Inquiry-Based Calculus III

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Chapter 1

Curves and Surfaces

Definition 1. *(Not really a definition). The **dimension** of a geometric object is the number of degrees of freedom on the object, or the number of numbers you need to determine a point on the object.*

Definition 2. *If a geometric object X is a subset of $\mathbb{R}^n$, and has dimension* $\dim(X)$*, then the **codimension** of X is $n - \dim(X)$.*

Problem 1. *Sketch the graph of the line $L = \{(x,y) \in \mathbb{R}^2 : y = 3x+1\}$.*

Problem 2.

1. *What is the dimension of the line L in Problem 1?*
2. *What is the codimension of L?*

Problem 3. *Which of the following points are on the line L, and which are not? Explain your reasoning.*

1. $(10,31)$
2. $(1,2)$
3. $(0,0)$
4. $(0,1)$

Problem 4. *Give five examples of points that are on the line L that are not listed in Problem 3. Explain your strategy.*

Problem 5. *For each of the points in Problem 3 that is on L, find another line so that the point is the intersection of L with this line.*

Problem 6. *For the points in Problem 3 that are on L, can you find one curve that intersects L in all of those points, but nowhere else?*

Problem 7. *Let $G : \mathbb{R} \to \mathbb{R}^2$ be the function defined by $G(t) = (1+t, 4+3t)$. On one graph, plot $G(-2)$, $G(-1)$, $G(0)$, $G(1)$, and $G(2)$.*

Problem 8. *Prove or find a counterexample: If a is a real number, then $G(a)$ is on L.*

Problem 9. *Prove or find a counterexample: If Q is a point on L, then there is a number a such that $G(a) = Q$.*

Problem 10. *Compare the functions in Problem 1 and Problem 7. How are they the same? How are they different?*

Problem 11. *Let $H : \mathbb{R} \to \mathbb{R}^2$ be the function defined by $H(t) = (-t, 1-3t)$. On one graph, plot $H(-2)$, $H(-1)$, $H(0)$, $H(1)$, and $H(2)$.*

Problem 12. *Compare and contrast G and H. How are they the same, how are they different?*

Problem 13. *Find yet another parametrization of L. Explain your strategy.*

Problem 14. *Sketch and describe the graph of $P = \{(x,y,z) \in \mathbb{R}^3 : y = 3x+1\}$.*

Problem 15.

1. *What is the dimension of P in Problem 14? How do you know?*
2. *What is the codimension of P?*

Problem 16. *Which of the following points are on P, and which are not? Explain your reasoning.*

1. $(10,31,0)$
2. $(10,31,53)$
3. $(1,1,1)$
4. $(0,0,0)$
5. $(0,1,0)$
6. $(0,1,12)$

Problem 17. *Let $f : \mathbb{R} \to \mathbb{R}^3$ be the function defined by $f(t) = (t, 1+3t, t)$. On one graph, plot $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$.*

Problem 18. *Prove or find a counterexample: If a is a real number, then $f(a)$ is on P.*

Problem 19. *Prove or find a counterexample: If Q is a point on P, then there is a number a such that $f(a) = Q$.*

Problem 20. *How are f and P the same? How are they different?*

Problem 21. *Let $S : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by $S(u,v) = (1+u, 4+3u, v)$.*

1. *Graph the four points $S(-2,0)$, $S(2,0)$, $S(2,3)$, and $S(-2,3)$.*
2. *Now graph the curves $S(-2,v)$ and $S(2,v)$ for $0 \leq v \leq 3$.*
3. *Now graph the curves $S(u,0)$ and $S(u,3)$ for $-2 \leq u \leq 2$.*
4. *Describe and graph the surface.*

Problem 22. *What is range of f? What is the range of S? How are f and S the same? How are they different?*

Problem 23. *Summarize the relationship between the sets and functions in Problems 1, 7, 14, 17, and 21.*

Problem 24. *Let R be the rectangle $[-2,2]\times[-2,2]$, and let $T : R \to \mathbb{R}^3$ be the function defined by $T(u,v) = (u,v,4-u^2)$.*

1. *Graph the four points $T(-2,-2)$, $T(-2,2)$, $T(2,2)$, and $T(2,-2)$.*
2. *Now graph the curves $T(-2,v)$ and $T(2,v)$ for $-2 \le v \le 2$.*
3. *Now graph the curves $T(u,-2)$ and $T(u,2)$ for $-2 \le u \le 2$.*
4. *Describe the surface and build in using play dough.*

Problem 25. *The following definition is rather technical and difficult to read. Explain in your own words what it means. Draw pictures to help the reader understand the definition.*

Definition 3. *Let D be a set of points in the plane $\mathbb{R}^2$. Let P be a point in the plane.*

- *P is an* interior *point of D if and only if there is a (possibly very tiny!) positive real number r, so that the open disk $\{Q \in \mathbb{R}^2 : |P-Q| < r\}$ is contained entirely in D.*
- *P is a* boundary *point of D if and only if, no matter what positive real number r is, the open disk $\{Q \in \mathbb{R}^2 : |P-Q| < r\}$ contains at least one point that is in D and at least one point that is not in D.*

Problem 26. *Let D be the set $\{(x,y) \in \mathbb{R}^2 : -2 \le x \le 2 \text{ and } 1 \le y \le 3\}$.*

1. *Find the set of all interior points of D. Draw the interior points.*
2. *Find the set of all boundary points of D. Draw the boundary points.*

Problem 27. *(Extension Question) Let Q be the set $\{(x,y) \in \mathbb{Q}^2 : -2 \le x \le 2 \text{ and } 1 \le y \le 3\}$.*

1. *Find the set of all interior points of Q. Draw the interior points.*
2. *Find the set of all boundary points of Q. Draw the boundary points.*

Problem 28. *Sketch the region in $\mathbb{R}^2$ given by*

$$\{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le 1, y \le \sqrt{x}\}.$$

The boundary of this region consists of three curves. Write parametric equations for each of these curves.

Problem 29. *Consider the region D in $\mathbb{R}^2$ given by*

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

1. *Sketch D and its boundary.*
2. *What is the dimension of D?*

3. *What is the dimension of the boundary of D?*

4. *Find an implicit equation for the boundary of D.*

5. *Find a parametric equation for the boundary of D.*
Hint: Review the definition of $\sin(t)$ *and* $\cos(t)$ *using the unit circle.*

Problem 30.

1. *What is the intersection of two planes in* $\mathbb{R}^3$*? What is its dimension? What is its codimension? Draw pictures to explain the different possibilities.*

2. *What is the intersection of a plane and a line in* $\mathbb{R}^3$*? What is its dimension? What is its codimension? Draw pictures to explain the different possibilities.*

Problem 31. *Consider the sets*

$$S = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 6\}$$

and

$$T = \{(x,y,z) \in \mathbb{R}^3 : y-z = 2\}.$$

1. *Sketch S, T, and their intersection. What are their dimensions and codimensions?*

2. *Find an equation that describes the intersection.*

3. *Can you find* **one** *implicit function for the intersetion? Explain.*

Problem 32. *Consider the sets*

$$P = \{(x,y,z) \in \mathbb{R}^3 : z = 5\}$$

and

$$C = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 4\}.$$

1. *Sketch P, C, and their intersection. What are their dimensions and codimensions?*

2. *Find an equation for the intersection.*

Problem 33. *Sketch the region in* $\mathbb{R}^3$ *given by*

$$D = \{(x,y,z) \in \mathbb{R}^3 : -1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1\}.$$

Describe the boundary of this region. Write parametric equations for each of the pieces of the boundary.

