INQUIRY-BASED LEARNING AND THE ART OF MATHEMATICAL DISCOURSE

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Abstract: Our particular flavor of inquiry-based learning (IBL) uses mathematical discourse, conversations, and discussions to empower students to deepen their mathematical thinking, building on strengths of students in the humanities. We present an organized catalog of powerful questions, discussion prompts, and talk moves that can help faculty facilitate a classroom focused on mathematical discourse. The paper brings this discourse alive through classroom vignettes and explores various teacher moves and their impacts.

The mathematical theme of the classroom investigations, Maypole dance patterns, stems from the learning guide "Discovering the Art of Mathematics: Dance." Both authors are part of the NSF-funded project "Discovering the Art of Mathematics," which provides IBL materials for Mathematics for Liberal Arts courses, see `www.artofmathematics.org`.

Keywords: Inquiry-Based Learning, Mathematics for Liberal Arts, Mathematical Discourse, Mathematics and Dance, Discovering the Art of Mathematics, Maypole Dancing

1 INTRODUCTION

Inquiry-based learning (IBL) demands an entirely new approach to teaching. What the students do is different. What the teacher does is different. The tools are different. Transitioning from traditional lecture-style teaching to inquiry-based learning can be challenging. We find the following practices for the teacher to be crucial, see also [5]:

1. anticipate student responses to challenging mathematical tasks;

2. monitor and support students' work on and engagement with the tasks;
3. select particular students to present their mathematical work;
4. sequence the student responses that will be displayed in specific order; and
5. connect different students' responses and connect the responses to key mathematical ideas.

Our Mathematics for Liberal Arts (MLA) courses are run as inquiry-based learning classrooms with a strong emphasis on classroom discourse (conversations and discussions). We believe that mathematical discourse enhances students' thinking and reasoning skills.

In this paper, we take you right into our classrooms using short vignettes. Faced with student thinking, we will engage you in finding just the right question to ask in a conversation with one student or the whole class. Additionally, we will provide you with an organized toolkit of questions for your future mathematical conversations.

We start our paper with our definition of IBL in Section 2 and introduce the mathematical task in Section 3. Sections 4 and 5 focus on tools to monitor and support students' work (Practice 2 above), while Sections 6 and 7 focus on tools to connect students' responses (Practice 5 above).

Of course, many more tools are needed for a successful inquiry-based classroom. Our project "Discovering the Art of Mathematics" provides learning guides, teacher materials, and professional development workshops; see Section 9.

2 OUR DEFINITION OF INQUIRY-BASED LEARNING

Our pedagogy builds on guided discovery, a descendant of the ancient Socratic approach to teaching (Hadamard[7], Young[14], Bronowski[1], Moise[11], Koestler[8], Lakatos[9], Freudenthal[4]). Following Laursen[10] we define IBL in post-secondary education by the following characteristics:

- The main work of the course, both within and outside of class, is problem solving.
- The majority of class time is spent on student-centered activities.
- The course is driven by a carefully built sequence of investigations that guide rediscovery.
- The teacher's role is decentralized, acting as a coach instead of a knowledge dispenser.
- Students are empowered by playing active roles - determining how class time is spent, initiating communication, and taking responsibility for learning.
- Students use reflection as well as active communication, both verbal and written, to assimilate new modes of thought, new learning strategies, and new mathematical schema.

The conversations in the vignettes are broadly based on what actually happened in the classroom. The particular student names and what they say specifically is, however, modified for the purpose of this paper.

3 INVESTIGATIONS: MATHEMATICS OF MAYPOLE DANCING

Figure 1. Maypole Dancing in Ashfield, MA, and a ribbon pattern on our classroom Maypole.

We want students in Mathematics for Liberal Arts courses to appreciate the beauty of mathematics, to care about mathematics, to tap into their creativity as they inquire, and to experience the joy when things "make sense". We find it helpful to start with a topic from the liberal arts that we are excited about, raising a puzzling question that we hope will draw them in. This will likely look very different from the mathematics that they have been exposed to and the ways in which they have previously encountered mathematical ideas.

Let us look at the start of a chapter from "Discovering the Art of Mathematics: Dance" to see what that might look like.

