CHAPTER 3

Other Puzzles: Kakuro, Radon/Kaczmarz, and What's Inside of You

1. What made magic squares magical is that the sums of rows, columns, diagonals, and perhaps other groups of entries all have a common sum. What can you say about the sums of the columns of a Sodoku puzzle? The rows? The diagonals? Other groups of entries?
2. For a Latin square made with numbers, what can you say about the sums of the columns? The rows? The diagonals? Other groups of entries?
3. Use Investigation **1** to repharse the rules of Sodoku in terms of sums.

1. Kakuro

Kakuro is a puzzle with a history and popularity that resemble those of Sodoku. **Kakuro** "boards" are grids made of squares, looking much like a typical crossword puzzle board - many blank squares with a number of solid black squares. Taken together, all of the blank squares that are adjacent without interruption by a black square - in a given row or column are called a **run**. Like Sodoku, to solve a Kakuro puzzle you need to fill in all of the blank squares using only the numbers 1 - 9. The differences between the puzzles - other than their shape - are the rules and the clues:

- There is a clue for every run - the clue tells you what the ***sum of the terms in the run*** must be. No other clues are given; i.e. unlike Sodoku, no squares have been filled in.
- No number can be repeated in any run.

Like Sodoku puzzles, the solution to a Kakuro puzzle is expected to be unique.

4. Try to solve the Kakuro puzzle in Figure 1.
5. What did you find difficult about this puzzle? What did you find easy? What strategies did you use?
6. In Figure 2 is a *portion* of a Kakuro puzzle. How can you tell that it is only a portion of a puzzle?
7. Can you solve the portion of the puzzle in Figure 2? If so, can this portion of the puzzle be solved uniquely? Whatever your answer, prove that your result is correct.
8. Similarly, can you solve the portion of the puzzle in Figure 3? If so, can this portion of the puzzle be solved uniquely? Whatever your answer, prove that your result is correct.
9. Similarly, can you solve the portion of the puzzle in Figure 4? If so, can this portion of the puzzle be solved uniquely? Whatever your answer, prove that your result is correct.
10. Similarly, can you solve the portion of the puzzle in Figure 5? If so, can this portion of the puzzle be solved uniquely? Whatever your answer, prove that your result is correct.

11. Try to solve the Kakuro puzzle in Figure 6.
12. What did you find difficult about this puzzle? What did you find easy? Did you find any new strategies?

13. Can you solve the portion of the puzzle in Figure 7? If so, can this portion of the puzzle be solved uniquely? Whatever your answer, prove that your result is correct.

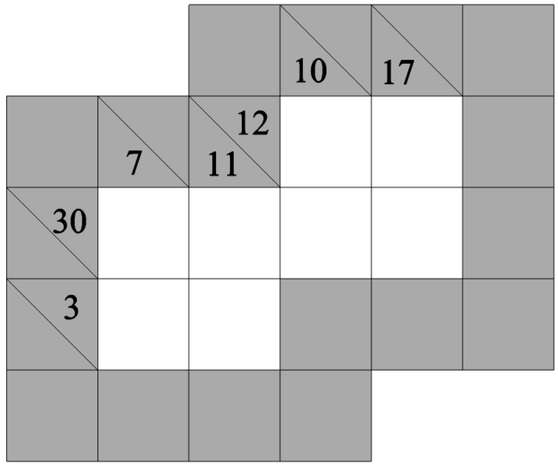

FIGURE 1. Junior Kakuro puzzle #1.

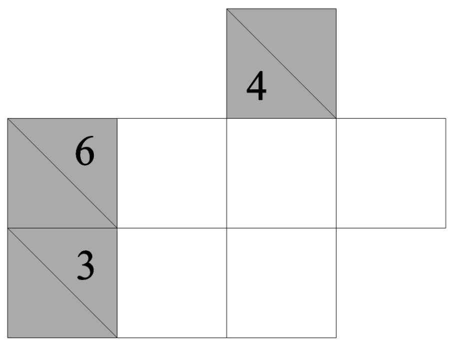

FIGURE 2. Kakuro excerpt #1.

By now you should be noticing that some clues are more valuable than other simply because the clue number can be broken up in only a few ways. For example, if our clue is a 7 for a run of three then the only possible combination is to use a 1, 2, and a 4. This is quite valuable. Frequent Kakuro players will have many of these examples at hand. For amateurs, like the authors, there are Kakuro Combination Charts available in many places on the Internet.

How many different possible combinations a number can take is actually an important mathematical topic. We call $7 = 1 + 2 + 4$ a ***partition*** of the number 7. There are 15 different partitions of the number 15. This includes many partitions that are not legal in Kokuro, since digits are repeated, like $7 = 2 + 2 + 1 + 1 + 1$. Partitions were first studied systematically by the great **Leonard Euler** (Swiss mathematician; -). Some of the most important progress was made by the great, but tragically short-lived **Srinivas Ramanujan** (Indian mathematician; -). For decades after Ramanujan's death the subject saw mainly minor advances. In 1999-2000, while studying Ramanujan's notebooks, **Ken Ono** (American mathematician; -) made a discovery that shocked the mathematical world - *partition congruences* must exist for every prime number, not just the small family they were thought to hold for. Shortly after his discovery that these partition congruences must exist, one of Ono's undergraduate students, **Rhiannon Weaver** (; -), found 70,000 partition congruences - mathematical patterns that were thought not to exist.

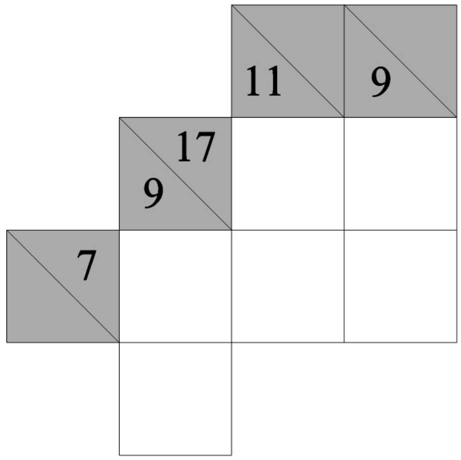

FIGURE 3. Kakuro excerpt #2.