Problem 34. *Imagine taking a parabola and rotating it around its central axis, see Figure 1.1. It sweeps out a surface called a* ***paraboloid****. (This shape is the traditional shape of car headlights. Nowadays with computer-aided design (CAD) systems, more complicated shapes are sometimes used, but they are based on this shape.) Build a paraboloid using play dough.*

1. *What is the dimension of a paraboloid? What is its codimension?*

2. *If we intersect the paraboloid shown with a horizontal plane, what shape will we get?*

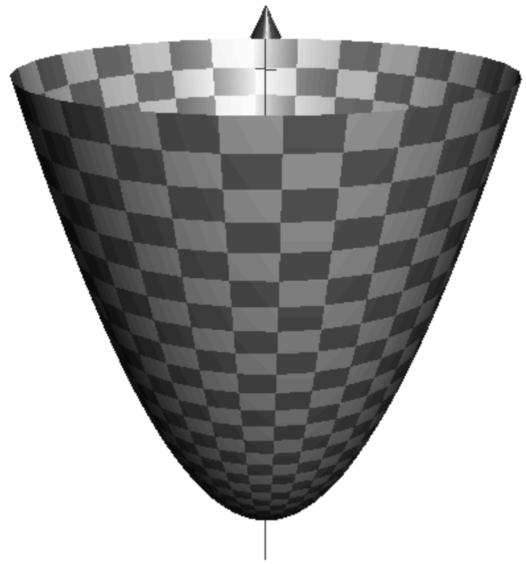

Figure 1.1: Paraboloid

3. *If we intersect the paraboloid shown with a vertical plane, what shape will we get?*

Problem 35. 1. *Write an implicit equation for a paraboloid.*

2. *Write another implicit equation for a different paraboloid.*

3. *Write a parametric equation for a paraboloid.*

4. *Write a another parametric equation for a different paraboloid.*

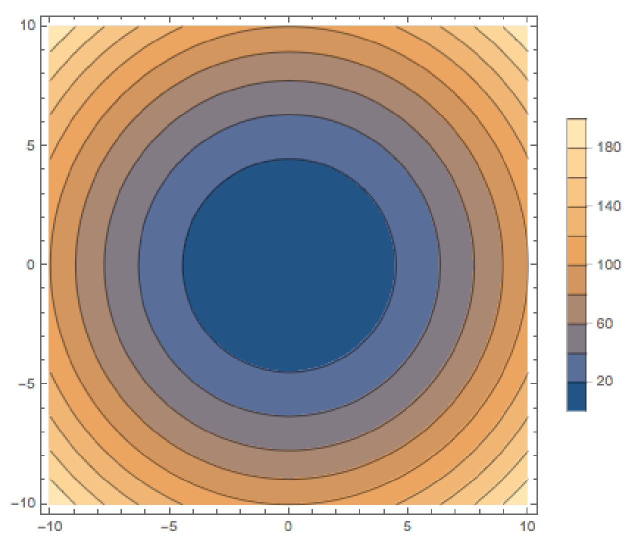

Figure 1.2: Contour Plot Paraboloid

Problem 36. *Figure 1.2 shows a so called* contour plot *of the paraboloid from Figure 1.1. Explain how the contour plot captures the information of the paraboloid.*

Problem 37. *Let $K : \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by $K(u,v) = (u\cos v, u\sin v, 5-u)$.*

1. *Sketch the range of K for $0 \leq u \leq 5$ and $0 \leq v \leq 2\pi$, also build it using play dough. What is its dimension?*
2. *What is the codimension of the range of K? Write a(n) implicit equation(s) for the range of K.*
3. *On the same graph, sketch and label the curves $K(u, \frac{\pi}{2})$ and $K(3,v)$.*
4. *What is the boundary of the range of K? Write implicit and parametric equations for the boundary.*

Problem 38. *Consider the set $H = \{(x,y,z) \in \mathbb{R}^3 : x^2 - y^2 + z^2 = 1\}$.*

1. *If we plug $y = 1$ into the equation for H, what do we get? What does this tell us about the geometry of H?*
2. *What happens if we plug in other constants for y? What does this mean about the geometry of H?*
3. *Same questions for x.*
4. *Same questions for z.*
5. *Draw the contour plot of H.*
6. *Sketch H and build it using play dough.*

Problem 39. *Consider the set $R = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 - y^2\}$.*

1. *If we plug $y = 1$ into the equation for R, what do we get? What does this tell us about the geometry of R?*
2. *What happens if we plug in other constants for y? What does this mean about the geometry of R?*
3. *Same questions for x.*
4. *Same questions for z.*
5. *Draw the contour plot of R.*
6. *Which of the graphs in Figure 1.3 shows R?*

Problem 40. *Describe in your own words the different representations of an object in $\mathbb{R}^3$ we have worked with so far: implicit equation, parametric equation, contour plot, play dough, and 3-dimensional graph. Use at least one example to illustrate your explanations. What do you struggle the most with? Why?*

Problem 41. *(Extension Question) How can we predict the codimension of the intersection of two objects? Make a conjecture.*

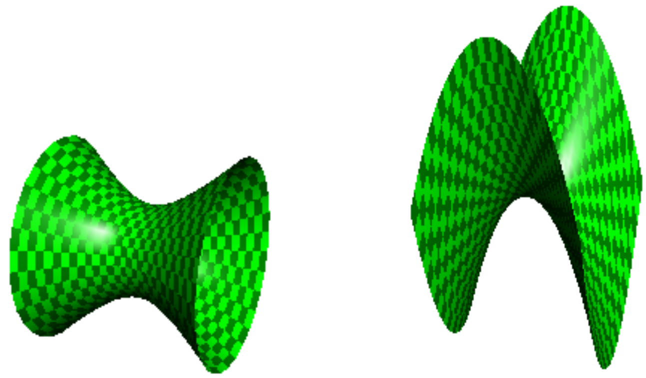

Figure 1.3: Mystery Graphs

1.1 Velocity, Speed and new Derivatives

Problem 42. *(Extension Question) Suppose that $G(t)$ represents the position of a giraffe t hours after noon, and $H(t)$ represents the position of a hippopotamus t hours after noon.*

1. *What is the speed of the giraffe? What is the velocity of the giraffe? What is the speed of the hippo? What is the velocity of the hippo?*
2. *Suppose that a cheetah is going down the same road as the giraffe, but three times as fast. Write a formula for a function $C : \mathbb{R} \to \mathbb{R}^2$ so that $C(t)$ represents the position of the cheetah t hours after noon.*

Chapter 2

Optimization

2.1 Derivatives

To describe and predict the world around us mathematical models can be used. Most models don't describe the real world perfectly but they can still be useful, and they are better than nothing.

A company manufactures two items that are sold in two separate markets where it has a monopoly. The quantities q_1 and q_2 demanded by customers, and the prices p_1 and p_2 (in dollars), of each item are related by $p_1 = 700 - q_1$, and $p_2 = 500 - q_2$. Thus, if the price for either item increases, the demand decreases. The company's total production cost is given by $C = 16 + q_1 q_2$.

Problem 43.

1. *Find an equation for the profit function $P(q_1, q_2)$.*
2. *What is a reasonable domain of $P(q_1, q_2)$?*

Problem 44.

1. *Create a table that shows values of P for different values of q_1 and q_2.*
2. *Using the table, predict for which values of q_1 and q_2 the profit is largest.*
3. *Using the table, predict for which values of q_1 and q_2 the profit is smallest.*

Problem 45.

1. *Assume that $q_1 = 0$ (what happened to the company??) and draw a graph of $P(0, q_2)$.*
2. *Assume that $q_2 = 0$ (what happened now??) and draw a graph of $P(q_1, 0)$.*
3. *Draw a graph of the function $P(q_1, q_2)$.*

Problem 46. *Use your knowledge from calculus 1 to find critical points of $P(0, q_2)$.*

Problem 47. *Recall that in calculus1 the derivative of a function represents the slope of a tangent line. Use that idea to visualize a tangent line to the graph of $P(q_1, q_2)$ (using play dough and a toothpick) that shows (and justifies) the critical point you just found. How can you be sure that there is no other critical point if $q_1 = 0$? Draw the result carefully or take a picture and describe it.*

Problem 48. *Use your knowledge from calculus 1 to find critical points of $P(q_1, 0)$. Visualize a tangent line to the graph of $P(q_1, q_2)$ (using play dough) that shows (and justifies) the critical point you just found. Draw the result carefully or take a picture and describe it.*

Problem 49. *Do you think you found a critical point of the function $P(q_1, q_2)$ yet? Why or why not?*

Problem 50. *Choose a different constant for q_1 (not zero this time) and compute the maximum of $P(q_1, q_2)$, as above.*

Problem 51. *Do problem 50 two more times, choosing different constants for q_1.*

Problem 52. *Choose a different constant for q_2 (not zero this time) and compute the maximum of $P(q_1, q_2)$, as above.*

Problem 53. *Do problem 52 two more times, choosing different constants for q_2.*

Problem 54. *Describe what you notice in the above process of finding maxima. Find a process that allows you to know for sure where the maximum of $P(q_1, q_2)$ is (now q_1 and q_2 are* **both** *variables). Now use a geometric argument to expalain why this process will always work.*

Problem 55. *Use partial derivatives to find the points (q_1, q_2) where P has vanishing partial derivatives, i.e. the partial derivatives are equal to zero. We call these points* critical points. *Do you know for sure that this critical point is a maximum? Explain.*

We want to be able to tell if our critical point is really a maximum or not.

