Discovering the Art of Mathematics: Proof as Sense-Making

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Abstract

We describe a model for mathematics for liberal arts courses where students are active participants in an authentic mathematical experience in which proof as sense-making plays a natural and essential role. At the center of this model is the pedagogical cycle: guided-discovery, group collaboration, whole-class discussion and synthesis into proof. With this cycle as the basis for the course, students begin to see proof and proof-writing as key components of sense-making, as extensions of narrative and rhetorical structures that are fundamental to human communication, and as complements to how we come to know and understand things in other areas of study. Having reframed proof in this broader context, we also share the benefits we have found in its use with mathematics majors.

1 Background and Context

Described here is an approach to proof that has grown out of the tradition of inquiry-based learning [3]. Originally used more prominently in Mathematical Explorations, one of our university's two mathematics for liberal arts (MLA) courses, where we have assessment data to support its success, it has grown to deeply inform our approach to proof in our courses for mathematics majors as well. Our description here encompasses both audiences.

1.1 Proof as Sense-Making

Proof is fundamental to mathematics. Yet its nature and role continues to be actively debated. Reuben Hersh and Philip Davis provide a lively caricature who embodies the many struggles of mathematicians to articulate what proof is. Named the Ideal Mathematician he/she:

> Can think of no condemnation more damning than to say of a student, "He doesn't even know what a proof is." Yet he [the Ideal Mathematician] is able to give no coherent explanation of what is meant by rigor, or what is required to make a proof rigorous. In his own work, the line between complete and incomplete proof is always somewhat fuzzy, and often controversial. [2]

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This is not a straw caricature. The firestorm caused by Jaffe and Quinn's "Theoretical Mathematics: Towards a Cultural Synthesis of Mathematics and Theoretical Physics" [7] is a perfect example. Zeilberger's "[Contemporary Pure] Math Is Far Less Than the Sum of Its [Too Numerous] Parts" [17] is a more recent broadside that is likely to renew sharp debate. While this current volume focuses on pedagogical aspects of proof, the fundamental, philosophical substance of these debates about proof can help us reconceptualize our approach to proof in the classroom.

In his paper [6] of the same name, Reuben Hersh argues that "Proving is Convincing and Explaining." We believe the active, process-oriented nature of this phrasing is valuable. Our purpose in this paper is to develop an expanded conception of proof: **proof as sense-making**.

The term sense-making has been used in mathematics education research for quite some time. (See, e.g., Schoenfled [15].) It has achieved a higher profile with the recent publication of "Focus in High School Mathematics: Reasoning and Sense Making" [11] by the National Council of Teachers of Mathematics. This publication advocates redirecting significant attention from the selection of appropriate topics for high school mathematics to a primary focus on "curricular emphases and instructional approaches that make reasoning and sense making foundational to the content that is taught and learned." **Sense-making** is defined as "developing understanding of a situation, context, or concept by connecting it with existing knowledge."(p. 4) The authors of this publication note, "In practice, reasoning and sense making are intertwined across the continuum from informal observations to formal deductions, . . . despite the common perception that identifies sense making with the informal end of the continuum and reasoning, especially proof, with the more formal end."

Sense-making, in part through explaining and convincing, is not foreign to our students. They do this in all of their *other* courses. By reframing the notion of proof, students' opportunities for sense-making are expanded and deepened. In settings where it has been lost or underutilized, which includes many MLA classrooms, proof can become a tool for reclaiming sense-making in mathematics classrooms.

1.2 Proof and the Liberal Arts

> An elegantly executed proof is a poem in all but the form in which it is written. ∼ Morris Kline [9]

In his beautiful response "On Proof and Progress in Mathematics" to Jaffe and Quinn, William P. Thurston [16] repeatedly uses the phrase "human understanding" and emphasizes psychological and social dimensions of proof, saying, "The measure of our success is whether what we do enables *people* to understand and think more clearly and effectively about mathematics." He declares "human language" to be the number one factor "important for mathematical thinking".

Communication in human language is one arena in which our liberal arts students are most comfortable. We lose critical connections between mathematics and the other disciplines if we only compare them in terms of what is generally referred to as "content," ignoring the issues of process, communication, and differing burdens of proof.

Instead, we suggest using the critical connections between mathematics and the liberal arts in the area of both communication and process to positively empower traditionally disenfranchised students to succeed in mathematics. These connections are particularly powerful when we consider the notion of proof in an inquiry-based setting [3].