In medieval village life, Maypole dancing was a ritual to celebrate May Day. The pagan tradition was meant to increase vitality and fertility. May Day is still celebrated in this way in many places in Europe and also in the hill towns of Massachusetts where the authors live; see Figure 1. The standard Maypole dance has a certain number of dance couples arranged in a circle around a high wooden pole. Colored ribbons of fabric are strung from the top of the pole. Each dancer holds the end of one such ribbon.

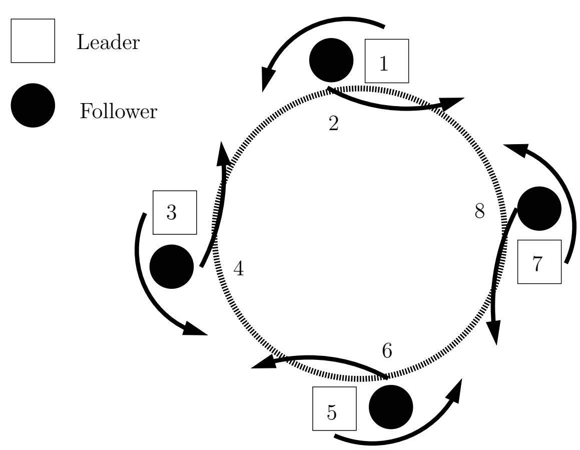

Figure 2. Starting position of a Maypole dance with four pairs of dancers. Leaders (indicated by boxes) dance in a counter-clockwise directions; followers (indicated by small circles) dance in a clockwise direction.

Figure 2 shows the starting position for a Maypole dance with four couples, as seen from above. The large dashed circle indicates the outline of the dance circle, where pairs of leaders and followers (indicated by squares and circles, respectively) are arranged for the dance. Leaders and followers move in opposite directions around the Maypole (leaders: counter-clockwise, followers: clockwise). For this initial pairing, leaders pass to the outside of the followers, as indicated by the arrows. When the dance starts, dancers will inter-weave with oncoming dancers by passing them on the inside and then outside in an alternating fashion. As the dance progresses, the colored ribbons wrap around the wooden pole, making patterns starting at the top of the pole and continuing down the pole as the dance progresses. You can watch the following video to see a Maypole dance in action: `http://www.youtube.com/watch?v=FxcIqMmlVOs&feature=related`.

Big Investigation: *How can we predict a pattern on the Maypole?*

Our students read this section as an introduction to the Maypole dancing chapter in our learning guide "Discovering the Art of Mathematics: Dance". Notice how the introduction culminates in a somewhat open and ill-posed problem, which is intentional. An important aspect of mathematics is deliberating and deciding what aspects of a problem are relevant and which we can disregard. We want to give students the opportunity to grapple with this important mathematical process right from the start. In Section 7, we describe tools for facilitating a discussion to support students in clarifying such questions as a whole class.

A bit of notation: In Figure 2 the leaders all hold white ribbons, while the followers hold black ribbons. We write WB-WB-WB-WB to notate the arrangement in couples: each couple consists of a white leader (written first) and a black follower (written second). The ribbon pattern (up to colors) for this particular arrangement looks like Figure 3: a checkerboard pattern.

Figure 3. Student recording her thinking after discovering a checkerboard Maypole pattern.

4 VIGNETTE: GROUP WORK

So, what happens when students are presented with such a challenge? In this section, we join a "Mathematical Explorations" classroom at Westfield State University, taught by Volker Ecke in spring 2013. Whenever we use the first person, this will be Volker Ecke talking about his experiences in the classroom.

4.1 Continuing the Conversation

All students sit in groups of 3-5 at group tables. I have just posed the following big investigation.

> **Big Investigation:** Given that we've seen that WB-WB-WB-WB creates a checkerboard pattern (see Figure 3), what pattern would you predict—without dancing—for the following ribbon assignments: BW-WB-BW-WB?

Figure 4. Students Discussing their Ideas.

I walk around, stop at a table, and sit down to listen to a conversation between the students at one table. See Figure 4.

As you read the vignette, imagine you are the teacher at this table. Think about what specific question *you* would ask this group next.