Problem 56. *Recall from calculus 1 the different possibilities for critical points of a function $f : \mathbb{R} \to \mathbb{R}$. What could f look like close to a critical point? Draw pictures of the graph of f for the different possibilities.*

Problem 57. *Now take play dough and create "all" the different ways the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ could look like around a critical point. Hint: think of a saddle for an octopus!!!*

Problem 58. *The temperature at the point (x, y) in a thunder storm region is given by*

$$T(x,y) = 50 + x^4 + y^4 - 4xy.$$

Find the point(s) (x, y) where the temperature is the lowest. What is the lowest temperature?

Problem 59. *The partial derivative f_x of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is again a function. What are the domain and codomain of f_x? Build a mountain with two peaks using play dough, this is your graph of f. Now try to build or draw the graph of f_x. Hint: Recall how you did this calculus 1 for functions $\mathbb{R} \to \mathbb{R}$...*

2.2 Second Derivatives

Problem 60. *Recall from calculus 1 several strategies that help you determine whether a critical point is a maximum, a minimum or neither. Explain why each strategy works.*

Problem 61. *Use second derivatives to decide whether the critical point of the function* $P(x,y) = x^2+y^2-xy$ *is a maximum, a minimum or neither. Explain your strategy in detail. Make sure to check your work.*

Problem 62. *Use second derivatives to decide whether the critical point of the profit function* $P(x,y) = x^2+y^2-4xy$ *is a maximum, a minimum or neither. Make sure to check your work.*

Definition 4. *Since a partial derivative is again a function we can take* again *the partial derivative (provided all functions behave nice and are differentiable at the right places...)! We call these new "double" derivatives* second partial derivatives *and denote them with* $f_{xx}, f_{xy}, f_{yx} f_{yy}$. *Here* f_{xy}, f_{yx}, *are called* mixed partial derivatives.

Problem 63. *Compute the mixed partial derivative* P_{xy} *of the function* $P(x,y) = x^2+y^2-xy$. *We want to visualize our answer by studying* P_x *and how it changes in y-direction. Start by looking at one particulat point in the* (x,y) *plane. Use play dough and toothpicks to explain your thinking.*

Problem 64. *Compute the value of* $D = P_{xx}P_{yy} - (P_{xy})^2$ *at the critical point of the function in problem 61.*

Problem 65. *Compute the value of* $D = P_{xx}P_{yy} - (P_{xy})^2$ *at the critical point of the function in problem 62.*

Problem 66. *Compute the value of* $D = P_{xx}P_{yy} - (P_{xy})^2$ *at the critical point of the function* $f(x,y) = -x^2-y^2+10$.

Problem 67. *Compute the value of* $D = P_{xx}P_{yy} - (P_{xy})^2$ *at the critical point of the function* $f(x,y) = x^4+y^4$.

Problem 68. *Compute the value of* $D = P_{xx}P_{yy} - (P_{xy})^2$ *at the critical point of the function* $f(x,y) = -x^4-y^4$.

Problem 69. *Compute the value of* $D = P_{xx}P_{yy} - (P_{xy})^2$ *at the critical point of the function* $f(x,y) = x^4-y^4$.

Problem 70. *Look back at problems 64, ... 69. Find additional examples in the book. Make a conjecture how you could use D to predict whether a critical point of a function is a maximum, a minimum or a saddle.*

Problem 71. *Let's assume that a function f has vanishing mixed partial derivatives. Make sense of (prove) your conjecture in Problem 70 geometrically.*

Problem 72. *Compute the critical points of the function*

$$f(x,y) := 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4.$$

Are they maxima, minima or saddle points? Check your work.

Problem 73. *(Extension Question) Using a few examples, compare the functions* P_{xy} *and* P_{yx}. *What do you noctice? Try to make sense of your conjecture.*

2.3 Tangent Planes

Problem 74. *Draw a picture of the linear approximation of a function* $f : \mathbb{R} \to \mathbb{R}$ *around a point* p.

Problem 75. *Now generalize this idea and explain what a linear approximation of a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *around a point* p *might be. (Use play dough or draw a picture.)*

One interesting question is if every function has a linear approximation. Let's look back to calculus 1 first.

Problem 76. *Recall from calculus 1 the definitions of a continuous functions, and of a differentiable function. Use the definitions to draw graphs of functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *that are*

- *not continuous at some point* a
- *not differentiable at some point* a
- *continuous but not differentiable at some point* a*. Explain how you can use the tangent line through the point* $(a, f(a))$ *to graphically show that* f *is not differentiable at* a.

Problem 77. *Create with play dough the graph of a function from* $\mathbb{R}^2$ *to* $\mathbb{R}$ *that is*

- *not continuous at some point* (a,b)
- *not differentiable at some point* (a,b)
- *continuous but not differentiable at some point* (a,b)*. Explain how you can use the tangent plane through the point* $(a,b,f(a,b))$ *to graphically show that* f *is not differentiable at* (a,b).

Problem 78. *Draw the graph of the function* $f(x,y) = \sqrt{x^2+y^2+1}$*. Find two vectors that are tangent to the graph of* f *at the point* $(2,3,f(2,3))$*. Hint: You can think in slices and use partial derivatives!*

Problem 79. *Find the equation of the tangent plane at the point* $(2,3,f(2,3))$ *in parametric representation. Check your work, for instance by drawing both the original graph and the plane that you found (using a computer algebra system).*

Problem 80. *Find the equation of the tangent plane at the point* $(2,3,f(2,3))$ *in implicit representation. Check your work.*

Problem 81. *Generalize your strategy from problem 78: Given* **any** *function* $f(x,y)$ *find the parametric and the implicit equation of the tangent plane at the point* (a,b).

2.4 Directional Derivatives

Problem 82. *It is winter in New England. Imagine a bug crawling along the floor of a room that is 7x7 square meters large. Figure 2.1 shows the contour plot of the heat of the floor in Fahrenheit.*

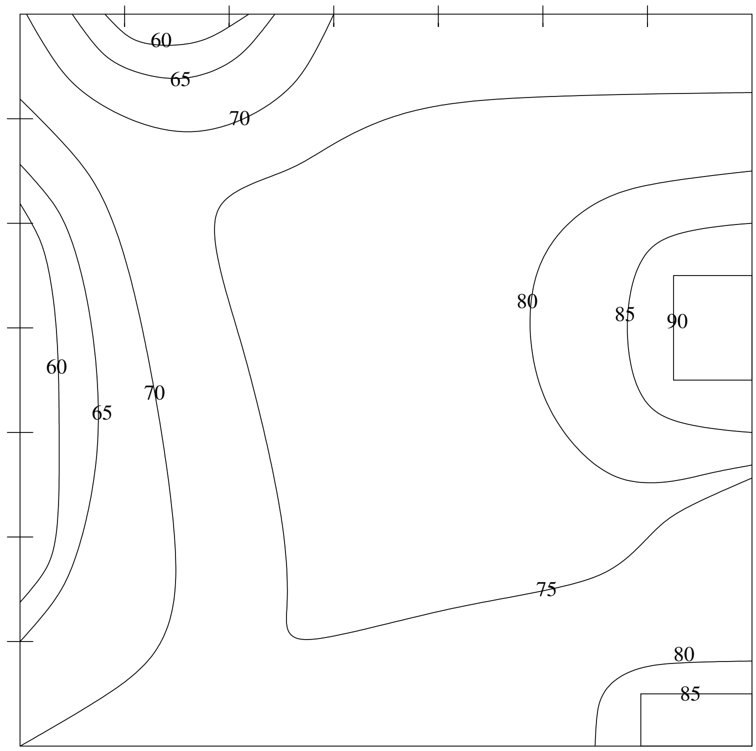

Figure 2.1: Heat Contour

1. *Where is the door to the outdoors in the room? Why?*
2. *Where is the windows? Why?*
3. *Where is the heater? Why?*
4. *What might be standing in the lower right corner of the room?*
5. *The bug would like to crawl from the window to the right wall along a path with the least amount of temperature changes. Draw two possible paths into the contour diagram.*

