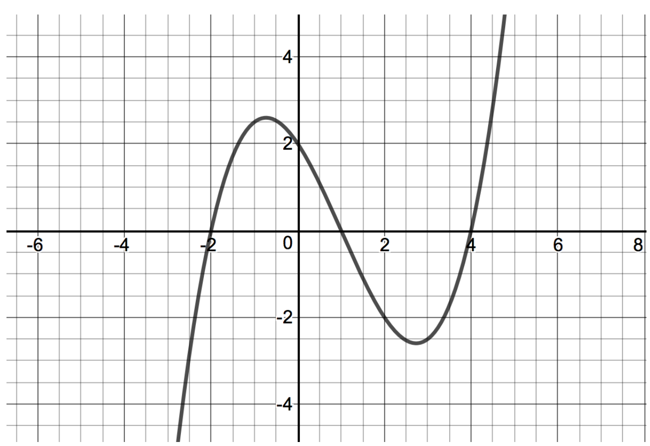
(a) Zeros: $1, 2, -3$

(b) Zeros: $-5, 0, 4, -1$

(c) Zeros: $\frac{1}{2}, 11, \frac{3}{4}$

(d) Zeros: $.25, \pi, e, 19, 6^{44}$

7. Find an equation for the polynomial given in the graph below.



8. Write the following polynomials in their factored form. Uses Desmos to help.

(a) $f(x) = x^4 - 12x^3 + 49x^2 - 78x + 40$

(b) $g(x) = 2x^3 + 5x^2 - 124x - 63$

(c) $h(x) = 5x^3 - 3x^2 - 5x + 3$

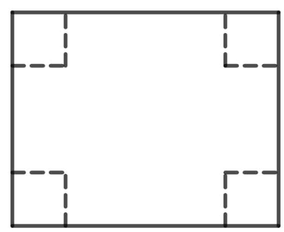
(d) $p(x) = 12x^4 - 107x^3 + 75x^2 + 114x - 80$

8.2 Modeling Activity: Box optimization

Materials: pen, 8.5×11 sheet of paper, scissors, ruler, tape, Desmos.

Goal: Model a volume scenario with a polynomial function. Observe how we consider restricted domains in real world situations.

Scenario: We are going to build a box out of a sheet of paper. The process for doing this is to cut congruent squares from the corners of the paper and then fold up the sides of the box. See the diagram below.



Question: Find the dimensions of the box that has the largest possible volume.

1. Begin by creating a box by cutting out the squares and folding up the sides. Record the length, width, and height of your box and find the volume. (Recall that Volume = length $\times$ width $\times$ height). Prepare to share your answer with the class so we can collect some data.
2. Plot the data we have received in class in Desmos. Use the height of each box as the input and the volume as the output.
3. How is the height of your box related to the size of the square you cut out? If you know the height of your box can you determine the length and width? How do you do that? Write down equations that give you the length and width of the box from the height:

   Length =

   Width =

4. Based on your observations from the data create the box that you think has the largest volume.
5. Find a function that fits the data you have plugged into Desmos. You should use the formulas for length and width that you found above.
6. What is the minimum heigh of a box that you can physically create? What is the maximum height of a box that you can physically create? How does this affect the model that you created in Desmos?
7. What are the dimensions that produce the maximum volume? What is the maximum volume?

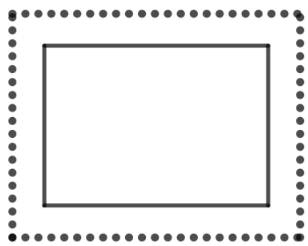
8.3 Modeling Activity: Framing a picture

This activity was inspired from: `http://fawnnguyen.com/got-beg/`

Materials: pen, 8.5×11 sheets of paper (2 colors), scissors, ruler, Desmos.

Goal: We want to use our knowledge of polynomials to help us frame a picture.