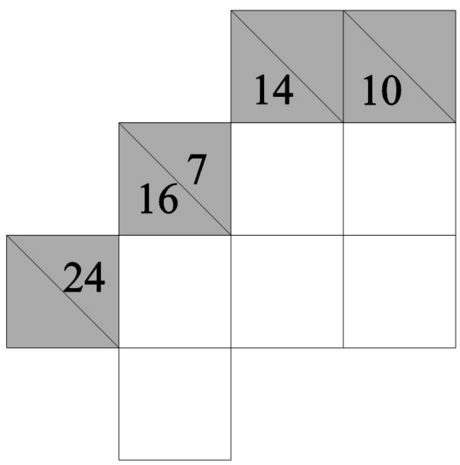

FIGURE 4. Kakuro excerpt #3.

Partitions make up the focus of two chapters of Discovering the Art of Number Theory in this series and the interested reader is encouraged to look there. Indeed, it was the discovery of Ono that motivated much of that book. And ironically, as this section of Kokuro was being written, Ono and several colleagues had another shock for the mathematical world. Partitions, counting the number of ways a given whole number can be written as the sum of other whole numbers, are inherently fractal in nature! Their discovery came in a remarkable fashion, as an Emory University press release tells us:

> A eureka moment happened in September, when Ono and Zach Kent were hiking to Tallulah Falls in northern Georgia. As they walked through the woods, noticing patterns in clumps of trees, Ono and Kent began thinking about what it would be like to walk amid partition numbers. "We were standing on some huge rocks,

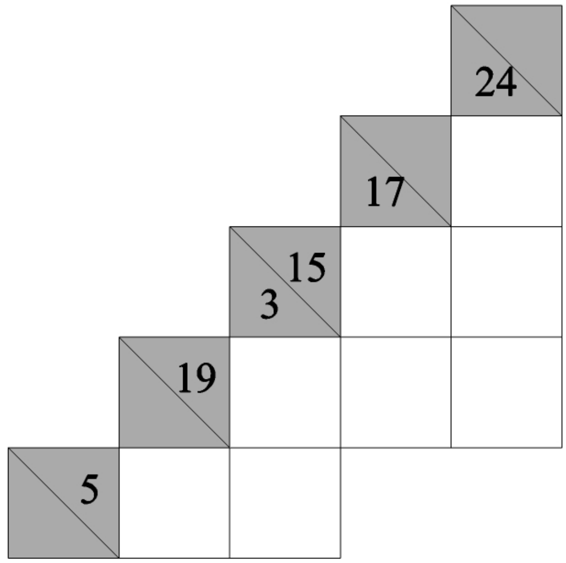

Figure 5. Kakuro excerpt #4.

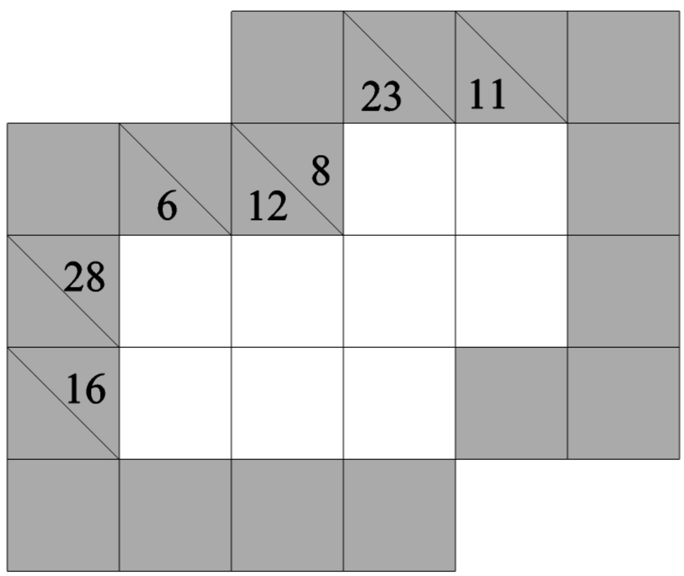

Figure 6. Junior Kakuro puzzle #2.

where we could see out over this valley and hear the falls, when we realized partition numbers are fractal," Ono says. "We both just started laughing."[1]

Never underestimate the power of your subconscious.

14. Similarly, can you solve the portion of the puzzle in Figure 8? If so, can this portion of the puzzle be solved uniquely? Whatever your answer, prove that your result is correct.

[1] "New theories reveal the nature of numbers," by Carol King available at the `http://esciencecommons.blogspot.com/2011/01/new-theories-reveal-nature-of-numbers.html`.

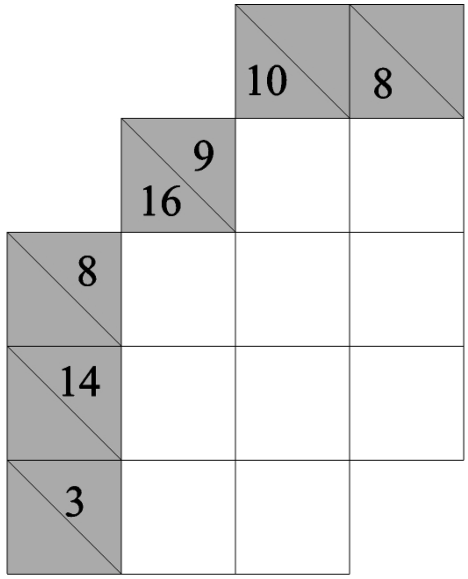

Figure 7. Kakuro excerpt #5.

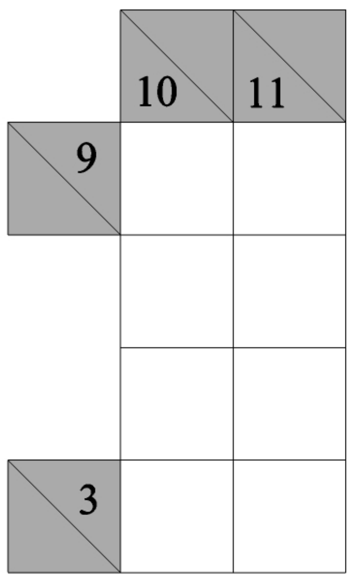

Figure 8. Kakuro excerpt #6.

15. There are many Kakuro puzzles available online - of all different levels of difficulty. Find one that you thinks is near the top end of your ability level. Solve this puzzle. Describe the major challenges.

2. Radon/Kaczmarz Puzzles

The next puzzles are our own invention and we have chosen to name them **Radon/Kaczmarz puzzles**, or **RK puzzles** for short, in honor of the mathematicians **Johann Radon** (Austrian mathematician; 1887 - 1956) and **Stefan Kaczmarz** (Polish mathematician; 1895 - 1940) whose work we will describe later.[2]

Like Sodoku and Kakuro puzzles, RK puzzles involve filling in grids with numbers constrained by certain rules and satisfying certain clues. In RK puzzles the numbers entered into the grid are required to be whole numbers 1 - 9, only now numbers *can* be repeated. The clues for RK puzzles are sums of terms, like in Kakuro, only here the sums take many different forms. We will refer to these sums as ***aggregates***. Aggregate data will be given visually, as in Figure 9 below. In this figure the aggregate data and puzzle grid are given on the left, the solved puzzle on the right.