1.3 Proof as Narrative

> In deductivist style, all propositions are true and all inferences valid. Mathematics is presented as an ever-increasing set of eternal, immutable truths. . . Deductivist style hides the struggle, hides the adventure. The whole story vanishes. ∼ Imre Lakatos [10]

How can mathematicians describe what proofs are and what they look like to general-education students? The landscape of proof is fraught with many different methods, types and styles; underlying mathematics that may not be well-enough understood; and a style of communication that is dense and often full of technical notation. While this style of communication is typical of working mathematicians, it need not be what we teach our students, especially not our general education students. Instead, we can rely on their working knowledge of several types of discourse.

In "Narrative Structure in Inquiry-Based Learning" [8] C. Kinsey and T. Moore describe how narrative structure offers a parallel for inquiry-based learning in mathematics:

> Let the students explore and discover mathematics by introducing a setting and group of [mathematical] characters, posing a question to create conflict, and leading them to resolve the conflict with carefully chosen exercises.

Notice the very rich way that the process of mathematical discovery is phrased in narrative terms.

> The students learned to look at a new topic as a new cast and setting and to look at new questions as new plot points... They expected to be involved in the conflict of the story. They looked at the given examples as foreshadowing and hints to resolve the conflict and clarify the complications. Most importantly, they expected a resolution that made sense.

Notice how the students play a central role in this process. They are at the center of the learning process because they have been involved in it as part of the entire narrative - a structure they are used to.

We can expand this role of narrative in proof-writing via narrative's *dramatic arc*. In inquiry-based learning there is exploration; this is the *rising action* in the dramatic arc. In the dramatic arc there is a *climax* and then *falling action*. In an inquiry-based mathematics classroom this is the student discovery, the "a-ha moment," the statement of a conjecture. In the dramatic arc the final stage is the *denouement*, the resolution. The parallel in mathematics is the explanation of human sense-making, the communication of justification, proof. Compare this arc with the technique in 2.2 which is the focus of this paper.

Narrative provides an important, concrete context for students to view their mathematical sense-making and their proof-writing. Additionally, the analogy with dramatic arc foreshadows the pedagogical technique that is the focus of this paper.

1.4 Proof as a Rhetorical Situation

> Mathematics - this may surprise or shock some - is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions. He arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof... The deductive stage, writing the result down, and writing its rigorous proof are relatively trivial once the real insight arrives; it is more like the draftsman's work not the architect's. $\sim$ Paul Halmos [5]

The role of proof in mathematics is, to borrow language from our colleagues in composition, a rhetorical situation - authors convincing an audience of the legitimacy of their message. In an inquiry-based setting the students are clearly cast as the authors. At each stage we can help them learn to see their discoveries, their "a-ha moments," and their conjectures as the analogues of thesis statements in expository or persuasive papers. Students are used to framing and then defending a thesis statement. By repeatedly reminding them of this connection, we help them understand the process of doing mathematics. They begin to see that the use of rhetoric as a key part of this process is not so different than it is in many other areas of human discourse.

Thus, rhetoric provides important context for our students to view the role of proof and proof-writing. We also find historical evidence connecting the pedagogical technique that is the focus of this paper to discourse.

A. Szbaó and A. Ungar argue [12] that mathematics and the transition to formal axiomatic foundations "grew out of the more ancient subject of dialectic." As a counterpart of rhetoric, dialectic arose from the Socratic dialogues which can be seen as a precursor to inquiry-based learning. We are re-forming the natural and important connections between proof, inquiry, and discourse.

1.5 Setting

The approach described in this paper was developed by the authors and our colleague Philip K. Hotchkiss for the MLA course Mathematical Explorations at Westfield State University as well as for the National Science Foundation funded project *Discovering the Art of Mathematics* for which the same group serves as principal investigators. Our university is a comprehensive university of about 5,000 full-time undergraduate students and 700 graduate students. Mathematical Explorations is one of several courses offered to satisfy the two-course general-education requirement in mathematics. Class size is typically 30 - 35 with some smaller honors sections. More than 60 sections of this course have been taught by us using the approach described here. *Discovering the Art of Mathematics* provides freely available curriculum materials (see Section 2.2) to support inquiry-based learning in MLA [4]. These materials have been used by a number of faculty at other institutions and are available for both classroom use and beta-testing.