Scenario: You are tasked with framing a picture with a gold frame. The frame should go all the way around the picture and be even on all sides. The gold comes in full sheets that you will have to cut into the appropriate sized pieces. See the image below.



1. Begin with a 5in $\times$ 7in picture and an 8in $\times$ 8in sheet of gold. Cut up the 8×8 sheet to fit evenly around the 5×7 picture. What is the thickness of your picture frame? How did you find the thickness?
2. The next picture you have to frame is 4in $\times$ 6in and you have the same 8in $\times$ 8in sheet of gold. What is the thickness of your picture frame? How did you find the thickness?
3. The final picture you have to frame is 8in $\times$ 10in and you have a full 8.5in $\times$ 11in sheet of gold. What is the thickness of your frame? How did you find the thickness?
4. Discuss the method you used for finding a suitable frame for your pictures. How did your knowledge of polynomials help in solving this problem? Prepare for an in class discussion.
5. Extension: Generalize this problem to any size picture and any size sheet of gold. That is, what equations would you use if you have an $n \times m$ picture and a $s \times t$ sheet of gold?

8.4 Extension: Factoring polynomials by division

Book Reference: Section 3.2

The factor theorem showed us that the roots of a polynomial are related to its factors. Due to this theorem we will be interested in factoring polynomials to find the zeros. The procedure that we will use for this is called **polynomial division**. This works just like long division of real numbers. Here is an example where we are dividing $x^3 + 4x^2 - 5x - 14$ by $x - 2$.

$$
\begin{array}{r|l}
 & \quad x^2 + 6x + 7 \\ \hline
x-2 & x^3 + 4x^2 - 5x - 14 \\
 & -(x^3 - 2x^2) \\ \cline{2-2}
 & \qquad 6x^2 - 5x \\
 & \quad -(6x^2 - 12x) \\ \cline{2-2}
 & \qquad\qquad 7x - 14 \\
 & \qquad\quad -(7x - 14) \\ \cline{2-2}
 & \qquad\qquad\qquad 0
\end{array}
$$

The division tells us that

$$x^3 + 4x^2 - 5x - 14 = (x - 2)(x^2 + 6x + 7)$$

and so $a = 2$ is a zero of the polynomial $x^3 + 4x^2 - 5x - 14$. If the remainder of the division is not zero then $x - c$ is not a factor of the polynomial, and c is not a zero.

1. Use polynomial division for the following computations:
   (a) $(5x^3 - 2x^2 + 1) \div (x - 3)$
   (b) $(x^3 + 8) \div (x + 2)$
   (c) $(3x^4 - 2x^3 + x^2 - x + 1) \div (x^2 - 1)$
   (d) $(x^3 + 2x^2 + 3) \div (x^2 - 3x + 2)$

2. When dividing polynomials by a term $(x - c)$ we can streamline polynomial division into a process called **synthetic division**. Review the handout from our book and prepare for a class discussion surrounding synthetic division.

3. Use synthetic division for the following computations:
   (a) $(5x^3 - 2x^2 + 1) \div (x - 3)$
   (b) $(x^3 + 8) \div (x + 2)$
   (c) $(x^4 - x^2 + 3) \div (x - 1)$
   (d) $(x^3 + x^2 - 2x + 7) \div (x + 4)$

Use synthetic division for the following computations:

4. $(x^3 + x^2 - 6x + 4) \div (x - 1)$
5. $(x^3 + x + 2) \div (x + 1)$
6. $(3x^5 - 2x^3 - 1) \div (x - 1)$
7. $(x^4 + 9x^3 + 9x^2 - 9x + 2) \div (x + 2)$
8. $(x^4 - 2x^3 - 2x^2 - 2x - 3) \div (x - 3)$
9. $(x^3 + x^2 + 3x + 9) \div (x + 2)$

Use polynomial division to find all of the zeros of the following polynomials. One or more of the zeros are given:

10. $x^3 - 6x^2 + 11x - 6;\ c = 1$
11. $x^3 + 2x^2 - 3x - 6;\ c = -2$
12. $2x^4 + 11x^3 - 23x^2 - 11x + 21;\ c = 1, -1$
13. $x^3 + 4x^2 - 49x - 196;\ c = -7$

$$
\begin{array}{rrrrr}
 & x^2 & +\,6x & +\,7 & \\ \cline{2-5}
x-2 \big) & x^3\,+\,4x^2 & -\,5x & -14 & \\
 & -x^3+\,2x^2 & & & \\ \cline{2-2}
 & 6x^2 & -\,5x & & \\
 & -6x^2 & +12x & & \\ \cline{2-3}
 & & 7x & -14 & \\
 & & -7x & +14 & \\ \cline{3-4}
 & & & 0 &
\end{array}
$$

Next, observe that the terms $-x^3$, $-6x^2$ and $-7x$ are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we 'bring down' (namely the $-5x$ and -14) aren't really necessary to recopy, so we omit them, too.

$$
\begin{array}{rrrr}
 & x^2 & +\,6x & +\,7 \\ \cline{2-4}
x-2 \big) & x^3+4x^2 & -\,5x & -14 \\
 & 2x^2 & & \\ \cline{2-2}
 & 6x^2 & & \\
 & & 12x & \\ \cline{2-3}
 & & 7x & \\
 & & & 14 \\ \cline{3-4}
 & & & 0
\end{array}
$$

Now, let's move things up a bit and, for reasons which will become clear in a moment, copy the x^3 into the last row.

$$
\begin{array}{rrrrr}
 & & x^2 & +\,6x & +\,7 \\ \cline{2-5}
x-2 \big) & x^3 & +4x^2 & -\,5x & -14 \\
 & & 2x^2 & 12x & 14 \\ \cline{2-5}
 & x^3 & 6x^2 & 7x & 0
\end{array}
$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial. This means that we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$
\begin{array}{r|rrrr}
-2 & x^3 & +4x^2 & -\,5x & -14 \\
 & & 2x^2 & 12x & 14 \\ \hline
 & x^3 & 6x^2 & 7x & 0
\end{array}
$$

We've streamlined things quite a bit so far, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, $6x$ and 7, respectively, by the -2 in $x-2$, then by -1 when we changed the subtraction to addition. Multiplying by -2 then by -1 is the same as multiplying by 2, so we replace the -2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & 0 \end{array}$$

We have constructed a **synthetic division tableau** for this polynomial division problem. Let's re-work our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3+4x^2-5x-14$ by $x-2$, we write 2 in the place of the divisor and the coefficients of $x^3+4x^2-5x-14$ in for the dividend. Then 'bring down' the first coefficient of the dividend.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was 'brought down' to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \boxed{0} \end{array}$$

Chapter 9

Rational Functions

When we divide polynomial functions we get a new type of function called a rational function. In this chapter we are interested in answering the following questions:

- How do I find the zeros of a rational function?
- What information can I get when the denominator is equal to zero?
- What happens when the numerator and denominator share a zero in a rational function?
- What are asymptotes and how do I recognize them from in a rational function? Specifically we will study
  - vertical asymptotes;
  - horizontal asymptotes;
- What are scenarios that can be modeled with rational functions?

9.1 Introduction to Rational Functions

Book Reference: Section 4.1

In the previous chapter we studied polynomial functions. If you add, subtract, or multiply two polynomials the result is another polynomial, but when you divide two polynomials you get something new. This new function, called a **rational function** is the focus here.

> **Rational function**
>
> A rational function is a function of the form
>
> $$r(x) = \frac{p(x)}{q(x)}$$
>
> where $p(x)$ and $q(x)$ are polynomials.