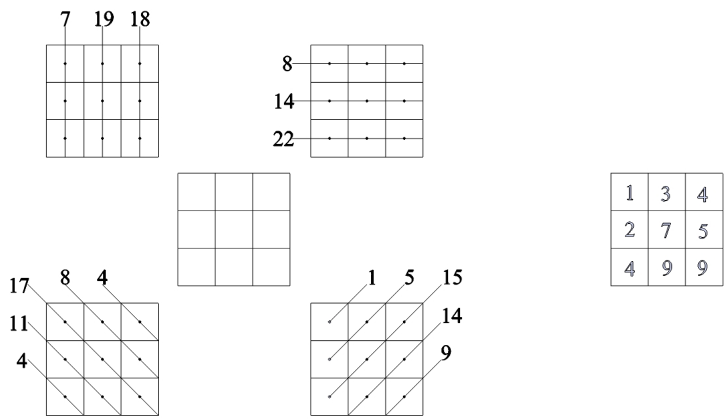

FIGURE 9. RK puzzle board and aggregate data (left) and solved puzzle (right).

16. Can you solve the RK puzzle #1 in Figure 10 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
17. Can you solve the RK puzzle #2 in Figure 11 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
18. Can you solve the RK puzzle #3 in Figure 12 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
19. Can you solve the RK puzzle #4 in Figure 13 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
20. Can you solve the RK puzzle #5 in Figure 14 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
21. Can you solve the RK puzzle #6 in Figure 15 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
22. Find simple conditions that are necessary for an RK puzzle to have a solution. These are conditions that can be checked without having to try to explicitly solve the puzzle. These conditions should be robust enough that they explain why any of the insolvable RK puzzles above are in fact not solvable. Prove that these conditions are necessary.
23. Are the conditions in Investigation **22** sufficient conditions for an RK puzzle to have a solution? I.e. if the conditions are satisfied does this guarantee that there is a solution to the RK puzzle? Prove that your result is correct.
24. Make and then prove a *positive result* about the solution of *arbitrary* 3 by 3 RK puzzles where the aggregate data includes the 3 vertical aggregates, the 3 horizontal aggregates, and both the 5 left and 5 right diagonal aggregates.

[2]After their development, we learned of the *Challenger puzzle* which is quite similar to a 4 by 4 R/K puzzle and which appears in a number of newspaper puzzle columns.

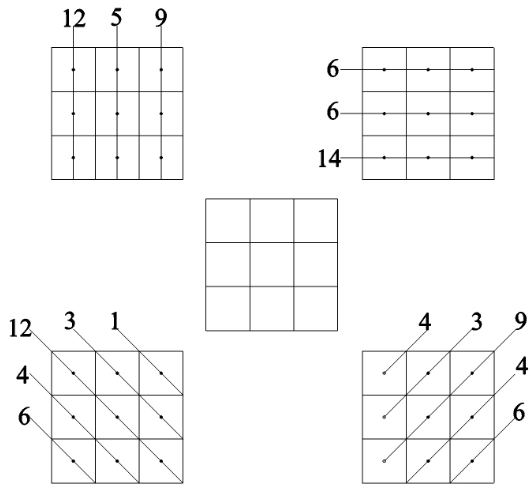

Figure 10. RK puzzle #1.

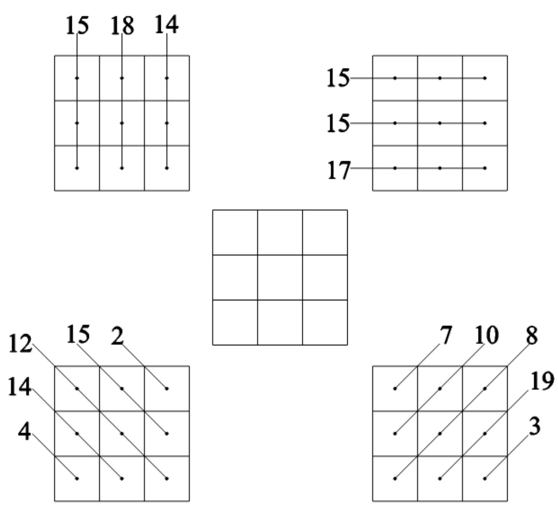

Figure 11. RK puzzle #2.

We are now going to move on to larger RK puzzles.

25. Can you adapt the conditions you found in Investigation **22** to 4 by 4 puzzles? If so, explain how. If not, explain why not. What about RK puzzles that are larger than 4 by 4? Explain.

26. Can you solve the RK puzzle #7 in Figure 16 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

27. Can you solve the RK puzzle #8 in Figure 17 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

28. Can you solve the RK puzzle #9 in Figure 18 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

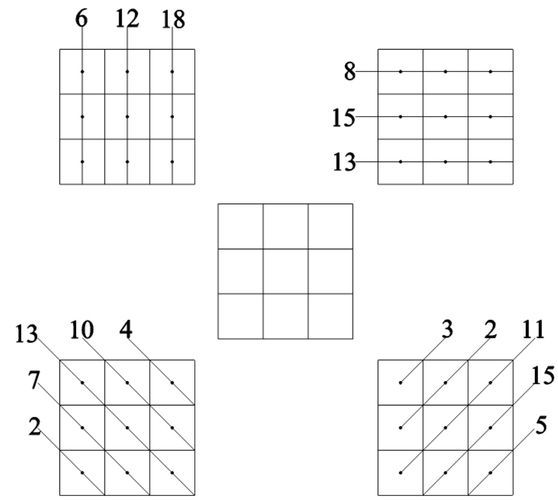

Figure 12. RK puzzle #3.

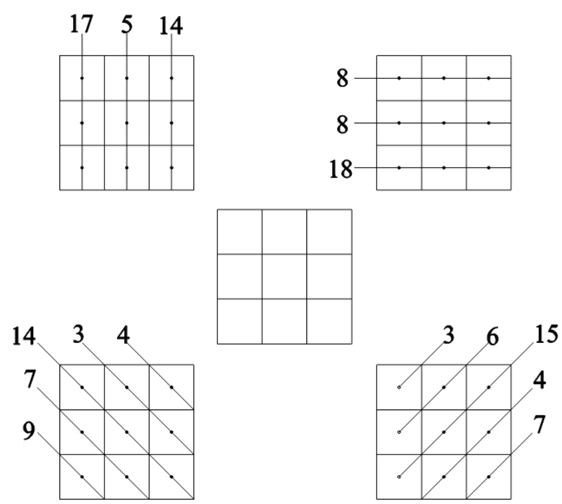

Figure 13. RK puzzle #4.