Sam: Well, I think we'll get bigger squares, like a 2×2 checkerboard?
Chris: Huh?
Sam: You see it's like doubling the situation we had before: instead of one B and one W, we have two of each.
Chris: Oh, that makes sense.
Kaitlyn: Wait, do you mean a square or do you mean a diamond on the Maypole?
Sam: A diamond, I think, just like with the original checkerboard, only larger.

What *specific question* would you ask this group next?

We emphasize the word *question* here, because we really don't want to *tell* the students what to do next or evaluate their thinking. We want the students to keep the power of deciding what to do and think and to evaluate their results themselves. It is not easy to limit yourself to just using questions, but we find it a valuable practice.

Returning to the group, here are some of Volker Ecke's thoughts:

I know that this is a typical misconception. In reality, the first black leader will meet the followers W, B, W, B, passing over the first and the third (both W's), so there will be at least one black band on the Maypole running diagonally. So big diamonds are out, see Figure 5. For an example of this pattern on a Maypole, see Figure 1.

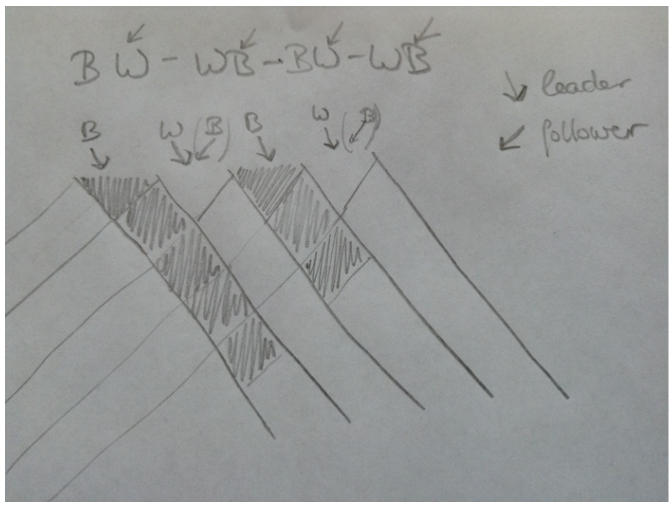

Figure 5. Student work: The ribbon pattern for BW-WB-BW-WB.

I can think of the following list of questions and am now wondering which one to choose:

1. *Can you convince me of your reasoning? Maybe a picture could help?*
   The first question will hopefully get the students to think more deeply about their answer and find the misconception themselves.[1] I hope that mentioning the picture will get them to use some representation to check their work. Maypole patterns are difficult to follow in your head with no visual representation.

[1]If I use these questions only for "incorrect" answers, the students will know that I think their answer is incorrect. But if I use these questions in general it will not give away my evaluation of their conjecture.

2. *Hmm, that other group over there was saying that they got stripes. Could you go over there and compare ideas?*

   Instead of me evaluating which group found the correct pattern, I let the students work it out themselves. This way they have to create convincing arguments when they discuss their answers and the contradiction will motivate them to do so. If I had given them the correct answer the students would be less likely to "want to be right". Letting the students communicate directly with each other also helps promote a sense of classroom community. Since both groups will have seen two ways of thinking about the problem my question also helps students to "zoom out" and become more flexible in their thinking.

3. *I'm wondering about the first leader. Doesn't the first leader meet the followers W, B, W, B, so you will only see black in that direction? [Drawing a supporting picture, similar to Figure 5.]*

   With this question I am really modeling a correct way to the students. I use this move very rarely. When no group is on a correct path to the solution after some time of working I might infuse some groups with an idea like this. Or if a group or student needs a special moment of success to become more confident this might be a way to lead them to the correct answer that they can then share out proudly.

4. *So, you're saying that you'd get a 2×2 diamond pattern, because the letter pattern is just BW doubled. I see. Could you prepare a clear presentation to convince the class of your conjecture?*

   This seems similar to the first question, but has an additional idea behind it. I know that this group has an incorrect conjecture but I want to "save it" as a gem for the whole class discussion. It is very helpful to start a discussion with a contradiction between two groups, asking the whole class to find out which answer and reasoning is correct.