2 Description and Implementation

> [In this course] I was encouraged and guided to engage in making the discoveries and understandings for myself. The techniques for how I found my solutions were very clear to me because, I was the one who found a pattern and applied the pattern to generate the solution. I have gained my own personal experience with mathematics that has changed my view on the subject completely. ∼ *Discovering the Art of Mathematics* student

Proof is an essential part of the complete cycle of mathematical sense-making: begin with an exploration of a mathematical object or pattern; collect data and observe connections; use informal and/or inductive reasoning to make conjectures about these connections; verbalize definitions, assumptions and/or axioms; develop logical, rigorous arguments to establish the validity of the result. As described above, and illustrated below, for MLA the proofs are expressed in natural language to a much greater degree than the ways mathematicians typically communicate. But they are no less proofs.

2.1 Our Technique: A Look Into Our Classroom

While there are differences in our teaching, we are united in our use the technique to be described. We describe it here in first-person narrative as scenes like this are typical of each of our classrooms each time we meet with our students.

It's still seven minutes before class starts when I walk into the classroom. Each of the nine tables in the classroom has 2 or 3 students seated at it, most with their notebooks open already doing mathematics. I begin to tour the room to see what the different groups of students are collaborating on. It is generally quite a mix. They all started from the same guided-discovery investigations designed to help them solve some larger problem or (re-)discover a significant mathematical result. Some groups remain in earlier phases - collecting data, finding patterns, and recording ideas. Some are in slightly later phases - making guesses, formulating conjectures and articulating connections. All of these activities are part of the continuum of sense-making.

By the time class is supposed to "start", every student is doing mathematics. I've said nothing to the class as a whole, given no direction for them to start working, yet each student/group has picked up their mathematical explorations where they previously left off - either from the last class or from subsequent work outside of class. I may make a few announcements. I may help organize a whole-class discussion. Many students/groups will ask questions, to which I will generally respond with my own questions. I will not lecture. My role is to facilitate the students' work.

The focus of student work is the big investigation posed the previous class:

Determine all possible values generated by the Diophantine equation $3a + 5b$ *when* $a, b \geq 0$.

The students also have a short sequence of investigations as finer prompts:

1. *Choose whole number values for* $a, b \geq 0$. *For these values, evaluate the expression* $3a + 5b$.
2. *Repeat Investigation 1) for a different pair of values for* a, b.
3. *Repeat Investigation 1) for another dozen different pairs of values for* a, b.
4. *Can every positive whole number be obtained as the result of evaluating* $3a + 5b$, *or are there some numbers that cannot be made from this expression?*
5. *Continue investigating until you feel comfortable making a conjecture which identifies exactly which, all infinitely many of them, positive whole numbers can be made by evaluating the expression* $3a + 5b$ *for* $a, b \geq 0$ *being whole numbers.*
6. *Prove your conjecture.*

A typical interaction with a group of students is as follows. . . As I observe I realize Chelsea and Mike have been working together on one strategy while their group-mates Micaela and Joe have collaborated on a different approach. Chelsea begins explaining their approach:

Chelsea: I think you can get any positive number. You just take away a multiple of 3 and then it ends in zero. That's easy.

Mike: How would I do that for 86?

Chelsea: 86 minus 6 is 80, so uhm, $a = 2$ and, uhm, $b = 16$.

Mike: *(computes)* OK. So ending in zero means that five goes into it, right?

Joe: I don't get what you're doing. Why are you looking at 86? Did you make the whole list up to 86?

Micaela: We've only gotten to 24 so far. *(She points to a list which is a precursor of Figure 2 in Appendix A.)*

Chelsea: You don't need the whole list.

Joe: Yes, you do. You have to find all solutions.

Professor: Joe, Micaela, do you agree with Chelsea and Mike's conjecture that you would get any positive whole number?

Micaela: There are some missing. You can't get 4.

Chelsea: So for 4, we would subtract ... What's a multiple of 3 that ends in 4? Uhm, 24 works. So $4 - 24$ is negative 20, which ends in zero, so it's 4 times 5.

Mike: No, negative 5.

Joe: But negative numbers are not allowed.

Mike: Oh, ok. *(thinking)*

Professor: So it sounds like Chelsea and Mike may want to adjust their conjecture.

Professor: *(turning to Micaela and Joe)* Could you convince me that 86 is going to be in your list?

Micaela: That's too much to write all of them down.

Professor: Could you convince me that 86 would eventually appear on your list without writing them all down?