29. Can you solve the RK puzzle #10 in Figure 19 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

30. Can you solve the RK puzzle #11 in Figure 20 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

31. Suppose that you are given the additional ***shallow left diagonal*** aggregate data, shown in Figure 21, for RK puzzle #11 in Investigation **30**. Can you solve the puzzle? If so, is the solution unique? If so, do you need all of the additional data? Whatever your answer, prove that your result is correct.

32. Make and then prove a positive result about the solution of *arbitrary* 4 by 4 RK puzzles where the aggregate data includes the 4 vertical aggregates, the 4 horizontal aggregates, both the 7 left and 7 right diagonal aggregates, and as few shallow left diagonal aggregates as possible.

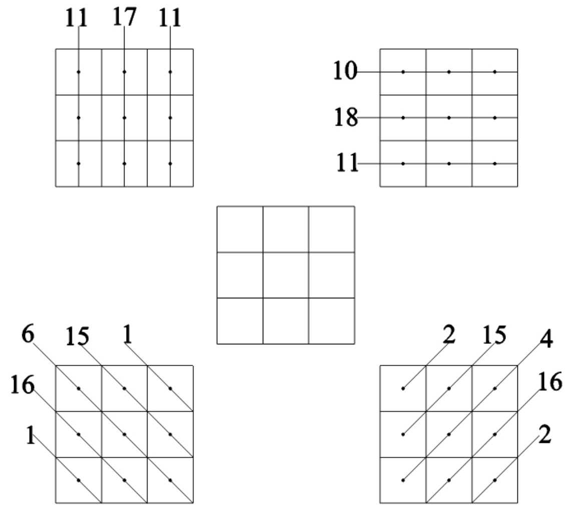

FIGURE 14. RK puzzle #5.

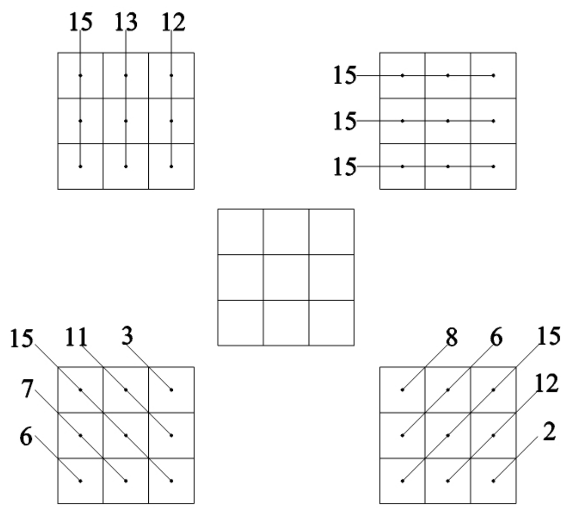

FIGURE 15. RK puzzle #6.

33. Make your own 4 by 4 RK puzzle whose given aggregate data is: 4 vertical, 4 horizontal, 7 left diagonal, 7 right diagonal, and 10 shallow left diagonals. Explain carefully how you made the puzzle. (Note: Templates are included in the appendix.)

Sodoku, Kakuro, and RK puzzles all assume that there is a unique solution to be had - and one expects sufficient clues to be given or it is not a valid puzzle. In each, our job is to *reconstruct* the solution from incomplete data. Such reconstruction makes puzzles like this part of the important class of problems known as *inverse problems*.

34. Make your own 5 by 5 RK puzzle whose given aggregate data is: 5 vertical, 5 horizontal, 9 left diagonal, 9 right diagonal, and 13 shallow left diagonals. Explain carefully how you made the puzzle. (Note: Templates are included in the appendix.)

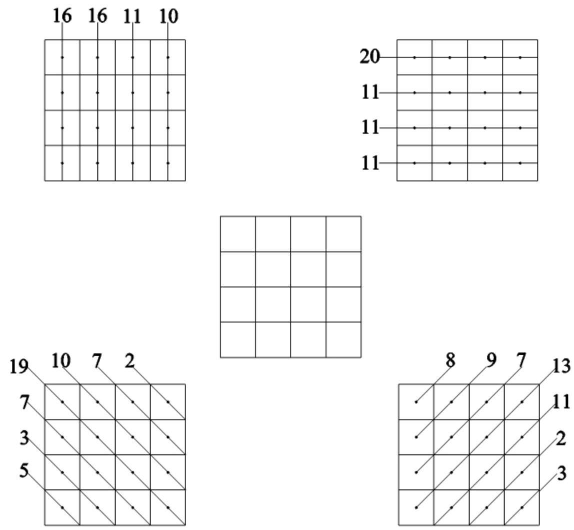

Figure 16. RK puzzle #7.

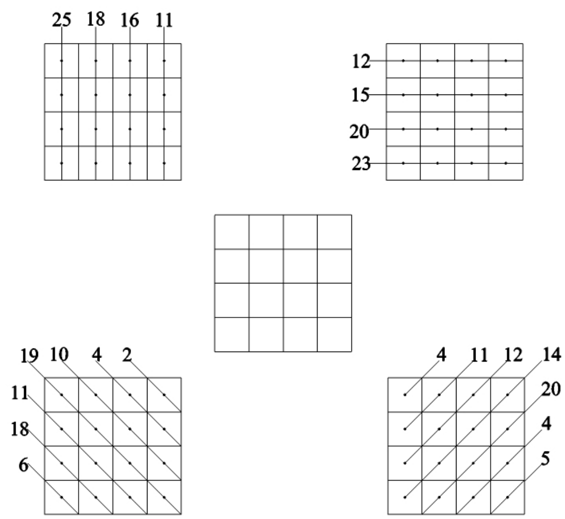

Figure 17. RK puzzle #8.

35. Switch 5 by 5 RK puzzles with a partner - giving them only the aggregate data. Were you able to solve the puzzle they gave you with the given aggregate data? Were they?

36. Make and then prove a positive result about *arbitrary* 5 by 5 RK puzzles where the aggregate data includes the 5 vertical aggregates, the 5 horizontal aggregates, both the 9 left and 9 right diagonal aggregates, and the 13 shallow left diagonal aggregates.