Here are some examples of rational functions:

a) $f(x) = \dfrac{x}{x+1}$

b) $g(x) = \dfrac{1}{x-1}$

c) $h(x) = \dfrac{x^2-1}{2x+5}$

d) $m(x) = \dfrac{x^3+2x-3}{x^2-x^7}$

1. Evaluate the four functions above at $x = 0$ and $x = 1$. What do you notice? Did you run into any problems when you plugged in these numbers? When did those problems occur?

2. Think about the following prompt for a few minutes and share with your group to prepare for an in class discussion.

   > For a given rational function which points are we going to be interested in finding? What do you think happens at these points?

   Try graphing the four functions above in Desmos to help you answer these questions.

3. Use Desmos to find the zeros of the following functions. What do you notice?

   a) $r(x) = \dfrac{(x+2)(x-3)}{x+1}$

   b) $s(x) = \dfrac{(x-3)(2x-7)}{x^2+8}$

   c) $t(x) = \dfrac{x^3-4x}{5x-9x^2}$

   d) $w(x) = \dfrac{x^2-3x+2}{x^2+4x+1}$

4. Write down a relationship between the zeros of a rational function $r(x)$ and its numerator $p(x)$ and denominator $q(x)$. Which one is the most important for finding the zeros? Is there one that we can ignore?

Our explorations on the previous page should have led us to conclude that the numerator and denominator are both important for determining the zeros of a rational function. The numerator gives us possible zeros but we have to check them against the denominator. These ideas are made formal in the following property:

Zero-quotient property

Given any two real numbers A and B, if

$$\frac{A}{B} = 0$$

then $A = 0$ and $B \neq 0$.

5. Solve the following rational equations by factoring the numerator and denominator and then using the zero-quotient property.

   (a) $\dfrac{(x+2)(x^2-4)}{x+12} = 0$

   (b) $\dfrac{2x^3+5x^2-23x+10}{x^2-2x+5}$

   (c) $\dfrac{x^2+11x-26}{(x-2)(x-3)}$

   (d) $\dfrac{x^2-3x+2}{x-2} = 0$

   In part (d) above you should note that the when you plug in the value $x = 2$ you get an output of the form $\frac{0}{0}$. We say that this is undefined and we represent this graphically as a **hole** in the graph.

6. Create some rational equations that have holes in them. Can you create a function that has the form $\frac{0}{0}$ at $x = 2$ but does not have a hole in the graph? Prepare to share your examples with the class.

7. Identify any zeros and holes in the following functions.

   (a) $f(x) = \dfrac{3x^2+11x-4}{x^2-16}$

   (b) $h(x) = \dfrac{x^3+x^2-17x+15}{x^2+7x-8}$

   (c) $g(x) = \dfrac{(2x-1)(x+7)(3x-4)}{(6x-8)(x+1)}$

   (d) $r(x) = \dfrac{9x-x^3}{x^2-5x}$

8. Compute the average rate of change of the following functions on the given intervals. Use your Desmos graphs from question 3. to think about the average rate of change graphically to prepare for an in class discussion.

   a) $r(x) = \dfrac{(x+2)(x-3)}{x+1}$ on $[1, 3]$

   b) $s(x) = \dfrac{(x-3)(2x-7)}{x^2+8}$ on $[4, 8]$

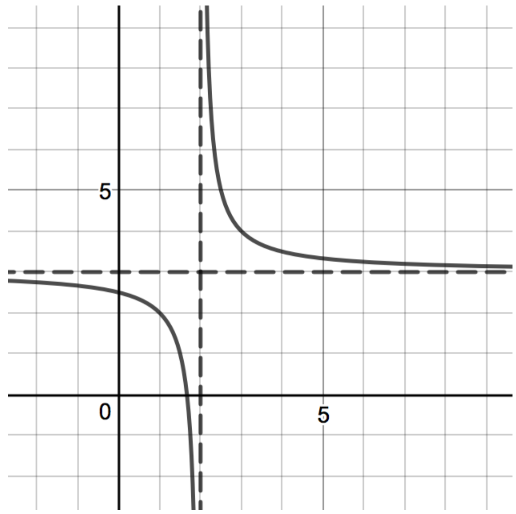
   c) $t(x) = \dfrac{x^3-4x}{5x-9x^2}$ on $[-4, -3]$