Micaela, Joe: Hmmm ... *(thinking)*

To gain a richer appreciation of students working on this particular proof, and to see the final results of their work, see our blog at `www.artofmathematics.org/blogs/jfleron/3a5b-proofs` which contains both written proofs and classroom videos. Micaela and Joe's proof is included in Figure 2 in Appendix A and as "Proof 4" online while Chelsea's idea appears online in the video "3a+5b-proof-0006-III-CvR".

2.2 Our Technique

We hope that this vignette has given you a picture of our technique for actively involving MLA students in a sense-making exploration of mathematics. The central focus of this technique is the repeated application of the following cycle:

1. Student exploration via guided-discovery.
2. Group collaboration and discussion for sense-making.
3. Whole-class discussion focusing on big ideas, strategies, and finding misconceptions.
4. Synthesis of results into proof.

Notice the essential role of communication in each of these areas - from students' reading and interpreting the prompts which spark the investigations through the final reporting via proof. Details about each of these stages include:

Guided-Discovery: We utilize the inquiry-based materials of the National Science Foundation supported project *Discovering the Art of Mathematics* [4]. A central component of this project is a library of 11 freely available guided-discovery learning texts, each with sufficient material to support a semester-long, three credit MLA course.[1]

Group Collaboration: The students work in groups on these guiding questions, the teacher circulates the room observing, encouraging and supporting the students. We find it helpful not to answer student questions directly but to respond instead with questions that will deepen their thinking; see [13].

Whole-Class Discussion: We facilitate whole-class discussions to make students aware of other strategies, possible misconceptions, connections between ideas, and to talk about communal standards of rigor. We rely heavily *Classroom Discussions: Using Math Talk to Help Students Learn, Grades K-6* [1]. Such whole-class discussion can take place at various stages of the students' explorations. We like to use it to compare student conjectures, refine the precision of students' claims, and to work out the details of a mathematical explanation.

Synthesis: As the students clarify their thinking, they are asked to write, explaining their results and the validity of these results. The proof-writing tasks vary. Sometimes they are fairly narrow, providing justified answers to a series of shorter investigations. Some tasks result in proofs that are more narrative in nature, describing the students' investigations as well as their results. Other tasks result in proofs which more closely resemble typical mathematical writing with conjectures followed by proofs. While in each phase the language and style used is more informal than what typically finds in mathematical writing, the level of rigor remains high.

2.3 Sample Learning Cycle and Proof

In this section we illustrate the approach concretely with a focus on the latter part of the cycle. The investigations which set the context are directly from the book *Discovering the Art of Mathematics - Music* [14]. We encourage you to try the problems on your own before reading the solution.

[1]All materials are available online at `http://www.artofmathematics.org/`. This project also provides faculty resources, videos illustrating inquiry-based learning in action, professional development workshops, and a host of other resources.

Previously, guided investigations prompted students to find the number of ways you can create a rhythm with k beats on a drum if there are n counts in a measure. Students worked collaboratively trying to find organizational structures that allowed them to make sense of the data; having found the three rhythms with 2 beats on a measure of 3 counts (110, 101, and 001) the students searched to find connections between this new piece of data and the earlier data. As part of a whole-class discussion more than one group's organizational strategy resembled Pascal's triangle and this approach was adopted as the class' method of organizing their data. (A sequence of guided investigations was available in reserve had this not been raised by students as part of the discussion.) Subsequently, the learning cycle began again with students working on the sequence of investigations below.

Independent Investigation: *In your group, study Pascal's triangle and find at least three different patterns. Be creative! Then share the patterns with your class.*

One of the patterns you probably found is the addition pattern: when adding two adjacent numbers the result will be right beneath the two numbers that you added. Mathematicians love finding patterns but they also wonder why the patterns occur and how you can be sure that they will continue to happen. Our goal now is to understand why *the addition pattern in Pascal's triangle occurs and to make sure that it will always happen.*

1. *In Figure 1, fill all possible rhythms in the respective boxes.*

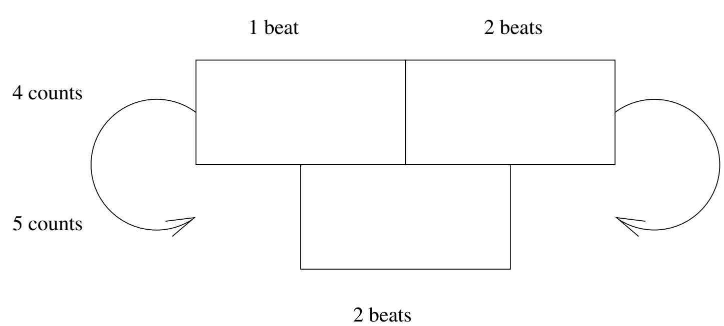

Figure 1: Part of Pascal's Triangle.