In general, having 6 vertical, 6 horizontal, 11 left diagonal, 11 right diagonal, and 16 shallow left diagonal aggregates is not sufficient information to solve a 6 by 6 RK puzzle.

37. Given that such a 6 by 6 RK puzzle is not uniquely solvable, what other aggregates can you think of that might be useful as clues?

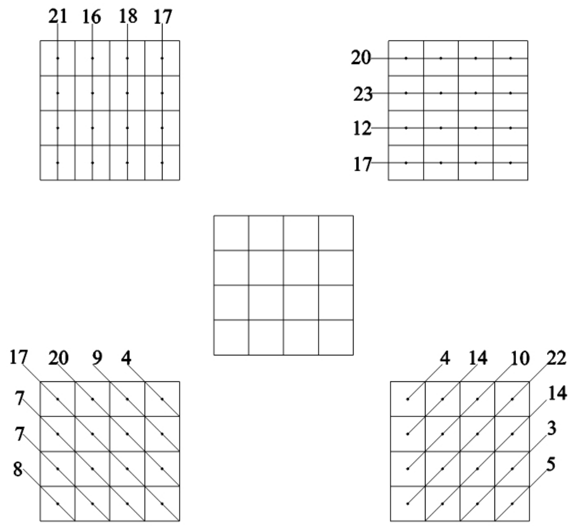

FIGURE 18. RK puzzle #9.

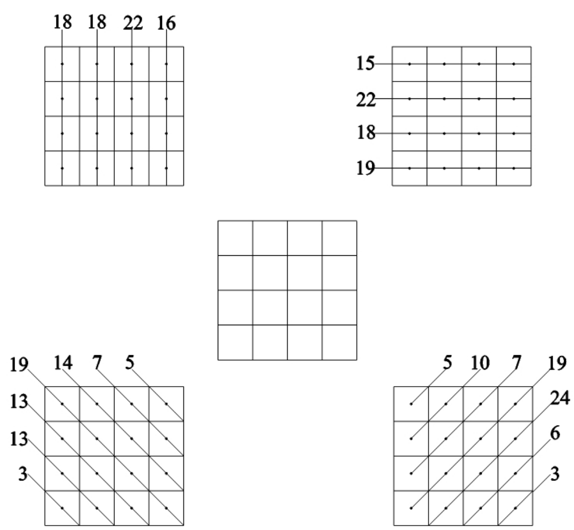

FIGURE 19. RK puzzle #10.

38. Can you solve the RK puzzle #12 in Figure 22 with the given aggregates? If so, solve it. If not, explain why it cannot be done.

39. Can you solve the RK puzzle #13 in Figure 23 with the given aggregates? If so, solve it. If not, explain why it cannot be done.

As the size of RK puzzles increase, it is interesting - and important - to know how many aggregates may be required to uniquely solve any solvable puzzle. This is a question that arises in algebra often - *systems of equations* that do not have enough data to be solved uniquely are called *under-determined systems* while those that have enough data to be solved uniquely are called *over-determined systems*.

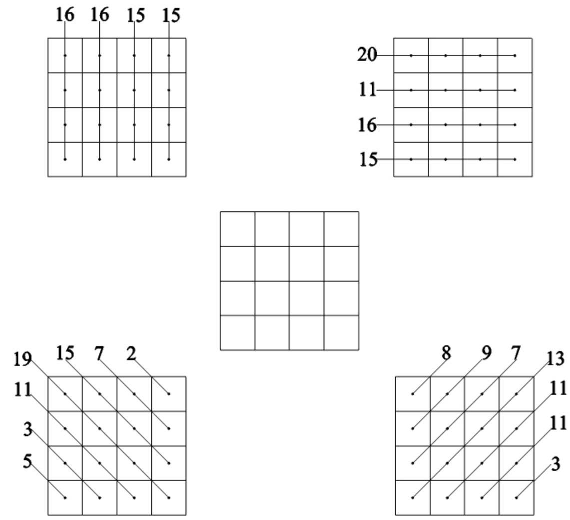

Figure 20. RK puzzle #11.

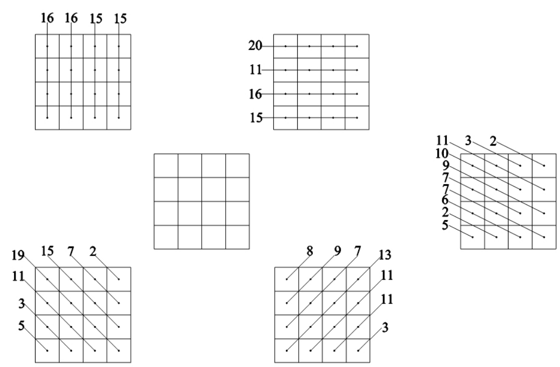

Figure 21. Shallow Left aggregates for RK puzzle #11.

40. Let's begin by considering 2 by 2 RK puzzles. Can you find the smallest number of aggregates that will insure that there is a unique solution to an *arbitrary* solvable puzzle of this size? Explain.

41. You have collected some data in your investigations that may help suggest how many aggregates are necessary. Complete the following table based on your investigations above. Here "Number of Variables" means that total number of pieces of data that must be recovered to solve the RK puzzle and "Number of Clues Needed" is an upper bound on the number of aggregates necessary to insure that a solvable puzzle has a unique solution.

Size of RK Puzzle	Number of Variables	Number of Clues Needed
2 by 2	4	
3 by 3	9	
4 by 4		
5 by 5		
6 by 6		
9 by 9		

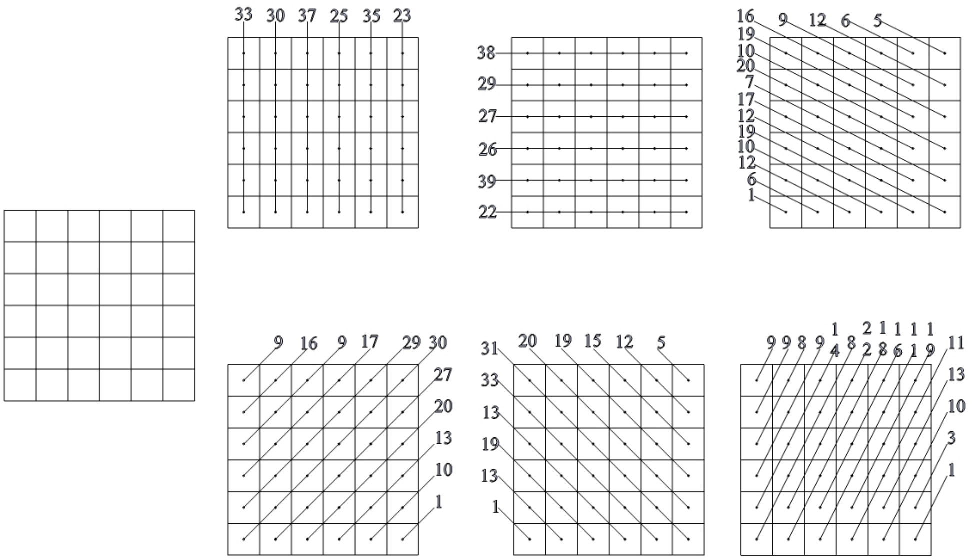

FIGURE 22. RK puzzle #12.