   d) $w(x) = \dfrac{x^2-3x+2}{x^2+4x+1}$ on $[0, 1]$

9.2 Vertical and horizontal asymptotes

Book Reference: Section 4.1

In this section we are going to develop tools for describing what happens when our *outputs* get really big, or when our *inputs* get really big. We use the word **asymptote** to describe this behavior. Graphically we represent them as dashed lines in a graph like in the graph below.



The vertical dashed lines are called **vertical asymptotes** and the horizontal dashed lines are called **horizontal asymptotes**. We will have a more formal definition later.

1. Use Desmos to graph the following rational functions and identify any vertical asymptotes or holes in the graph. What are you looking for to decide when a function has a vertical asymptote or a hole, and how can you tell the difference? How can you tell without looking at the graph?

   (a) $f(x) = \dfrac{1}{x+1}$

   (b) $g(x) = \dfrac{1}{(x+1)(x-2)}$

   (c) $h(x) = \dfrac{x+1}{(x+1)(x-3)}$

   (d) $p(x) = \dfrac{(x-1)^2}{(x-1)(3x+4)}$

   (e) $q(x) = \dfrac{x+1}{(x+1)^2(x-3)}$

   (f) $r(x) = \dfrac{-2x^2}{-3x^2+3}$

   (g) $s(x) = \dfrac{5x^4 - x^3}{x^4+3}$

   (h) $t(x) = \dfrac{x^3+1}{x^2+1}$

2. For each of the functions above determine whether or not they have a horizontal asymptote. If they do, write down what it is. What are you looking for to decide when a function has a horizontal asymptote? How can you tell without looking at the graph?

3. Before moving on write down your own definition of vertical asymptote and horizontal asymptote. To do this, describe the behavior of the function at an asymptote.

Some new notation

When we are thinking about the behavior of a function we often want to know what happens when are inputs are close to a particular number. If we want to consider x values close to a number c we use the notation $x \to c$. To consider arbitrarily large numbers we use the notation $x \to \infty$, or $x \to -\infty$ for arbitrarily large negative numbers.

Vertical and horizontal asymptotes

Consider the function $f(x)$.

- The vertical line $x = c$ is a vertical asymptote for $f(x)$ if $f(x) \to \pm\infty$ when $x \to c$.
- The horizontal line $y = c$ is a horizontal asymptote for $f(x)$ if $f(x) \to c$ when $x \to \pm\infty$.

4. Read the formal definitions with new notation above and discuss them with your group. Make connections with our previous examples of asymptotes.

5. Create a summary sheet in your notebook that describes how to find important characteristics about a rational function. An outline is given below:

   Let
   $$r(x) = \frac{p(x)}{q(x)}$$

   - $r(x)$ has a zero when...
   - $r(x)$ has a hole when...
   - $r(x)$ has a vertical asymptote when...
   - $r(x)$ has a horizontal asymptote when...
     further, the horizontal asymptote is zero when...
     and the horizontal asymptote is ______________ when...

6. Find any zeros, holes, vertical asymptotes or horizontal asymptotes of the following functions. First try to identify them without graphing, and then use Desmos to verify your answers.

   (a) $f(x) = \dfrac{x-9}{(x+4)^2}$

   (b) $g(x) = \dfrac{x^2}{x^3 + 2x^2 - 15x}$

   (c) $h(x) = \dfrac{x^2 + 4x + 4}{2x^2 - 2x - 12}$

   (d) $r(x) = \dfrac{x^3 - 1}{x^2 - 1}$

9.3 Two classic optimization questions: boxes and cans

Book Reference: Section 4.3

Question 1 - Optimizing a box
A classic calculus optimization problem. In place of Calculus we will use Desmos to help us out.