5. *Could you dance the pattern to see if your conjecture is true?*

   Once the students have settled on a conjecture, we allow them to dance to check their work. Having one actual representation of the Maypole for the later class discussion will also be very helpful. When I support the different groups I already know what aspect of their work I want each group of students to share with the whole class. And I know that having one group with an incorrect idea to launch the discussion, one group with a correct pattern on the pole, and one group with a helpful picture representation would be a great asset to have available.

6. *What would you need to do to be absolutely sure that you're right?*
   This question is leading a student or group to find a proof for their conjecture. While I know that they will not be able to do so (since their conjecture is incorrect) the question will help them think more deeply about the problem. Hopefully they will find their misconception and at the same time be able to find a solid argument for the correct pattern.

7. *Kaitlyn, what do you think about Sam's idea? Do you agree or disagree?*
   This question will help with group dynamics. Maybe Kaitlyn has a different (correct) idea but was not willing or able to share it with her group. Sam seems to dominate the discussion, and this question would change that dynamic.

I am now joining the conversation started in the vignette on page **7**, *using my first choice of the above list of questions.*

Teacher: Can you convince me of your reasoning? Maybe a picture could help?
Kaitlyn: Maybe we can use the representation of the ribbons we used before *(points at a picture similar to Figure 5 which they created for the checkerboard pattern)* and just see what we get? Let's see – the first leader is black, ... *(Starting to color the crossings correctly.)*
Teacher: *(Walks away quietly since the students seem to be thinking and working.)*

4.2 A New Experience, for Student and Teacher Alike

For most of our students, this mathematics class is very different from their K-12 mathematics experience. Here, the students are learning mathematics by actively doing mathematics, rather than just listening to a teacher's explanations.

We witness a community of learning in progress. This vignette highlights the fundamental role of social interaction in the development of cognition and in "making meaning" (Vygotsky[13]). A safe environment is key for such a community to develop, so the discussions and interactions can happen in a non-judgmental way. Based on their prior experience, students may initially relate to the teacher as the source of "right" or "wrong". The questions we outline above offer some suggestions for how to engage with the students in a way that can return the authority back to them. It is important to use the same language whether the student is correct or incorrect, otherwise the students will quickly figure out that you are actually evaluating their ideas! For example, we often use "I am wondering, ..." at the beginning of a sentence, no matter the situation or content.

The teacher can set the tone for discussions in the classroom by explicitly modeling talk moves in his or her interactions with the students. It can also help to have an explicit conversation about group norms with the whole class early in the semester.

Students—and teachers—often arrive with the idea that incorrect thinking is bad, when in reality, for many of us, it is an integral and natural part of making sense of a new and complex situation. Incorrect thinking is so much more interesting sometimes than correct thinking and has an important role to play in motivating the class to "figure this out" as a community. We often think of some student misconceptions as "beautiful mistakes". Consider, for example, Sam's reasoning in the vignette on page **7**: He notices, astutely, that instead of one B and one W, there are now two of each. His idea is reasonable that this doubling will somehow show up in the ribbon pattern, leading to the creative conjecture of a 2×2 checkerboard of "diamonds". Astute, reasonable, creative. And when you try it out, it's not what you see, leading everybody to think more deeply. How beautiful!

Many of the skills we witness are higher-level skills: critical thinking, taking responsibility, justifying your opinion, questioning others, and using tools strategically. We feel that these skills are crucial in allowing our students to develop into successful, creative individuals and actively involved citizens.

5 TALK MOVES FOR MATHEMATICAL CONVERSATIONS

Given the many different situations a teacher will encounter when interacting with students, what are the kinds of questions you can ask? We need a toolkit of questions to choose from.

Inspired by Suzanne Chapin's talk moves for whole class discussions, [2], (see also Section 7), we decided to identify and organize talk moves for a conversation with just one student (or a small group). Our categories are chosen so that a teacher can easily decide which groups of questions to select from in a given situation. We recommend you pick one group at a time, find phrasings that feel natural to you, and try them out with your students.