Problem 83. *Imagine the bug was crawling along the walls. Draw the graphs of the temperature functions ($\mathbb{R} \to \mathbb{R}$) along each wall of the room. Are you able to draw precise graphs? Explain.*

Problem 84. *Figure 2.2 shows the line segment that the bug decided to crawl along. Find the average rate of change of the temperature between points P and Q that the bug experiences. What are the units?*

Problem 85. *Estimate the instantaneous rate of change of the temperature at point P that the bug experiences when it is preparing to crawl towards Q. How can you improve your estimate? Which information would you need to make your estimate even better?*

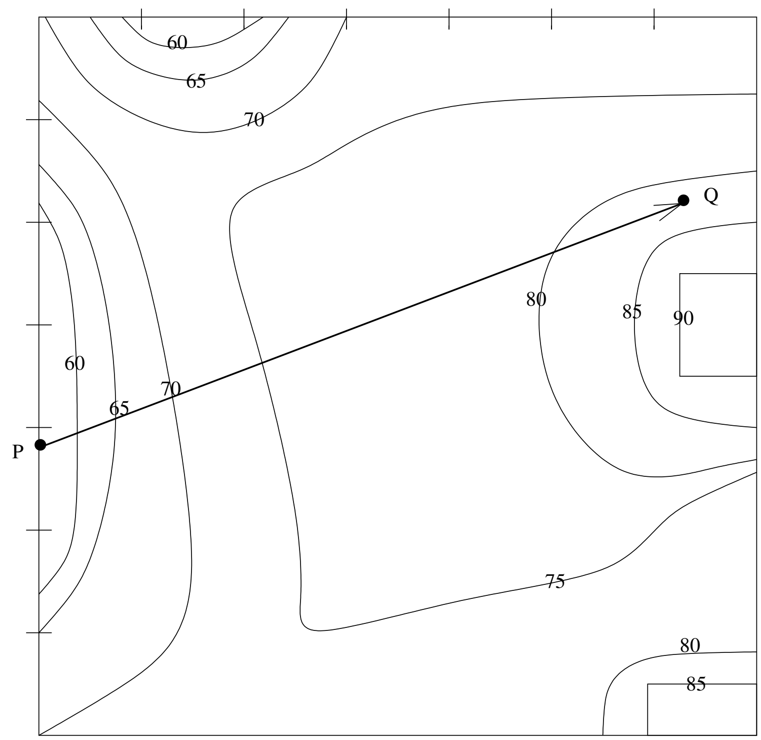

Figure 2.2: Bug on the Floor

Problem 86. *Find a limit expression* $\lim_{h\to 0}\cdots$ *that computes the instantaneous rate of change of the temperature* T *at point* $P=(a,b)$ *that the bug experiences (when it faces direction* $Q-P=(u_1,u_2)$*).*

Problem 87. *How can the directional derivative help the bug find a path with the least amount of temperature changes?*

Problem 88. *To play a bit with the definition, compute the directional derivative of* $f(x,y)=4x^2+y$ *at the point* $(a,b)=(1,2)$ *in the direction* u *for each* u *defined below.*

1. $u=(0,1)$
2. $u=(1,0)$
3. $u=(1,1)$
4. $u=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$

Problem 89. *The directional derivative is not easy to compute using the definition you just invented (especially by a bug...). We need to find an expression that is easier to compute. Use the tangent plane* $f(x,y)\approx f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$ *with* $x=a+hu_1$*,* $y=b+hu_2$ *to show that*

$$f_u(a,b)\approx\frac{f_x(a,b)u_1+f_y(a,b)u_2}{|(u_1,u_2)|}.$$

Problem 90. *Suppose you (or the bug) are computing a directional derivative. Why is it easier to use the expression in problem 89 than to use the definition?*

Problem 91. *The bug is crawling along the floor of a room from point* $(2, \frac{\pi}{4})$ *in direction of the vector* $v = (1,1)$*. The temperature on the floor is given by the function* $f(x,y) = 60 + (x-2)\sin(xy)$ *where* $0 \leq x, y \leq 3$*. Find the instantaneous rate of change of temperature that the bug is experiencing. Use a contour diagram to check if your answer is reasonable.*

Problem 92. *To simplify the expression in problem 89 (and make it easier to remember) rewrite it using vectors and a dot product.*

(f_x, f_y) is called the *gradient* of f, denotated by ∇f and we will now explore what other properties the gradient has.

2.5 Gradient

Problem 93. *Explore the relationship between the gradient and the level curves in the contour graph of $f(x,y) = x^2 - y^2$, see Figure 2.3. The gradient ∇f depends on the point (x,y). Choose a point (x,y) that lies on a level curve, compute the gradient at the appropriate point and draw the gradient vector into the contour graph so that it starts at the respective point (x,y). Repeat this process (very* precicely*!) until you noctice a pattern. Make a conjecture. (We call the set of all the gradient vectors a* vectorfield*)*

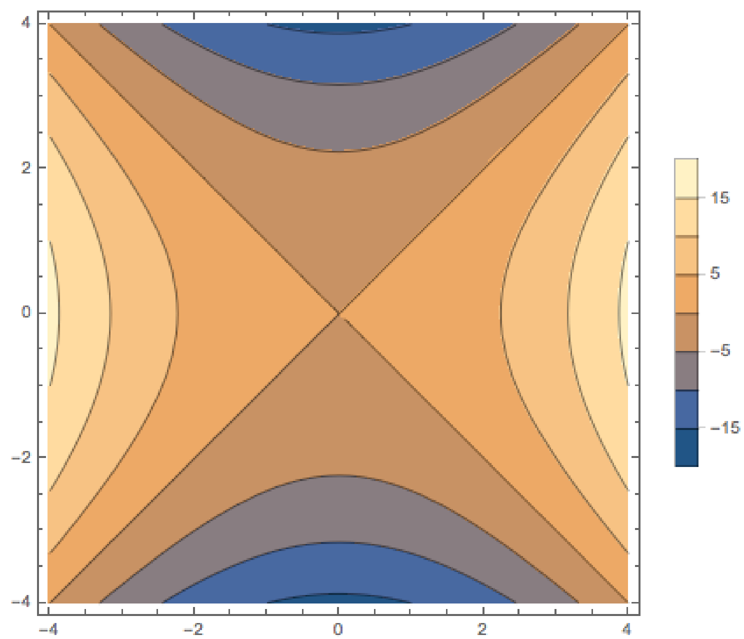

Figure 2.3: Contour Graph of $x^2 - y^2$

Problem 94. *It is winter and the bug would like to get to the heater in the room in Figure 2.1 as fast as possible. It entered the room through the window. In which direction does the bug need to go? Draw a possible bug path into the contour graph.*

Problem 95. *If $p = (x,y)$ is a point on a level curve of f and u is a vector that is tangent to the level curve at (x,y), what can you say about the directional derivative $f_u(p)$?*

Problem 96. *Prove your conjecture from problem 93. Hint: Use properties of dot product.*

Problem 97. *You want to build a "compass" for the bug in the room. Our compass will always point in the direction of the gradient vector. Answer the following questions to learn how to orient yourself in the room.*