2. *Now look at the rhythms you filled into Figure 1; can you see any structure that suggests how the rhythms in the upper boxes are connected to the rhythms in the box below? Explain the structure you found.*
3. *Go to another place in Pascal's triangle and choose three similarly positioned boxes. Fill them with rhythms and see if your structure applies here as well.*
4. *Does your structure apply to the top of the triangle?*
5. *Explain in your own words why the addition pattern in Pascal's triangle occurs, using the structure you found. Be specific in your arguments.*

The cycle of student exploration - group collaboration - whole-class discussion - continued as the students developed increasingly robust ways of making sense of the underlying structure. For the final synthesis the expectation is not an algebraic proof but rather organized sense-making expressed in the students' own words. The following proof is written in a language that is typical for our students.

> We focused on an example to see why the pattern worked - 4 counts and 1 beat, and 4 counts and 2 beats. There are 4 rhythms with 4 counts and 1 beat: 1000, 0100, 0010, 0001. There are 6 rhythms with 4 counts and 2 beats: 1100, 0110, 0011, 1001, 1010, 0101. We had to understand why it makes sense that there are exactly 4+6=10 rhythms with 5 counts and 2 beats.
>
> What we found is a method to create all the rhythms with 5 counts and 2 beats from the row above. The key idea is: *Any rhythm has to start with either a 0 or a 1.*
>
> Here's how we created all of the rhythms that have 5 counts and 2 beats and start with a 0. The last 4 counts of each of the rhythms will exactly come from the box on the top right, since those are *all* rhythms with 4 counts and 2 beats. We can not use rhythms with 4 counts and 1 beat or more than 2 beats, since we need to have exactly 2 beats. So only one box in the upper row provides what we need. So now we have 6 rhythms.
>
> Then we created the rhythms with 5 counts and 2 beats starting with a 1. Again, there is only one box that provides what we need, since we now want rhythms with 4 counts and 1 beat. The desired box is on the top left. This adds the remaining 4 rhythms.
>
> This explains why we get *all* rhythms, but we need to make sure we don't get anything twice. The rhythms coming from different boxes start on different digits (0 or 1), so they're different. Rhythms from the same box already had to be different somewhere so the new rhythms formed are also different.
>
> If we pick any other pair of adjacent boxes our method will work in the same way.

A number of student proofs and classroom videos which highlight the cycle described here, in several different curricular areas, are available at `http://www.artofmathematics.org/classroom/proof-as-sense-making`.

3 Outcomes

The necessity of an adequate foundation based on sense-making for successful proof-writing should be clear from our description above. For MLA students this includes: context for what proofs are and how they relate to other forms of writing, communication, collaboration, active engagement, exploration and discovery, creativity, enjoyment, sense-making, and personal responsibility. Without these key ingredients there is little substance upon which proofs can be created and, subsequently, no possibility for meaningful proof-writing to take place.

Preliminary data from the *Discovering the Art of Mathematics* project provides positive evidence for the effect?iveness of our approach. The study is based on pre- and post-survey data from several sections taught by four different professors between 2010 and 2012 at Westfield State University.[2] (Unless indicated otherwise, results are based on a total of $n = 192$ matched pre/post responses.) Specifically, students report:

I learn mathematics best when ... *I explain ideas to other students* ($p < .001$, effect size: .33). *I work on problems in a small group* ($p < .01$, effect size: .26). We see this as validation that the major activity in the classroom, making sense of mathematical ideas in a small group, contributes to creating mathematical proofs. Explaining your thinking is a high-level tool for clarifying and deepinging your own reasoning.

In order to solve a challenging math problem, I need ... *To try multiple approaches to constructing a solution* ($p < .01$, effect size: .24). *To have freedom to do the problem in my own way* ($p < .01$, effect size: .21) We believe this shows that students are discovering that creativity, imagination, and exploration are key requirements for making sense of mathematics. They are letting go of the novice conception that there's just a single, direct way of doing every mathematics problem, allowing them to think and act more like expert mathematicians.