5.1 Stages of Problem Solving

1. Understanding the Problem: *Can you read the investigation out loud? What is this investigation asking you to do?*
2. Devising a Plan: *Can you predict what will happen? Can you guess and check? Can you draw me a picture? Is there a simpler version of this problem? Is there a pattern? What do you know about ...? What is the relationship of ... to ...? What would happen if ...? What if the numbers were ...? Are any of your previous strategies helpful? When stuck, what has helped you before?*
3. Carrying out a Plan: *So if I use your strategy for ..., what would happen? Is there another way to solve the problem so you can check your answer? Can you start a new page and organize all your information?*

4. Looking Back: *Is there another solution? How do you feel about your answer? How do you know your answer is reasonable?*
5. Explaining: *How can you convince me (us)? Why do you think that? Why did you ... ? Can you write an explanation for next year's students? What do you think would happen if . . . ? What is the difference between . . . and . . . ? How are . . . and . . . similar? What is a possible solution to the problem of . . . ? How does . . . affect . . . ? In your opinion, which is best, . . . or . . . —and why? Do you agree or disagree with this statement: . . . ? (Support your answer.)*
6. Generalizing: *Does . . . always work? Can you make a general claim or conjecture? What else would you like to know? What conclusions can you draw about . . . ?*

Here the reader may note that we have categorized our questions in a manner similar to Stenmark[12] and Polya, and we have created two additional categories: "Explaining" helps students find gaps in their reasoning and understand their own ideas and strategies in greater depth. Additionally, explaining a solution helps the teacher assess what a student actually understands. "Generalizing" is an important aspect of mathematical (and critical) thinking. It forces students to look at a bigger picture and think hard. Generalizing will also lead naturally to further mathematical questions that you might want to pursue in class.

5.2 General Moves to Clarify Thinking

The following talk moves can be used at any time when a student is lost or stuck. They can also help the teacher understand what a student is doing and thinking. (You might learn something new here!) Two of Suzanne Chapin's general talk moves work well for a one-on-one conversation as well (revoicing & wait time), so we list them here and in Section 7 .

1. Clarifying: *I am wondering about . . . Can you help me understand? Can you tell me what you are thinking about? Can you tell me what you mean? Tell me more.*

2. Grappling with a contradiction: *What do you* think*? What is happening here?*
3. Activating Prior Knowledge: *How would you use ... to ... ? What is a new example of ... ? How is ... related to ... that we studied earlier?*
4. Revoicing: *What I hear you say is ...*
5. Wait time: *Do you want more time to think?*
6. Modeling: *I think I'll see what's going to happen if I use your strategy for ..., ok? Let me see, how about trying ... ?*

"Modeling" is helpful if a student or group is completely stuck or attached to a strategy that will not work out, see choice 3 in the classroom vignette on page **9** for an example.

5.3 General Moves for Emotional Support

Doing inquiry-based mathematics can be new and unsettling to students. Supporting them emotionally is often as important as supporting them mathematically. If they are unmotivated it can help to draw them in by showing them what *you* think is interesting—maybe using a video, a drawing, a pattern, showing a connection with history or an application, etc. We also use tools from the Nurtured Heart Approach [6]. This approach, among other things, uses positive reinforcement to encourage persistence, sharing, curiosity, creativity, etc.

1. Student is insecure and fishing for evaluation: *I am not saying that you are right or wrong; I am just wondering. / What do* you *think?*
2. Student is frustrated: *I know it's hard. / I love your persistence, I can see that you are not giving up. / Why don't you check in with ... ?*
3. Student is anxious: *You can do it! / Take a breath. / Get coffee. / Look back at ... I know you could do this before. / What would help you right now?*
4. Student is unmotivated or unfocused: *Which problem are you thinking about? / Let me show you something cool (maybe unrelated). /*

Back to math / Let's focus. / I think this is so interesting because . . .

5.4 Ending a Conversation

In conversation with a student, we often just walk away when we believe a student has reached the point of being capable to continue without further guidance. This conveys to the students that you trust them to figure it out for themselves. The conversation usually does not end with the solution of the problem, since then we have probably "given too much away".

1. Walk away: *You can do this, I'll check in later.*
2. Walk away silently (if students are engaged in their work).

6 VIGNETTE: WHOLE CLASS DISCUSSION

So, what does that look like when we bring the whole class together to discuss each group's ideas? What actions can the teacher take in the moment to help create an environment where sharing everybody's ideas (and confusion) is valued? What kinds of talk moves are instrumental for deepening student thinking? Let's listen in.