1. *Is the gradient vector 2-dimensional so we can actually build a "normal" compass?*
2. *How can we use the compass to stay just on one level curve? Explain.*
3. *How can we use the compass to walk towards warmer (the warmest?) temperature? Explain.*
4. *How can we use the compass to walk towards colder (the coldest?) temperature? Explain.*

Problem 98. *By changing the direction of u we can make the directional derivative larger and smaller. Can we make it arbitrarily large and small? If not, can you predict a maximum or minimum? Prove your conjecture. Hint: Use the following property of the dot product, with ϕ being the (smaller) angle between u and v:*

$$u \cdot v = \|u\| \|v\| \cos(\phi).$$

Problem 99. *Draw the graph (or contour graph) of a temperature function in a square room (with no physical obstructions) in which the following situation can take place: If the bug uses the gradient compass to find the warmest spot in the room, it might never reach it. Explain how this can happen.*

Problem 100. *You are standing on a mountain given by the equation $z = 15 - x^2 - 3y^2$. At $(1, -2, 2)$, you want to erect a sign-post that points up in the direction of steepest ascent. Find the 3-dimensional tangent vector that points in the direction of steepest ascent.*

We can define the gradient in the same way for functions of 3 (or more) variables $\nabla f(x,y,z) = (f_x, f_y, f_z)$. All the properties of the gradient stay the same in more dimensions. Let's play with that:

Problem 101. *Our bug is now flying around in the room and would like to get to the hottest spot. Since there is a hot light bulb hanging in the room the bug doesn't know where the hottest spot would be. The bug is currently at point $(x,y,z) = (5,4,4)$ and it knows that the gradient of the temperature is $\nabla(T) = (0,0,10)$ at its location. What should the bug do next? Why?*

Problem 102. *(Extension Question)*

1. *Let $f(x,y) = 100x^{0.3}y^{0.7}$. Find the direction in which f is increasing most rapidly at the point $(500, 1500)$.*

2. *Suppose your uncle runs a print shop and (somehow) knows that if he has \$1000x invested in labor and \$1000y invested in equipment, his shop can produce $f(x,y) = 100x^{0.3}y^{0.7}$ units of printed materials. He currently has \$500,000 invested in labor and \$1,500,000 invested in equipment. He wins \$1000 in the lottery that he plans to invest in the print shop. He knows you've been studying calculus and comes to you for advice on how to allocate the money. Would you advise him to put more of it into labor, more into equipment, or equal amounts into each? Why?*

3. *Why is the gradient of f* **not** *giving you the best direction here? Hint: Plot the points on $x + y = 1$ and plot the points that the gradient considers is its decision.*

2.6 Boundaries, Think Global

Problem 103. *Recall from calculus 1 the definitions of local and global maximum and minimum. Then do the following:*

1. *Draw the graph of a function* $f : D \subset \mathbb{R} \to R \subset \mathbb{R}$ *that has 2 local maxima but no global maximum.*
2. *Draw the graph of a function* $g : D \subset \mathbb{R} \to R \subset \mathbb{R}$ *that has a global maximum at* x *with* $f'(x) \neq 0$.
3. *Draw the graph of a function* $h : D \subset \mathbb{R} \to R \subset \mathbb{R}$ *that has infinitely many global minima.*

Problem 104. *Let's think about what happens if you have two variables.*

1. *Draw the contour graph of a function that has 2 local maxima, one of them being the global maximum.*
2. *Draw the contour graph of a function that has 2 local maxima, but there is no global maximum.*
3. *Draw the contour graph of a function that has 2 local maxima, but the global maximum is a point on the boundary of the domain.*
4. *Draw the contour graph of a function* f *that has a global maximum at* (x,y) *with* $\nabla f(x,y) \neq 0$.

Problem 105. *The profit of a company is given by* $P(x,y) := -(x-8)^3/3 + 5(x-8)^2 - y^2/2 + 8y$ *for* $0 \leq x \leq 20$, $0 \leq y \leq 15$. *Find the global maximum and minimum of P if they exist. Remember to check what's happening on the boundary of the domain.*

Problem 106. *Let* $f(x,y) = x^2(y+1)^3 + y^2$. *Show that f has only one critical point, and that point is a local minimum but not a global minimum. Contrast this with the case of a function with a single local minimum in one-variable calculus.*

2.7 Lagrange Multipliers

Problem 107. *The profit of a company is given by $P(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}}$ with $x,y \geq 0$. For technical reasons the quantities x and y can only be produced if $x+y=4$. Draw the constraint $x+y=4$ into the same graph on the contour graph of P, see Figure 2.4, . Use the picture to estimate the global maximum of P.*

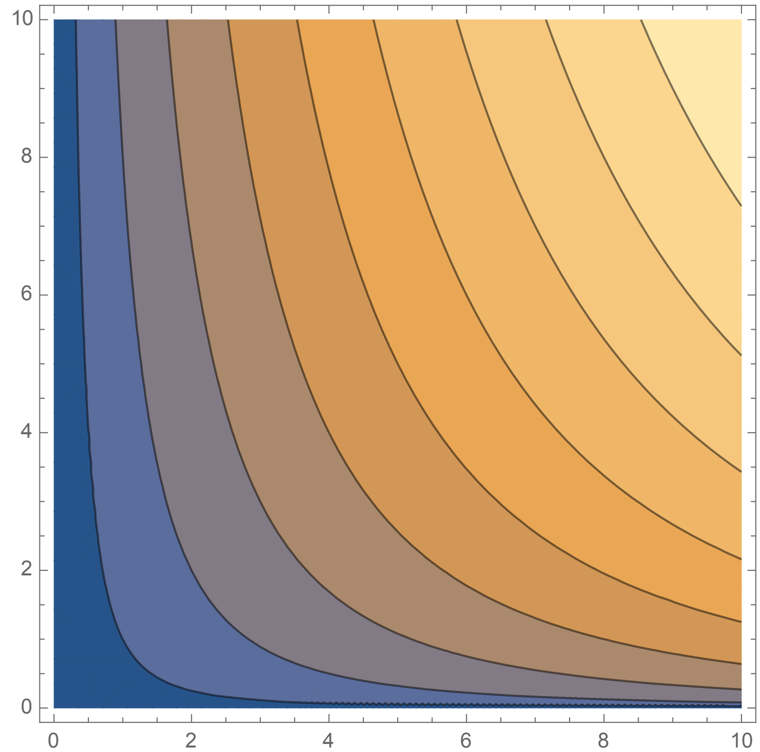

Figure 2.4: Contour Graph of $P(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}}$

Problem 108. *Compute the global maximum in Problem 107.*

We want to find a (different) geometric approach to find the maximum:

Problem 109. *Draw the gradient of P into the contour graph but use only vectors starting at the constraint $g(x,y) = x+y = 4$. Then draw the gradient of g into the same picture, again using only vectors starting at the constraint line. What do you notice happening at the maximum?*

Problem 110. *Use the above example to invent a general procedure to find points that are potential maximums or minimums. Explain why this procedure makes sense.*

Problem 111. *The profit of a company is given by $P(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}}$. For technical reasons the quantities x and y can only be produced if $x+y=4$. Use the procedure you developed above to find the global maximum of P.*

Problem 112. *Compare the procedures (and the answers) of Problem 111 with Problem 108. What are the advantages and disadvantages of the two procedures?*

Problem 113. *Find the maximal and minimal temperature in the region $\{(x,y) \in \mathbb{R}^2 | x^2+y^2 \leq 45\}$ if the temperature is given by the function $T(x,y) = (x-1)^2+(y-2)^2$. Is your answer geometrically reasonable?*

2.8 More Variables

Problem 114. *We want to think about how can we represent and visualize functions. Give examples of functions in several representations for the following cases:*

1. $f : \mathbb{R} \to \mathbb{R}$,
2. $f : \mathbb{R}^2 \to \mathbb{R}$,
3. $f : \mathbb{R}^2 \to \mathbb{R}^2$,
4. $f : \mathbb{R}^3 \to \mathbb{R}$,
5. $f : \mathbb{R}^4 \to \mathbb{R}$.