Materials: pen, paper, scissors, ruler, tape, Desmos.

Goal: Find the optimal dimensions for a box.

Scenario: Your task is to make a box with no top out of paper. The only conditions on this box is that it should have a square base and a volume of 500 in^3. Since we don't want to waste paper your goal is to minimize the amount of paper that is used. What are the dimensions of the box that minimize the amount of paper?

1. Use the materials provided and create a box with the given specifications in your group. Record your data to share with the class. From this data we will make a guess at the optimal dimensions of the box.

2. In order to find the minimum amount of paper necessary we will need to create some equations. Let the box have a width of x and height h. Determine the following as functions of x if possible:

   - Volume:

     $V(x,h) =$

   - Height:

     $h(x) =$

   - Surface Area:

     $A(x) =$

3. What values make sense to plug in for x? Can x be a positive number, or should it be negative? Is there a maximum or minimum value for x? What happens to the surface area when x gets really big? What happens when it gets really small? Prepare for an in class discussion.

4. Graph the function for surface area in Desmos on the appropriate domain and adjust the window to an appropriate scale, so you can see the function. Use this graph to determine the optimal dimensions of the box. What is the minimum surface area?

5. Extension: Suppose we want to make the box out of two types of paper, a heavier one for the base of the box. The heavier paper costs \$.0005 per square inch while the regular paper costs \$.0002 per square inch. What are the dimensions that minimize the cost of the box? What is the minimum cost of the box?

Question 2 - Optimizing a can
Another classic calculus optimization problem. In place of Calculus we will use Desmos to help us out.

Materials: pen, paper, scissors, ruler, tape, Desmos.

Goal: Find the minimum cost of making a soup can.

Scenario: Your task is to find the minimum cost of constructing a soup can. The can is to be made out of two materials, the cost of the lid material is \$.03 per square inch and the cost of the side material is \$.02 per square inch. The only restriction on the soup can is that it must hold a volume of of 15 in^3.

1. Use the materials provided and create a soup can with the appropriate volume. Record the dimensions of the can and its cost to share with the class. Describe how you found the cost of the soup can.
2. In order to find the minimum cost we will need to create some equations. Let the can have a radius of r and height h. Determine the following as functions of r if possible:
   - Volume:

     $V(r,h) =$
   - Surface Area:

     $A(r) =$
   - Height:

     $h(r) =$
   - Cost:

     $C(r) =$
3. What values make sense to plug in for the radius r? What happens to the cost when the radius gets really big? Really small?
4. Graph the cost function in Desmos on the appropriate domain and adjust the window to an appropriate scale, so you can see the function. Use this graph to determine the minimum cost of the box. What are the dimensions that achieve the minimum cost?

Appendices

Appendix A

Formula sheet for exams

When giving exams for precalculus I emphasize to the students that I do not want them to spend time memorizing the formulas. Instead I want them to understand the meanings behind the formulas and how to use them to solve problems. To accommodate this emphasis I give all of the formulas to the students on the exams. While teaching this course I have collected the formulas in the book and surveyed students as to which formulas they want. The following pages are the result and are provided to students.

Precalculus reference

January 12, 2019

Westfield State University
Department of Mathematics

Linear functions

Slope: $\dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\text{rise}}{\text{run}}$

Point-slope form: $y - y_0 = m(x - x_0)$

Slope-intercept form: $y = mx + b$

Quadratic functions

General form: $f(x) = ax^2 + bx + c$

Equations of vertex: $h = -\dfrac{b}{2a} \qquad k = f(h)$

Vertex form: $g(x) = a(x - h)^2 + k$

Quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Polynomials and rational functions

Polynomial: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

Degree: n; Leading coefficient: a_n; Constant term: a_0

Rational function: $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Rules for exponents