In the above Vignette we already talked about how I have the whole class discussion in mind when I ask my students questions. Let's say Groups 1 and 2 (of my 4 groups) are still thinking that we will get large diamonds, while Kaitlyn's group (Group 3) is now convinced that we get alternating stripes. Group 4 is still undecided, and did neither dance nor draw a picture to help themselves.

> *Teacher*: Ok, let's get back together for a moment, because I have a problem I need your help with. Group 1 convinced me of the fact that we get big diamonds but Kaitlyn has a picture that shows alternating stripes. I would like to have both ideas shared out in detail and then we can decide as a whole class what to do, ok? Jody, could you please start with your thinking?

Jody: Well, the patterns is just like the BW-BW-BW-BW pattern, just double, so it has to be the same.
Teacher: Can someone from Group 4 rephrase in their own words what Jody is thinking? Or ask her a question?
Ted: I think Jody believes it to be diamonds because we got diamonds before and this pattern is very similar, but there are two W's and two B's now in each step, so the diamonds are double in size.
Teacher: Thank you. Now let's look at Kaitlyn's idea.
Kaitlyn: So here's my picture. *(Points at her picture on the ELMO, see Figure 5 on page* **8***.)* You can see how the first black leader crosses over all the white followers and under the other black followers, so all you can see is black. The same happens when we start with a white leader, just then all white followers are on top. So it has to be stripes.
Teacher: Wow, that was a lot of information. Can someone restate Kaitlyn's thinking or ask her a question? *(Wait time)*
Ted: I can follow Kaitlyn's thinking about the stripes, but I still don't understand why they are not diamonds. Which color combination would give us big diamonds then?
Teacher: Before we think about Ted's question, how many of you can follow Kaitlyn's reasoning? Show me a "thumbs-up" if it totally makes sense, a "side thumb" if you have questions, and a "thumbs-down" if this makes no sense to you. *(Most students give a thumbs-up, some a side thumb, no one a thumbs-down.)*
Teacher: Ok, take a minute in your group to digest what just happened.

I am circling around, listening or helping. After about 5 minutes, I realize that all of the groups now agree with Kaitlyn's thinking.

Teacher: Let's get back together, I would love to get back to Ted's question. He was wondering, since we now think we don't get big

diamonds for BW-WB-BW-WB which combination of leaders and followers would give us big diamonds? Does anyone have an idea about this?
Sam: How about BB-WW-BB-WW? Now they really come in doubles.
Teacher: I can see what you are thinking. Why don't you all use Kaitlyn's representation to check if Sam's pattern gives us big diamonds.

Notice that I only moved on after the students seemed comfortable with Kaitlyn's idea. But to me, the most interesting part is the question that **naturally** *arose for the class: Given a pattern, can I predict the dance combination? This is another great* Big Investigation *that leads to lots of interesting mathematical thinking and it came out of the curiosity of the students, not because I decided to present it*[2].

7 TALK MOVES FOR WHOLE CLASS DISCUSSIONS

So, we want our students to share their thinking with the whole class. We want them to understand each other's approaches, critique each other's reasoning, argue why they agree or disagree, and in the process deepen their own understanding. How to make this happen? How to keep it going? Both what you say and how you say it are critical.

This section provides you with some teacher talk moves that you can use in whole class discussions. It draws heavily from "Five Productive Talk Moves" in *"Classroom Discussions: Using Math Talk to Help Students Learn"* by Suzanne H. Chapin, Catherine O'Connor and Nancy Canavan Anderson [2, 3]. It's a book we strongly recommend.

Revoicing. ("So you're saying that you will get big diamonds?")

[2] As it turns out, 2×2 diamonds are not possible as a pattern with the regular Maypole dance, no matter how many dancers we have and no matter how we assign black and white ribbons to them. If you follow one dancer's ribbon as it alternately weaves above and below the other ribbons, its color will show in every other square. Yet on a 2×2 diamond, these squares show different colors.