Problem 115. 1. *Let $f : \mathbb{R}^3 \to \mathbb{R}$. In how many dimensions does the graph of f live? How do you know?*

2. *Let $f : \mathbb{R}^3 \to \mathbb{R}$. In how many dimensions does the contour graph of f live? How do you know?*

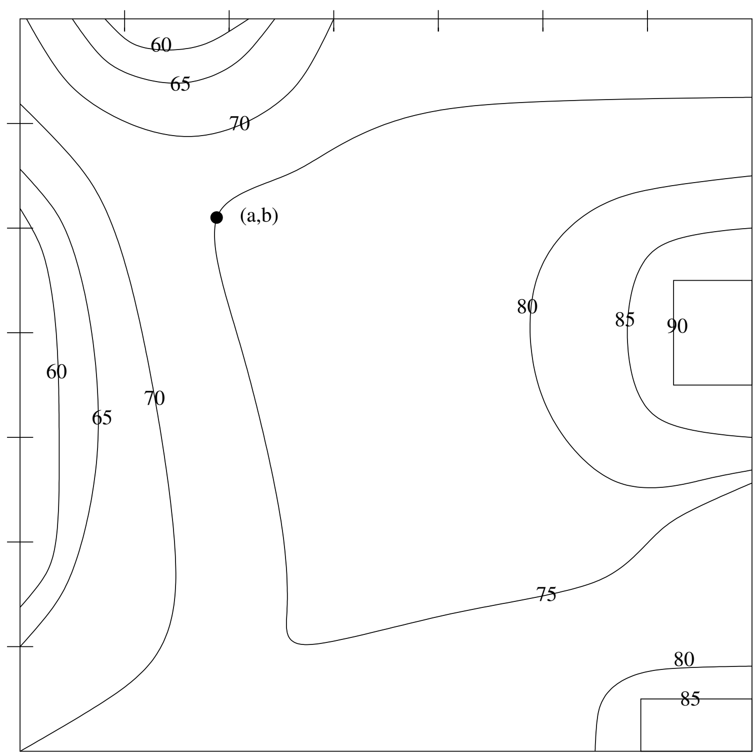

Figure 2.5: Contour Graph of f

Problem 116. *To become more flexible with contour graphs, imagine the bug is standing at point (a,b) in the contour plot of $f : \mathbb{R}^2 \to \mathbb{R}$ in figure 2.5. Draw the contour plot of the tangent plane at $(a,b,f(a,b))$ into the same picture. Use different colors for the different level curves of the tangent plane.*

1. *The bug starts out crawling from point (a,b) in direction $\nabla f(a,b)$. Describe what the bug sees.*

2. *The bug starts out crawling from point (a,b) perpendicular to the direction $\nabla f(a,b)$. Describe what the bug sees.*

Problem 117. *For functions $f(x,y)$ we used tangent planes to approximate the function around a point (a,b). What is the equivalent of a tangent plane, if we have 3 or more input variables?*

Problem 118. *For functions $f(x,y)$ we used level curves in a contour graph to visualize the function. What is the equivalent of a level curve if we have 3 or more input variables?*

Problem 119. *Imagine the bug is flying through the point (a,b,c) in the contour plot of some heat function $f : \mathbb{R}^3 \to \mathbb{R}$. Imagine the contour graph of the tangent space at $(a,b,c,f(a,b,c))$ around the bug. Again, the different level surrfaces of the tangent space have different colours.*

1. *The bug starts out flying from point (a,b,c) in direction $\nabla f(a,b,c)$. Describe what the bug sees.*

2. *The bug starts out flying from point (a,b,c) perpendicular to the direction $\nabla f(a,b,c)$. Describe what the bug sees.*

Problem 120. *Describe how the tangent space is a linear approximation of a function $f : \mathbb{R}^3 \to \mathbb{R}$.*

Problem 121. *Now we have some sense of how to visualize tangent spaces. But sometimes we want to compute the linear approximation. Use partial derivatives to find a (parametric or implicit) equation of the tangent space of a function $f(x,y,z)$ at some point (a,b,c). Hint: look back at the tangent plane equations.*

Problem 122. *Find any critical points of the following function*

$$f(x,y,z) := \frac{5}{6}x^2 + 4x + 16 - \frac{7}{3}xy - 4y - \frac{4}{3}xz + 12z + \frac{5}{6}y^2 - \frac{4}{3}zy + \frac{1}{3}z^2.$$

Can you tell if they are maxima, minima or saddle points? Which of your prior techniques seem to generalize and which don't?

Chapter 3

Integration

Problem 123. *Recall the definition of the definite integral using Riemann sums, from first year calculus. Draw a picture to explain the meaning of the definition.*

Problem 124. *How could we generalize the above definition for functions of two variables? Draw a picture that explains the meaning of your definition. As a start estimate the volume over the rectangle* $\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2$ *and* $0 \leq y \leq 2\}$ *and under the graph of the function* $f(x,y) = x+y+1$.

Problem 125. *Now imagine that we have a rectangular cracker spread unevenly with peanut butter. Let* $PBD(x,y)$ *represent the density (in grams per square centimeter) of the peanut butter x centimeters from the left edge of the cracker and y centimeters from the bottom of the cracker. Use a technique similar to what you did in Problem 124 to write the total amount of peanut butter (what units?) on the whole cracker in terms of* $PBD(x,y)$.

What you have just developed is the definition of the *double integral* of the function f over the rectangular region R. The notation we use for this is

$$\iint_R f\, dA.$$

Double integrals represent the relationships between quantities in many different contexts. One example would be if we made a rectangle of varying thickness or density of material. If we consider the density as a function, its double integral over the rectangle will represent the total mass of the shape. Similarly, if we had a metal plate with a varying electric charge density, the integral of the charge density over the plate would give us the total charge of the plate.

Problem 126. *Recall from single variable calculus that we could compute the area of a region using a 'dx' integral or a 'dy' integral.*

1. *Use an integral with respect to x to compute the area of the triangle with vertices* $(0,0)$, $(4,0)$, *and* $(4,3)$.
2. *Compute the same area, this time using an integral with respect to y.*
3. *Check your answer using* $\frac{1}{2}bh$*!*

The way we have set up the integral suggests that the following theorem is true. Actually proving the theorem requires showing that you can move sums and limits around in a way that is subtle and dangerous.

Theorem 1. *If D is the rectangle $\{(x,y) \in \mathbb{R}^2 : a \le x \le b$ and $c \le y \le d\}$, then*

$$\iint_D f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx.$$

Problem 127. *Explain in your own words what Theorem 1 says and why it is useful.*

Problem 128. *Let D be the rectangle $\{(x,y) \in \mathbb{R}^2 : 1 \le x \le 3$ and $2 \le y \le 5\}$. Evaluate*

$$\iint_D x^2 - 3y\, dA.$$

Problem 129. *Evaluate $\int_0^4 \int_0^1 x^2 e^{2y} + y\sqrt{x}\, dy\, dx$ and sketch the region of integration.*

Problem 130. *Evaluate $\int_{\frac{\pi}{2}}^{\pi} \int_0^{\frac{\pi}{2}} \cos(x-y)\, dx\, dy$ and sketch the region of integration.*

Problem 131. *Evaluate $\int_0^3 \int_0^2 \int_0^6 x+y+z\, dz\, dy\, dx$ and sketch the region of integration.*

Problem 132. *Evaluate $\int_{-3}^3 \int_{x^2}^9 y+2x\, dy\, dx$ and sketch the region of integration.*

Problem 133.