[2] A summary of the results will appear soon at `http://www.artofmathematics.org/about/goals-and-evaluation`.

Students show increased agreement with the following statements: *A proof is something you have to construct based on your own understanding* ($p < .01$, effect size: .33, $n = 41$). *I am confident in doing mathematics* ($p < .001$, effect size: .28). We believe this confirms the link between sense-making, proving, and understanding in mathematics that is the focus of this paper. For many students in the liberal arts, mathematics has not been previously experienced as something that could make sense, resulting in a lack of confidence in.

Students also report increased enjoyment in discovering new mathematical ideas ($p < .01$, effect size: .41), in rigorous reasoning in a math problem ($p < .01$, effect size: .21), and in working on challenging mathematical problems ($p < .01$, effect size: .17). Without these key ingredients there is no context for proof and no possibility for meaningful proof writing to take place.

We have seen positive changes in our students' abilities to make logical arguments and craft meaningful proofs. Their initial anxiety of searching for the "proof the professor wants" are assuaged by the whole-class discussions where the students see multiple approaches validated. They begin to think independently. They think creatively. Their "proof stories" richly illustrate their thinking and provide a context for rigorous reasoning. We are often surprised by the novel proofs of our students, approaches that we would not have thought of on our own and that enrich our understanding of the mathematics at hand.

Many statistically significant, positive changes in affect and meta-cognitive awareness have also been found. While these changes may not correlate precisely to proof and proof writing, they are thought to correlate with the more active involvement of students in the learning process of which proof is part. These results can be found online with those described above.

Extended to mathematics majors, as described below, our technique has helped our students think more critically and with greater reflection. They are driven less by procedural thinking, especially in proof situations, and more by sense-making. The disjointed and incomplete proof attempts we had become used to have been replaced by much richer "proof stories." The students' thought processes are much clearer and their ability to reason without gaps has improved dramatically.

4 Extending the Method

Our focus so far in this paper has been MLA courses. We now indicate how mathematics majors may benefit from this approach.

A typical introduction-to-proof course for majors is often structured around a number of different proof techniques. Students often see these methods as a checklist to be gone through in serial order as they try to find the "correct" one to "apply" when asked to prove something. That is, a formal structure precedes any exploration, investigation or attempt at sense-making. We teach our major students proof by induction, yet have we seriously investigated the differences between inductive and deductive reasoning with these students? These students have little context within which to conceptualize proof and there is a chasm between the body of mathematics of which they have made sense of and the one ini which we expect them to skillfully proof-write in.

We suggest reframing the role of proof as described here at least in our first- and second-year courses for majors where the developmental work on proof is done. In these early stages of their undergraduate careers, our majors have read and created far more narrative and expository pieces outside of mathematics than they have read or created mathematical proofs. They too can benefit from understanding proof in the broader framework described here. This is not to preclude apprenticeship in typical methods of proof and proof techniques, but to provide a broader base with deeper connections to already-developed skills.

We can illustrate this concretely with a calculus activity we have used successfully. In our experience calculus students are unable to decipher the proof of the First Fundamental Theorem of Calculus in a typical textbook. However well-intentioned our lecture on this topic may be, it seems to develop little understanding in our students. In contrast, we have found that the cycle we've described enables students to develop their own proofs of this central result. The earlier trips around the cycle involve investigating the derivative as slope and linear approximation by tangent lines (in several different contexts including speed and distance) as well as approximation of areas by Riemann sums. Once the sense-making is developed to a high level the stage is set to have the students investigate the general problem of finding the area under the curve when the function has an antiderivative. While somewhat informal, the proofs the students develop of this essential result illustrate deep understanding and ownership.

Proof is sense-making and we expect students to sense-make at each stage - not just when a final, somewhat formal proof is called for. By developing expectations in our major students that they should be careful in their reasoning always - when communicating verbally with each other and with us, when they work on homework, when they write answers on exams, when they submit written work that is not so explicitly in a proof setting - we are developing their sense-making abilities. These efforts have resulted in marked increase in our major students' abilities to develop and write proofs.