42. For an n by n RK puzzle, how many pieces of data need to be recovered to solve the puzzle? Based on your table above, can you make a conjecture about the approximate number of aggregates that are needed to insure that there is a unique solution?[3] Explain.

43. Roughly, how many aggregates do you think may be necessary to solve a 100 by 100 RK puzzle? Explain.

44. Roughly, how many aggregates do you think may be necessary to solve a 10,000 by 10,000 RK puzzle? Explain.

3. Why Are We Doing This?

You have explored lots of different puzzles of similar types. We hope that you have found them of interest; that they challenged your intellect. At this point you may be ready to move on to something new. Certainly a 100 by 100 RK puzzle does not seem like it would be much fun. A few years ago after working through much of this material and then tiring of RK puzzles an exasperated student said, "Why are we doing this? You said this was important. It better be a cure for cancer or something, because I'm about done." He wasn't far off.

In 1917 **Johann Radon** (Austrian Mathematician; 1887 - 1956) developed what has become known as the *Radon transform.* Developed to solve geometric problems he was studying, this transform is a cousin the *Fourier transform* which, among many other critical things, is a central tool in the conversion of digital data into analog forms; i.e. it is how your iPod plays music! While the Fourier transform plays a central role in mathematical analysis, the Radon transform was, for a long time, not very widely recognized.

Radon had a long and distinguished career in mathematics.

In 1937 **Stefan Kaczmarz** (Polish mathematician; 1895 - 1940) developed what has become known as the *Kaczmarz method* for algebraically solving systems of linear equations. Like the Radon transform, this development was not very widely heralded.

In September, 1939 both Nazi Germany and the Soviet Union invaded Poland. Poland was quickly overrun. It's military leaders and officers were all imprisoned. Also imprisoned were many

[3] Notice the qualifying terms "conjecture" and "approximate." This is, as far as we know, an open problem. It may be a decent research problem for an advanced, undergraduate mathematics majors.

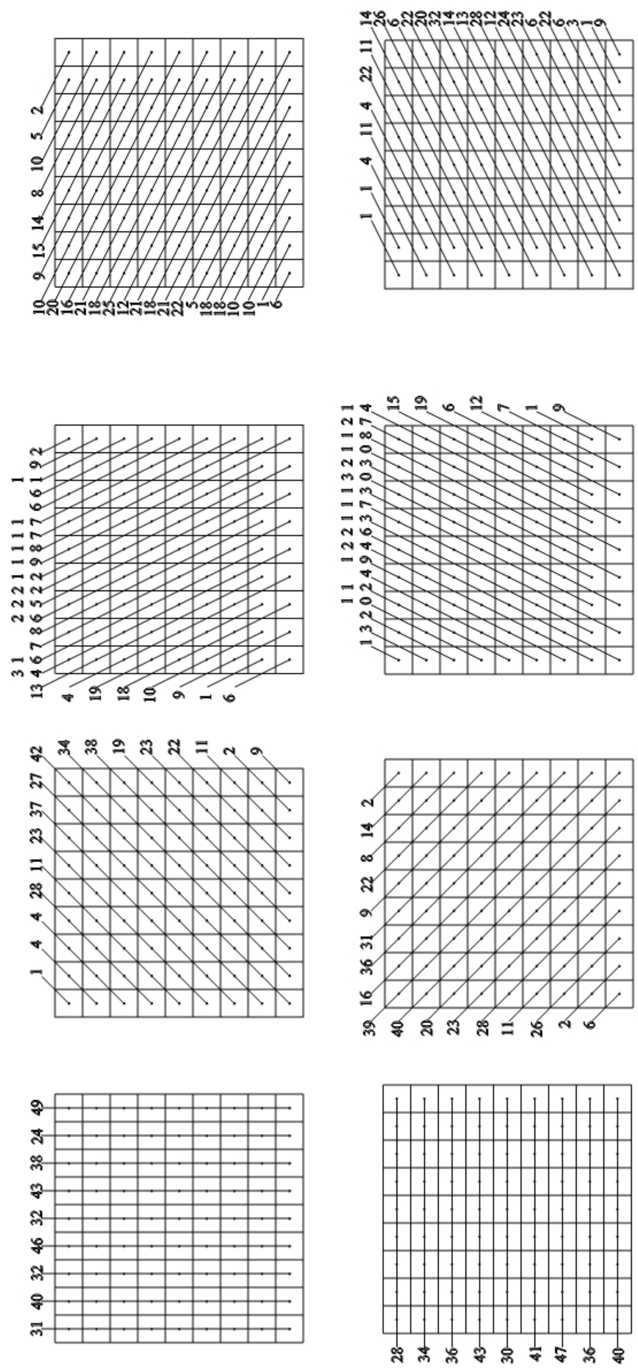

Figure 23. RK puzzle #13.

of its police, doctors, public servants, lawyers, priests and academics. A huge number of these prisoners of war were eventually killed. Some 22,000 were killed in the ***Katyn Forest Massacre*** in the forests near Katyn, Russia. Mass graves were discovered by the Nazis in 1943. While the Nazis were quickly blamed, many believed the massacre was carried out by the Soviet Secret Police.

FIGURE 24. Johann Radon, circa 1920.

In 1990, the Soviet Union finally took responsibility for this massacre. In 2010 **Vladmir Putin** (Russian Prime Minister; -) invited **Donald Tusk** (Polish Prime Minister; -) to a memorial to commemorate the 70th anniversary of the massacre. On 7 April, Tusk attended the memorial. In a horribly creul twist of fate, three days later a plane carrying **Lech Kaczyński** (Polish President; 1949 - 2010), his wife, and over 80 high ranking goverment and military officials to the memorial crashed, killing all on board. The location of the crash? Just outside the Katyn Forest.