1. *Evaluate $\int_{-1}^5 \int_0^4 x^2 - xy\, dx\, dy$ and sketch the region of integration.*
2. *Evaluate $\int_0^4 \int_{-1}^5 x^2 - xy\, dy\, dx$ and sketch the region of integration.*
3. *Compare and contrast your answers and make a conjecture about the oder of integration.*

Problem 134. *To look further into our conjecture, look at the following integrals:*

1. *Evaluate $\int_0^5 \int_x^5 y^2\, dy\, dx$.*
2. *Evaluate $\int_x^5 \int_0^5 y^2\, dx\, dy$.*
3. *Evaluate $\int_0^5 \int_y^5 y^2\, dx\, dy$.*
4. *Evaluate $\int_0^5 \int_0^y y^2\, dx\, dy$.*
5. *Compare and contrast your answers. Where possible, sketch the region of integration.*

Problem 135. *Evaluate $\int_0^4 \int_0^{\frac{3x}{4}} 1\, dy\, dx$. Compare and contrast with problem 126.*

Problem 136. *Looking at Problem 135, describe in general why can we use a function in two variables in an integral to compute an area. Draw a picture.*

Problem 137. *Evaluate* $\int_0^4 \int_0^{\frac{3x}{4}} y^2 \, dy \, dx$. *Compare and contrast with problem 135.*

Problem 138. *Evaluate* $\int_0^5 \int_0^4 \int_0^{\frac{3x}{4}} 1 \, dy \, dx \, dz$. *Compare and contrast with Problems 126 and 135.*

Problem 139. *Looking at Problem 138, describe in general why can we use a function in three variables in an integral to compute a volume. Can you draw a picture?*

Problem 140. *Let D be the plane region whose boundary consists of the curves $y = x$ and $y = x^2$. Compute* $\iint_D 1 + xy \, dA$ *two ways (fill in the blanks):*

1. $\int_{--}^{--} \int_{--}^{--} \_\_\_ \, dy \, dx$
2. $\int_{--}^{--} \int_{--}^{--} \_\_\_ \, dx \, dy.$

Problem 141. *Let R be the plane region bounded by $y = x^2$ and $y = 4$. Find the area of R three ways (fill in the blanks):*

1. *As a single integral,* $\int_{--}^{--} \_\_\_ \, dx.$
2. *As a double integral,* $\int_{--}^{--} \int_{--}^{--} \_\_\_ \, dy \, dx$
3. *As a double integral,* $\int_{--}^{--} \int_{--}^{--} \_\_\_ \, dx \, dy.$

Problem 142. *Evaluate* $\int_{-1}^1 \int_{-1}^1 \int_0^{x^2+y^2} z \, dz \, dy \, dx$ *and sketch the region of integration.*

Problem 143. *Evaluate* $\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1} \, dy \, dx$. *Hint: You need a good idea to make the integration easier, and you should use substitution eventually...*

Problem 144. *Sketch the region of integration for* $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} f(x,y,z) \, dz \, dy \, dx.$

Problem 145. *Sketch the region of integration for* $\int_0^2 \int_0^x \int_0^{4-x^2} f(x,y,z) \, dz \, dy \, dx.$

Problem 146. *Consider the tetrahedron T whose boundary consists of the planes $3x + 2y + z = 6$, $x = 0$, $y = 0$, and $z = 0$.*

1. *Sketch T.*
2. *How many ways are there to choose the order of integration for* $\iiint_T f(x,y,z) \, dV$*?*

Problem 147. *Find the limits of integration for all of the orders in Problem 146 and check your answers by computing the volume of the tetrahedron.*

Problem 148. *Find the integral expression to compute theb mass in grams of the solid cone above $z = \sqrt{x^2+y^2}$ and below $z = 5$, if its density in g/cm^3 is given by $\rho(x,y,z)$. Assume that x, y, and z are measured in cm.*

3.1 Polar Coordinates

Problem 149. *Compute* $\int_0^1 \int_0^{\sqrt{1-y^2}} 1 \, \mathrm{dxdy}$ *and use a geometric argument to check your answer. Hint: Use substitution and the double angle formula.*

Here is a new idea to make integrals like the one above easier: Like the more familiar rectangular (x-y) coordinate system, the **polar coordinate system** is a way to specify the location of points in the plane. Some curves have simpler equations in polar coordinates, while others have simpler equations in the rectangular coordinate system.

Definition 5 (Polar Coordinates)**.** *Given a point P in the plane we associate P with an ordered pair* (r,θ) *where r is the distance from* $(0,0)$ *to P and* θ *is the angle between the* $x-$*axis and the ray* $\overrightarrow{OP}$ *as measured in the counter-clockwise direction.*

Problem 150. *Suppose* (x,y) *is a point in the plane in rectangular coordinates. Write formulas (in terms of x and y) for*

1. *the distance r from the origin to* (x,y) *and*
2. *the angles between the positive x-axis and the line containing the origin and P.*

Problem 151. *Plot the following points in the polar coordinate system and find their coordinates in the rectangular coordinate system.*

1. $r = 1, \theta = \pi$
2. $r = 2, \theta = \frac{2\pi}{3}$
3. $r = 3, \theta = \frac{5\pi}{4}$
4. $r = -3, \theta = \frac{\pi}{4}$
5. $r = 2, \theta = -\frac{\pi}{6}$

Problem 152. *Suppose a point in the plane has r and* θ *as its polar coordinates and* (x,y) *are its coordinates in the rectangular coordinate system. Write formulas for x and y in terms of r and* θ*.*

Problem 153. *Sketch the region of the plane where* $1 \leq r \leq 2$ *and* $0 \leq \theta \leq \pi$*.*

Problem 154. *Express this plane region in terms of polar coordinates:* $x^2 + y^2 < 1$ *and* $x \geq 0$*.*

Problem 155. *Write the integral in Problem 149 in polar coordinates and compute it. What do you notice when you compare it to your prior answer? Look at the next problem to resolve any descepancies. Explain your reasoning.*

Problem 156 (Extension Question)**.** *The circumference of a circle of radius r is* $2\pi r$ *(this is basically the definition of* π *and really amazing). Now give a geomteric argument using Figure 3.1 to explain why the area of the circle is* πr^2*. Then use your reasoning to find the area of a sector of the circle spanning the angle* θ*.*

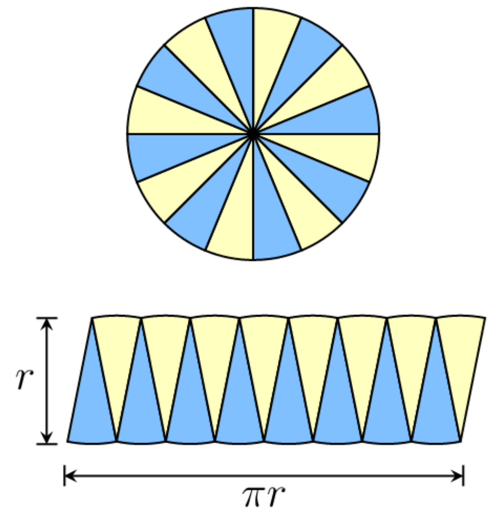

Figure 3.1: Area of a Circle

Problem 157. *Recall that the area of the sector of a circle with radius r spanning θ radians is $A = \frac{1}{2}r^2\theta$ (or do the problem before). Let $0 < r_1 < r_2$ and $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$. Sketch the region in the first quadrant bounded by the two circles $r = r_1$, $r = r_2$, and the two lines $\theta = \theta_1$, and $\theta = \theta_2$. Show that the area of the bounded region is $\left(\frac{r_1 + r_2}{2}\right)(r_2 - r_1)(\theta_2 - \theta_1)$.*

Problem 158. *Compute the area of a sector of rectangular coordinates: $[x_1, x_2] \times [y_1, y_2]$. Compare your result with Problem 157. How is area computation in polar coordinates different from area computation in rectangular cooprdinates?*

Theorem 2. *If f is integrable on a region D, then*

$$\int_D f \, dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta) \, r \, dr \, d\theta.$$

Problem 159. *Explain Theorem 2 in your own words. Then use it to compute (again) the integral in Problem 149. Does it match your prior results? Explain.*

Problem 160. *Convert to polar coordinates and compute* $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} \sin(x^2 + y^2) \, dx \, dy.$

Problem 161. *Use polar coordinates to find the volume of the solid under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leq 4$. Check your answer geometrically.*