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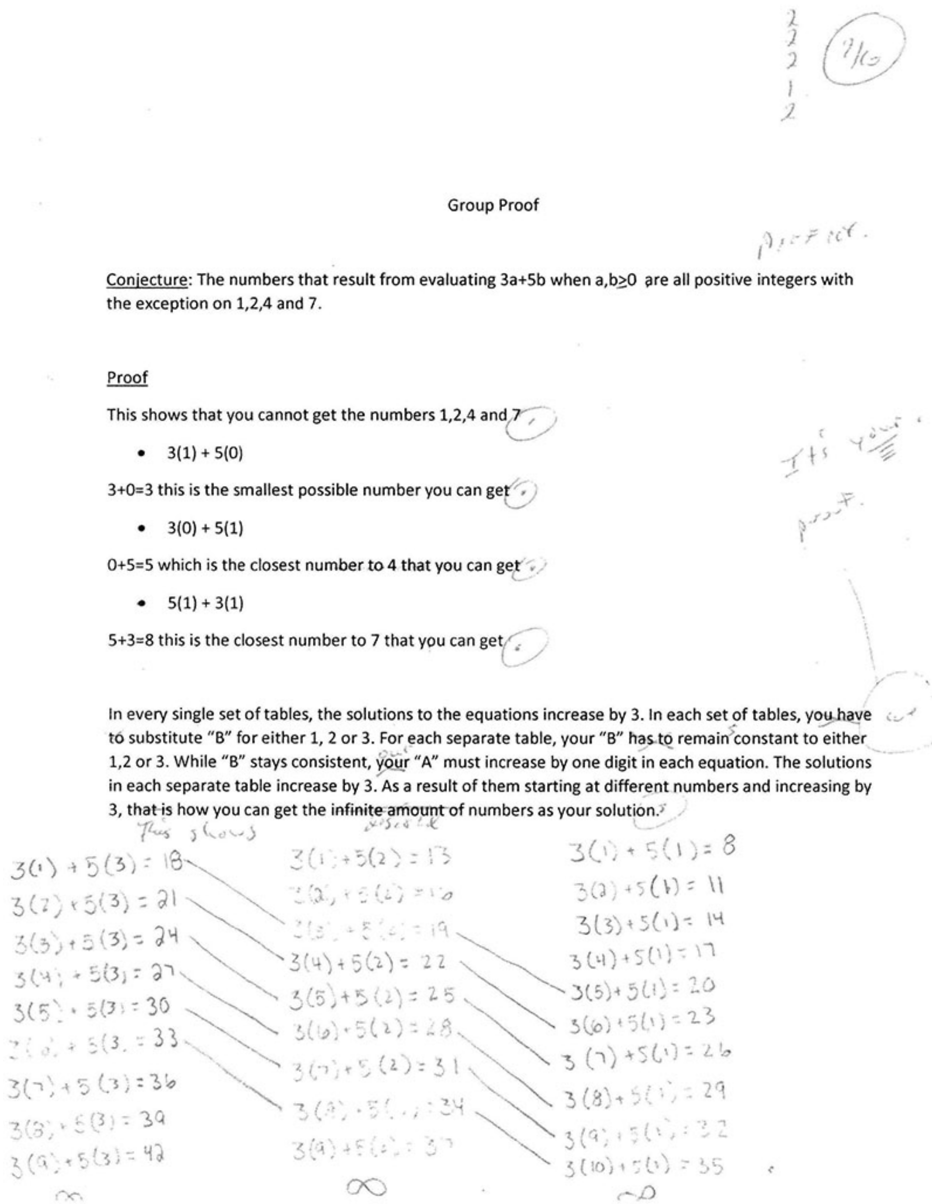
A Appendix: One Student Proof for $3a + 5b$



Group Proof

<u>Conjecture</u>: The numbers that result from evaluating 3a+5b when a,b≥0 are all positive integers with the exception on 1,2,4 and 7.

<u>Proof</u>

This shows that you cannot get the numbers 1,2,4 and 7

- 3(1) + 5(0)

3+0=3 this is the smallest possible number you can get

- 3(0) + 5(1)

0+5=5 which is the closest number to 4 that you can get

- 5(1) + 3(1)

5+3=8 this is the closest number to 7 that you can get

In every single set of tables, the solutions to the equations increase by 3. In each set of tables, you have to substitute "B" for either 1, 2 or 3. For each separate table, your "B" has to remain constant to either 1,2 or 3. While "B" stays consistent, your "A" must increase by one digit in each equation. The solutions in each separate table increase by 3. As a result of them starting at different numbers and increasing by 3, that is how you can get the infinite amount of numbers as your solution.

Figure 2: One way students prove which numbers can be generated by the Diophantine equation 3a+5b when $a, b \geq 0$.