$a^m \cdot a^n = a^{m+n}$

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$\frac{a^m}{a^n} = a^{m-n}$

$a^0 = 1$ when $a \neq 0$

$(a^m)^n = a^{mn}$

$a^{-n} = \frac{1}{a^n}$

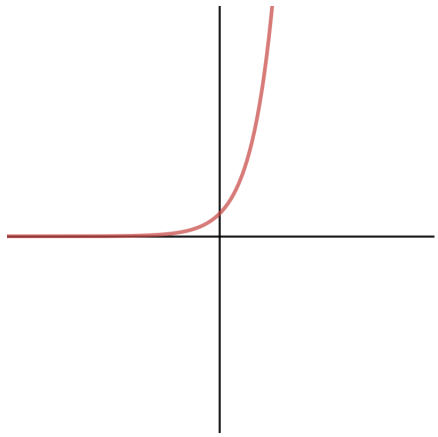
$(ab)^n = a^n b^n$

$a^{1/n} = \sqrt[n]{a}$

Exponential functions and logarithms

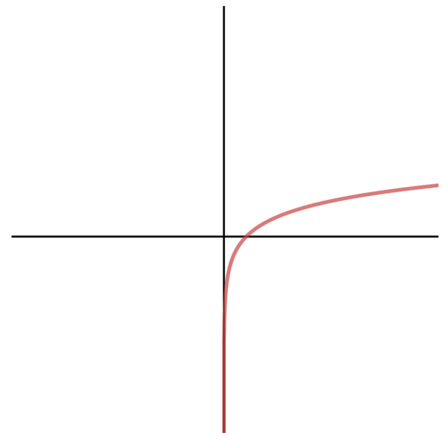
Exponential function

$f(t) = A \cdot b^t$ where $b > 1$



Definition of logarithm

$\log_b(x) = y \quad$ if and only if $\quad b^y = x$



Logarithm Rules

$\log_b(uw) = \log_b(u) + \log_b(w)$

$\log_b\left(\frac{u}{w}\right) = \log_b(u) - \log_b(w)$

$\log_b(u^w) = w \log_b(u)$

$\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$

Absolute value

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Area and volume formulas

Area of a rectangle: $A = l \cdot w$

Area of a circle: $A = \pi \cdot r^2$

Area of a triangle: $A = \frac{1}{2} b \cdot h$

Area of a trapezoid: $A = \frac{1}{2}(b_1 + b_2) \cdot h$

Circumference of a circle: $C = 2\pi \cdot r$

Volume of a box: $V = l \cdot w \cdot h$

Volume of a cone: $V = \frac{1}{3}\pi \cdot r^2 \cdot h$

Volume of a cylinder: $V = \pi \cdot r^2 \cdot h$

Volume of a sphere: $V = \frac{4}{3}\pi \cdot r^3$

Volume of a prism: $V = B \cdot h$

Trigonometry

Radians and degrees: 2π radians $= 180°$

Common angles in the first quadrant

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Formulas for reference angles: Given θ in $[0, 2\pi]$ these formulas give the reference angle.

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
θ	$\pi - \theta$	$\theta - \pi$	$2\pi - \theta$

Definitions of trigonometric functions

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

Sum and difference formulas

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

Fundamental trigonometric identity

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Inverse trigonometric functions

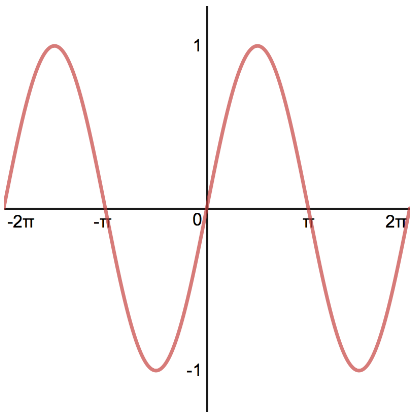
Arcsine: Domain: $[-1, 1]$, Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $y = \arcsin(x)$ if and only if $x = \sin(y)$