Stefan Kaczmarz is likely one of those Poles who was murdered in the Katyn Forest Massacre. His known mathematical accomplishments are few.

While the lives and careers of Radon and Kaczmarz were quite different, there share a critical historical relationship. Namely, the Radon transform and the Kaczmarz method are essential components of the development of CAT scans and other forms of medical imaging.

Ever had a CAT scan? An MRI? A PET scan? As early as 1975 it was suggested that this sort of medical imaging "may effect a revolution in medicine comparable to that brought about late in the 19th century by the introduction of anesthetics and sterile techniques.[4] It is safe to say that these forms of medical imaging are among the most important medical advances of the twentieth century. With the recent advent of *functional magnetic resonance imaging*, also known as fMRI, where we can watch pysiological changes in brain function in real time, the revolution continues still.

These medical imaging breakthroughs were made possible - in large part - by the Radon transform and the Kaczmarz method. These two mathematical methods are absolutely essential to the reconstruction of the images from collected data.

[4]From "Image Reconstruction from Projections," by Gordon, Herman, and Johnson, *Scientific American*, vol. 233, no. 4, 1975, p. 56.

FIGURE 25. Stefan Kaczmarz.

Remarkably, the work of Radon and Kaczmarz was so little known that these tools were reinvented as the need arose during the development of these medical imaging technologies! **Richard Gordon** (; -), **Bender** (; -), and **Gabor T. Herman** (; -) rediscovered Kaczmarz's method in 1970 and it is often called the *algebraic reconstruction technique*. **Allan M. Cormack** (South African doctor and physicist; 1924 - 1998) and **Godfrey Hounsfield** (English Engineer; 1919 - 2004) reinvented major aspects of the Radon transform.

So important were these discoveries that Cormack and Hounsfield shared the Nobel Prize for Medicine in 1979.

What would have happened if Radon and Kaczmarz work been remembered in this context? Gordon, Herman, and Johnson tell us "If Radon and the early tomographers had been aware of their common problems, many of the developments of the past few years might have been launched a half a century ago."[5] Remarkable!

Cormack's Nobel Lecture is beautiful and includes a wonderful reminder of the importance of intellectual engagement:

> What is the use of these results? The answer is that I don't know. They will almost certainly produce some theorems in the theory of partial differential equations, and some of them may find application in imaging with N.M.R. or ultrasound, but that is by no means certain. It is beside the point. Quinto and I are studying these topics because they are interesting in their own right as mathematical problems, and that is what science is all about.[6]

But what does all of this have to do with Puzzles and Games? The purpose of the Radon transform and the Kaczmarz method is to reconstruct data. They are highly technical approaches. The difficulties of practically utilizing them in a clinical setting are immense. Nonetheless, the way

[5] Ibid.

[6] Allan M. Cormack from his Nobel Lecture, December 8, 1979.

in which they enable us to create medical imaging technology like CAT scans shares a great deal of resemblence with the puzzles we have been solving. Honestly.

Tomography is from the Greek *tomos*, meaning slice, and *graphein*, meaning to write. It is used contemporarily to describe a type of imaging, often medical, where three dimensional objects are depicted by images that represent slices sectioned from the object - originally by means of an X-Ray, but now also by ultrasound, positron beams, sonar, or magnetic resonance.

An apple we can *cross section* with a knife, as shown in Figure 26. We can think of any three-dimensional solid as made up of its cross sections. Original *sliceforms* made by geometry students illustrate this idea in a nice way; see Figure 27. Of course, if we want to know what a living human being looks like inside - physically slicing is not "optimal." Instead, we let waves and beams do the slicing.

If you want more information on the relationship between three-dimensional objects and their two-dimensional cross sections, please see our Website "Navigating Between the Dimensions" at `http://www.wsc.ma.edu/ecke/flatland/Home.html` or the chapters "Navigating Between the Dimensions" and "Dimensional Interplay in Other Fields" from Discovering the Art of Mathematics - Geometry.

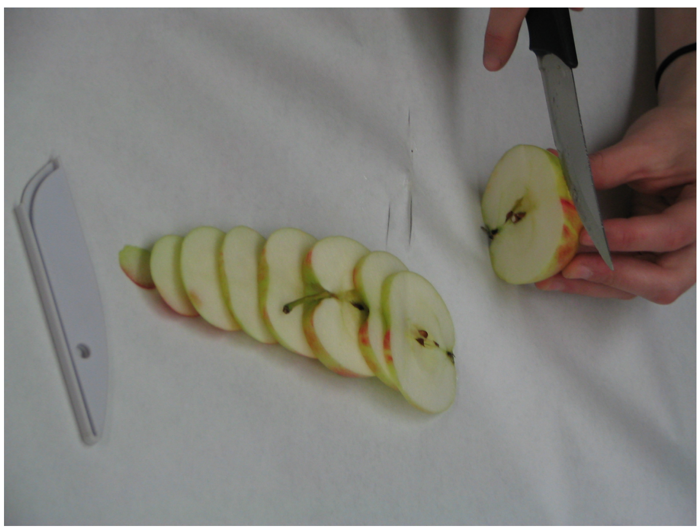

FIGURE 26. Cross sections of an apple.

So how do beams of waves or particals help us slice without damaging what they are helping us image? We'll illustrate using a CAT scan - ***computerized axial tomography*** - one of the most widely used forms.

The functional "guts" of a CAT scan are shown in Figure 28. The long axis of your body extends through the hole in a perpendicular direction. This is the direction the slices are generally taken in humans - the ***axial direction*** - contributing the "A" in CAT scan. Measured beams of X-Rays are released as illustrated by the red arrows. Those that pass through the body are measured on the opposite side. As the beams and detectors spin, many thousands of X-Ray beams pass through the axial plane of your body that is being imaged. These create one "slice." The detectors and beams then move up to take the next slice - perhaps a millimeter higher up. This happens with great speed so many thousands of slices are created.