When students express their thinking it may not be completely clear. This talk move allows us to maintain and deepen the conversation by asking the student whether what we heard was indeed what she or he meant. It focuses the attention on one particular aspect and gives the student the opportunity to clarify, correct, or provide further evidence for her or his thinking. Revoicing can also be helpful when we feel that other students may not have grasped what the student has said, thereby giving them another chance to understand.

Asking students to restate someone else's reasoning. ("Can you repeat what he just said in your own words?") Extending the idea of revoicing, we can also ask another student to put what she or he heard into her or his own words. First, this gives everybody another chance to follow the conversation and understand the speaker's point. Second, this move provides evidence that the other students could and did hear what was said. Finally, it again clarifies the claim the student is making, providing evidence that student thinking is taken seriously. Over time, students make efforts to make their contributions more comprehensible.

Asking students to apply their own reasoning to someone else's reasoning. ("Do you agree or disagree and why?") Once we are convinced that students heard a claim and had time to think about it, we want to learn about their reasoning about the claim. It is important for the teacher to refrain from supporting one or another position at this point, placing the authority in the community of learners. Asking "Why?" is critical in supporting students' mathematical reasoning and learning.

Prompting students for further participation. ("Would someone like to add on?") Since the teacher does not step in as the authority to declare what is correct and what isn't, it is important to create a space where differing viewpoints can be voiced.

Using wait time. ("Take your time . . . we'll wait . . .") Often this does not manifest as speech but as silence. It is valuable to wait at least ten seconds for students to think before calling on someone to answer. Similarly, students need to be given time to organize their thoughts. Many students have stopped participating in mathematics class because speed seemed to be a key criterion for answers in mathematics class. Shifting the value from speed to depth of thinking, with wait time, can help broaden participation.

8 ASSESSMENT

Teaching in an inquiry-based way using creative topics requires different assessment techniques. Our assessments are drawn from poster projects, oral presentations, written journals and proofs, written answers to all investigations, observations to justify participation grades, notebook quizzes and more. Describing all of them in detail would go beyond the scope of this paper. Please check our website `www.artofmathematics.org` for a future publication on assessment techniques.

9 OUR PROJECT: "DISCOVERING THE ART OF MATHEMATICS"

Our *Discovering the Art of Mathematics* project team consists of Professors Julian Fleron, Phil Hotchkiss, Volker Ecke and Christine von Renesse. We developed 11 learning guides, each for teaching a semester long inquiry-based Mathematics for Liberal Art (MLA) course, and each focusing on a particular content area, e.g. Dance, Number Theory, Music, Knot Theory, Geometry, Games and Puzzles, The Infinite, and Reasoning and Truth.

We are currently working on materials supporting teachers in using our learning guides and using inquiry-based teaching techniques. We are also offering traveling workshops (NSF funded) to help departments practice inquiry-based approaches in MLA and other classes. Please visit our website `www.artofmathematics.org` or email us at `artofmathematics@westfield.ma.edu` for all our free materials and more detailed information. We also having funding available for faculty to beta test our materials. Contact us.

As part of our project we evaluate student beliefs and attitudes about mathematics using pre and post surveys. The data shows positive changes in response to our materials and teaching methods. Details can be found on our web site.

Our project is based upon work currently supported by the Nation Science Foundation under NSF1225915 and previously supported by NSF0836943 and a gift from Mr. Harry Lucas.

ACKNOWLEDGEMENTS

We thank our colleague Judy MacKinnon for sharing her expertise with facilitating classroom conversations in mathematics. None of this work would be possible without the amazing support from our colleagues at Westfield State University, especially Julian Fleron and Phil Hotchkiss.

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BIOGRAPHICAL SKETCHES

C. von Renesse uses open inquiry techniques in all of her teaching. Her passion for music and dancing has been woven into her teaching as part of her inquiry-based approach to mathematics for liberal arts classes. Christine has advanced degrees in Elementary Education, Music and Mathematics from the Technical University Berlin, Germany and a Ph.D. in Computational Algebraic Geometry. She is now at Westfield State University.

V. Ecke loves diving into a mathematical inquiry with his students where they can discover their own power in making sense of mathematics, often for the first time. After undergraduate studies in mathematics and physics in Germany, Volker earned a Ph.D. in Computational Algebraic Geometry and Theoretical Computer Science before joining the faculty of Westfield State University.