Arctangent: Domain: $[-\infty, \infty]$, Range $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $y = \arctan(x)$ if and only if $x = \tan(y)$

Graphs of trigonometric functions

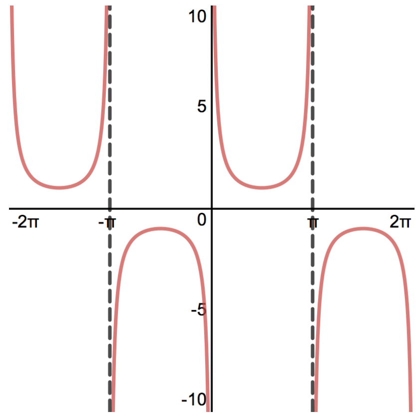
Sine

$f(x) = \sin(x)$



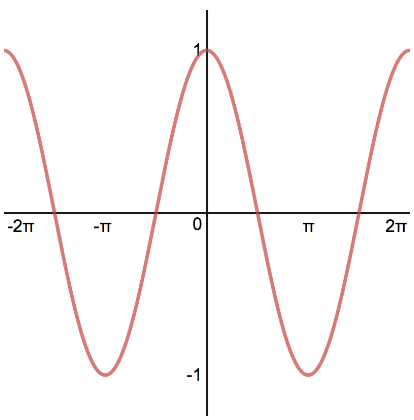
Cosecant

$g(x) = \csc(x) = \frac{1}{\sin(x)}$



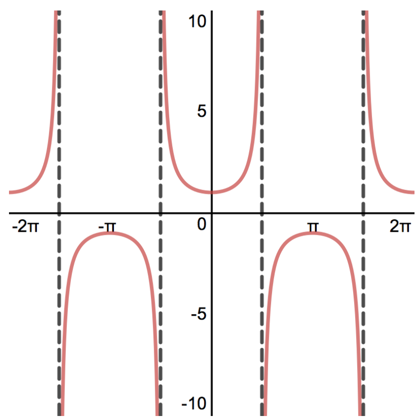
Cosine

$f(x) = \cos(x)$



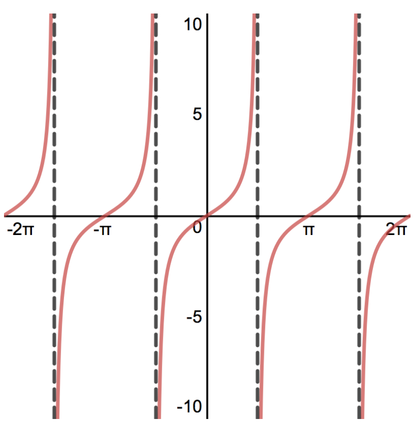
Secant

$g(x) = \sec(x) = \frac{1}{\cos(x)}$



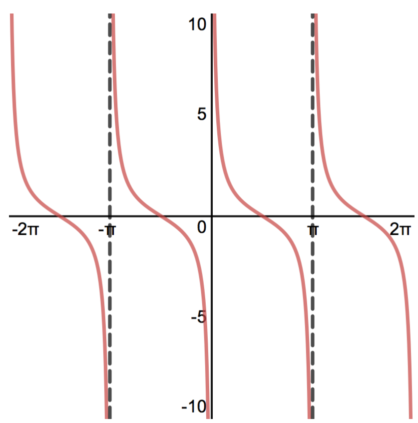
Tangent

$f(x) = \tan(x)$



Cotangent

$g(x) = \cot(x) = \frac{1}{\tan(x)}$



Appendix B

Links

- **Video Playlist**
- Chapter 1
  - **Card sort: Functions**
  - **Function Carnival**
- Chapter 2
  - **Pool Border**
- Chapter 4
  - **Circles**
  - **Radians Visually**
  - **Unit Circle**
- Chapter 5
  - **Facebook Data**
- Chapter 7
  - **Unit Circle**
  - **Metal Trough**