But how do the beams allow us to capture an image of the slice? Well, the intensity of the beams deminish as they pass through your body. They are ***attenuated***. How much they are attenuated depends on what they pass through. The *attenuation coefficients* are quite different for bone, soft

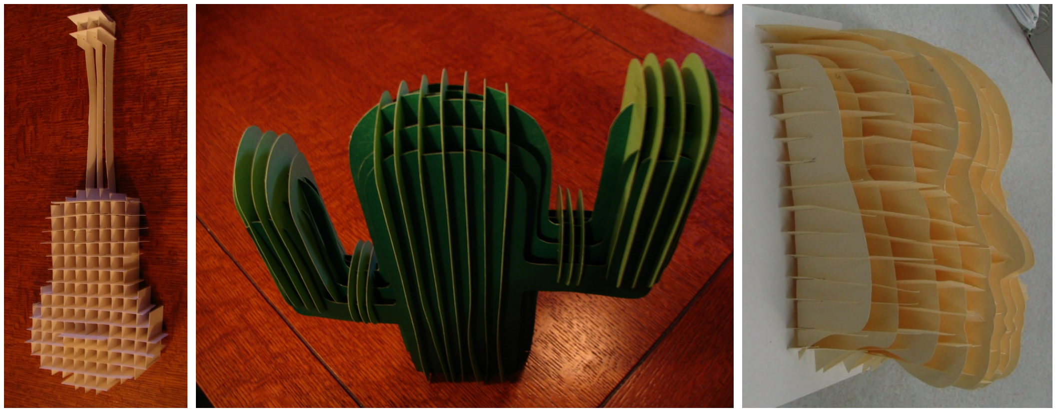

FIGURE 27. Original student sliceforms; "Guitar" by Katherine Cota, "Cactus" by Sharon Kubik-Boucher, and "The Face" the Lydia Lucia.

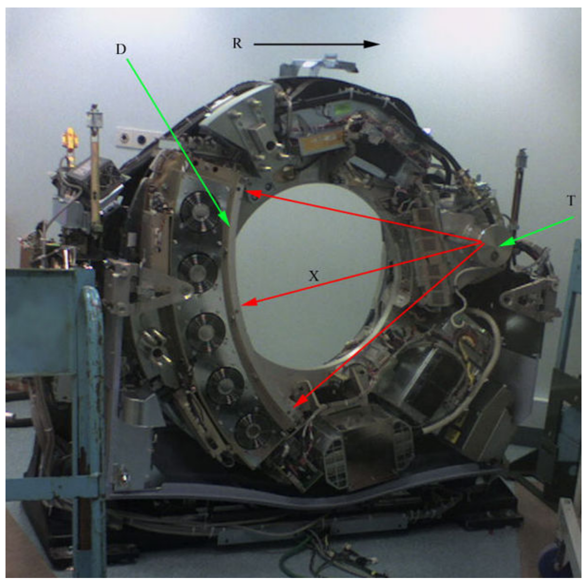

FIGURE 28. A modern (2006) CT scanner with the cover removed, demonstrating the principle of operation. Source: Wikimedia.

tissue, muscle, tumor, etc. Imagine the axial cross section of your body pixellated by a fine grid, as illustrated by Figure 29. Each arrow indicates an X-Ray beam. And each number represents the amount of radiation that was attenuated along the path of the ray. If we could determine how much radiation was attenuated by each pixel then we would know - from its attenuation coefficient - what it was; bone, soft tissue, muscle, tumor, etc.

And, this is exactly the type of problem that we were trying to solve when we were working on RK puzzles! Indeed, the work of Radon and Kaczmarz - reinvented later because it had not been broadly enough known - is what allows us to recover the data needed to image the slice from the individual measurements of thousands of beams.

Of course, there are significant technical issues that make the use - and advancements - in this type of imaging quite complex. Among other things, think of the amount of data that would be required to solve an RK puzzle whose resolution is about 10,000 by 10,000 - the typical size

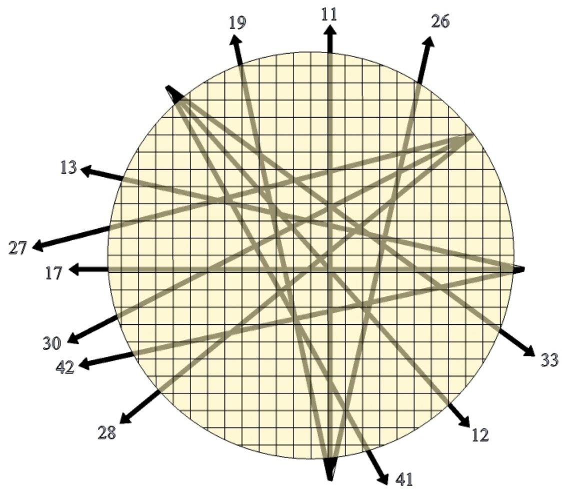

Figure 29. Illustration of a pixellated cross section bombarded with X-Ray beams and their attenuation coefficients.

of a decent image! Nonetheless, the underlying mathematical ideas at the heart of CAT scans are those ideas that you have discovered in solving RK puzzles. You understand the fundamental working components of critically important way of visualizing what is inside us - as illustrated by the remarkable images in Figure 30.

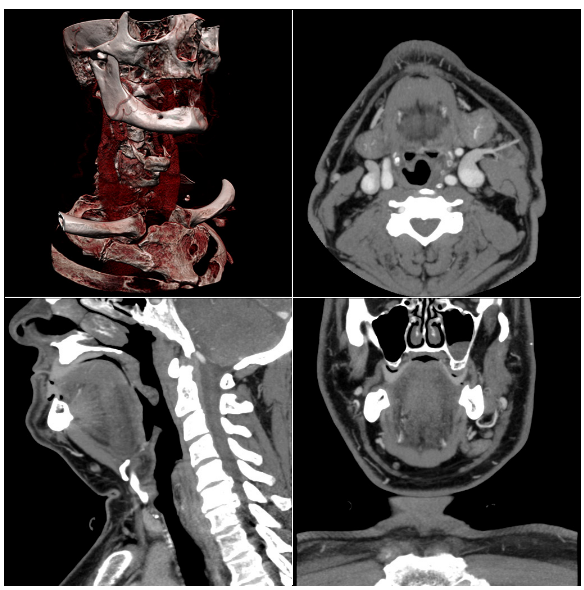

Figure 30. Typical screen layout of workstation software used for reviewing multi-detector CT studies. Clockwise from top-left: Volume rendering overview, axial slices, coronal slices, sagittal slices. Source: Wikimedia.

A wealth of information on imaging of this form is available online. Visible Human Program viewers available online. (E.g. the NPAC/OLDA Visible Human Viewer at `http://www.dhpc.adelaide.edu.au/projects/vishuman2/VisibleHuman.html` or the viewer at the Center for Human Simulation at `http://www.uchsc.edu/sm/chs/browse/browse.htm`.) The book Medicine's

New Vision has beautiful images and is quite readable. If you are interested in the hard-core mathematics required to actually put these ideas into place, The Mathematics of Computerized Tomography and Introduction to the Mathematics of Medical Imaging will quickly indicate what a substantial challenge this is.