Problem 162. *Find the volume under the paraboloid $z = 9 - x^2 - y^2$, and above the $z = 0$ plane in four ways: As a double integral in rectangular coordinates (use your calculator to do the integral), a double integral in polar coordinates, a triple integral in rectangular coordinates (calculator!), and a triple integral in "cylindrical" (polar with z) coordinates.*

Problem 163. *Compute the integral you found in Problem 148.*

Problem 164 (Extension Question)**.** *The function $f(x) = e^{-x^2}$ is very important in probability and statistics. It is sometimes called a bell curve, or a Gaussian or normal curve.*

1. *Graph f and explain how this kind of shape might be used to model, for example, how many midshipmen there are of a given height. Give an example of another situation where a curve of this shape would give a distribution of values.*
2. *In these two situations, what would a definite integral $\int_a^b e^{-x^2}\,dx$ represent?*

Problem 165 (Extension Question)**.** *Unfortunately, the function $f(x) = e^{-x^2}$ has no antiderivative in terms of functions we know and love! So we cannot use the Fundamental Theorem of Calculus to calculate its integral. Bummer. However...*

1. *Let D_R be the disk $x^2 + y^2 \leq R^2$. Evaluate $\int_{D_R} e^{-(x^2+y^2)}\,dA$. Call this number I_R.*
2. *What is $\lim_{R\to\infty} I_R$ and what does it represent?*
3. *Explain why $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.*

3.2 Integrals over Curved Surfaces

Recall that the general idea in integration is that $\int_{object} function\ d$size represents the number you get at the end of the following process:

1. Chop the object into pieces.
2. Pick one point on each piece and evaluate the function there to get a number.
3. Multiply this number by the size of its piece. ("Size" here might mean length, or area, or volume. The result of this multiplication should have the units you want to end up with.)
4. Add up the results for all pieces. This gives you an approximation of the final answer.
5. Take the limit as the number of pieces goes to infinity and the size of each piece goes to zero. Unless you are very unfortunate, this limit will exist and give you the number you're looking for.

Problem 166. *Explain how this process corresponds to the definitions we have already seen of* $\int_a^b f(x)\,dx$*,* $\iint_R f\,dA$ *in rectangular and polar coordinates, and* $\iiint_E f\,dV$*.*

Now we are going to do something that may or may not be tasty: spread peanut butter on a Pringle. The big question is: Suppose we have a curved surface, and something (peanut butter, electric charge, etc.) spread over it unevenly. We would like to set up an integral to represent the total amount of stuff (peanut butter, charge, etc.), based on the density function.
In general, if we have a parametric equation $S:\mathbb{R}^2\to\mathbb{R}^3$ for a surface, we can use the two parameters to divide the surface into pieces. Our first set of cuts leaves one variable constant; then we cut across those by leaving the other variable constant.

Problem 167. *Suppose that our surface is the hyperbolic paraboloid (a.k.a. Pringle, a.k.a. saddle surface) from Problem 39. Find a parametrization* $S:\mathbb{R}^2\to\mathbb{R}^3$ *for this surface.*

Problem 168. *Suppose our surface is a cylinder* $x^2+y^2=25$*. Find a parametrization of the cylinder. What would be a good way to chop the cylinder into pieces? What would the area of each piece be?*

Problem 169. *Explain why the following ideas are true (Extension Question) and how we can use them to find the area of a small piece:*

1. *The derivative of* S *with respect to one parameter will give a vector tangent to the surface. (Hint: Generalize the definition of derivative for vector functions* $S_1:\mathbb{R}\to\mathbb{R}^3$*. Draw a picture.)*
2. *The cross product of two vectors gives a vector whose magnitude is the area of the parallelogram with the original two vectors as sides. Hint: Linear Algebra?*

Theorem 3. *If* $S:[u_1,u_2]\times[v_1,v_2]\to\mathbb{R}^3$ *is a parametric equation for a surface* Q*, and* $f:\mathbb{R}^3\to\mathbb{R}$ *is a function, then*

$$\iint_Q f\,dA=\int_{v_1}^{v_2}\int_{u_1}^{u_2} f(S(u,v))\,\left|\frac{\partial S}{\partial u}\times\frac{\partial S}{\partial v}\right|\,du\,dv.$$

Problem 170. *Use Problem 169 to explain why Theorem 3 makes sense.*

Problem 171. *Suppose that we have an open-ended metal tube, the shape of a cylinder of radius 3cm and length 5cm, and there is an uneven electric charge on it. We choose coordinates so that the center of the tube is along the z-axis and the bottom of the tube is at $z = 0$. In this coordinate system, the electric charge density is $x+3$ Coulombs per square centimeter. What is the total electric charge on the tube? (Compute by hand)*

Problem 172. *Let P be the part of the paraboloid with equation $z = x^2 + y^2$ and $0 \leq z \leq 4$, and let $f(x,y,z) = x^2 + y^2 + z^2$. Use wolframalpha to compute $\iint_P f \, dA$.*

Problem 173. *Suppose that peanut butter is spread on a surface whose shape can be described by the equation $z = x^2 - y^2$ (see Problem 39), with $x^2 + y^2 \leq 1$. The density of peanut butter, in grams per square centimeter, is given by $PBD(x,y,z) = 2 - z$. Find the total amount of peanut butter on the surface (using wolframalpha).*

3.3 Integrals over Curved Lines

But what if we only have a curve instead of a whole surface? An example that would be good to think of is of a curved wire, with electric charge unevenly distributed along it. If $f(x,y,z)$ gives the charge density in Coulombs per centimeter, then $\int_C f\, ds$ represents the total charge (in Coulombs) of the whole wire.

Problem 174. *Suppose $\overrightarrow{c} : \mathbb{R} \to \mathbb{R}^3$ is a vector valued function which is differentiable on $[a,\ b]$, and $\overrightarrow{c}(t) = (x(t), y(t), z(t))$. Let s be the arc length of the curve $\overrightarrow{c}(t)$ from the point $(x(a), y(a), z(a))$ to $(x(b), y(b), z(b))$.*

1. *Let $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition of $[a,b]$.*
   Show that $s \approx \sum_{i=1}^{n} \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 + (z(t_i) - z(t_{i-1}))^2}$.
2. *Show that $s \approx \sum_{i=1}^{n} (t_i - t_{i-1}) \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}$.*
3. *Explain why this might lead you to believe Theorem 4, below.*

From the previous problem, the following theorem seems plausible. [1]

Theorem 4. 1. *If $\overrightarrow{c} : \mathbb{R} \to \mathbb{R}^3$ is a curve which is differentiable on $[a,\ b]$ so that $\overrightarrow{c}(t) = (x(t), y(t), z(t))$ then the* **arc length** *of $\overrightarrow{c}$ on $[a,\ b]$ is $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$.*

2. *If $\overrightarrow{c} : \mathbb{R} \to \mathbb{R}^3$ is a curve which is differentiable on $[a,\ b]$ so that $\overrightarrow{c}(t) = (x(t), y(t), z(t))$ and $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function then*

$$\int_C f\, ds = \int_a^b f(c(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

Problem 175. *A thin wire has the shape of the first-quadrant part of the circle with center the origin and radius a. If the charge density function is $f(x,y) = kxy$, find the total charge on the wire.*

Problem 176. *Suppose we want to use an integral $\iint_S f\, dA$ to find the area of a surface S. For example, we might need to know how much paint to buy to paint it. What function should we use for f?*

Problem 177. *Check your answer to the previous problem by proving that the surface area of a cylinder is $2\pi rh$.*

Problem 178. *Suppose we want to use an integral $\int_C f\, ds$ to find the length of a curve C. What function should we use for f? Check your answer by finding the circumference of a circle.*

[1] There is actually some subtle magic going on when the derivative limit and the sum limit are taken at the same time! Luckily the Mean Value Theorem you saw in first semester calculus actually tells you these limits can be taken at the same time, provided that $x(t)$ and $y(t)$ are well-behaved functions.

Chapter 4

Vector Fields

Imagine a large body of water, like a river or part of the ocean. The current in different parts of the water may flow faster or slower, or in different directions.

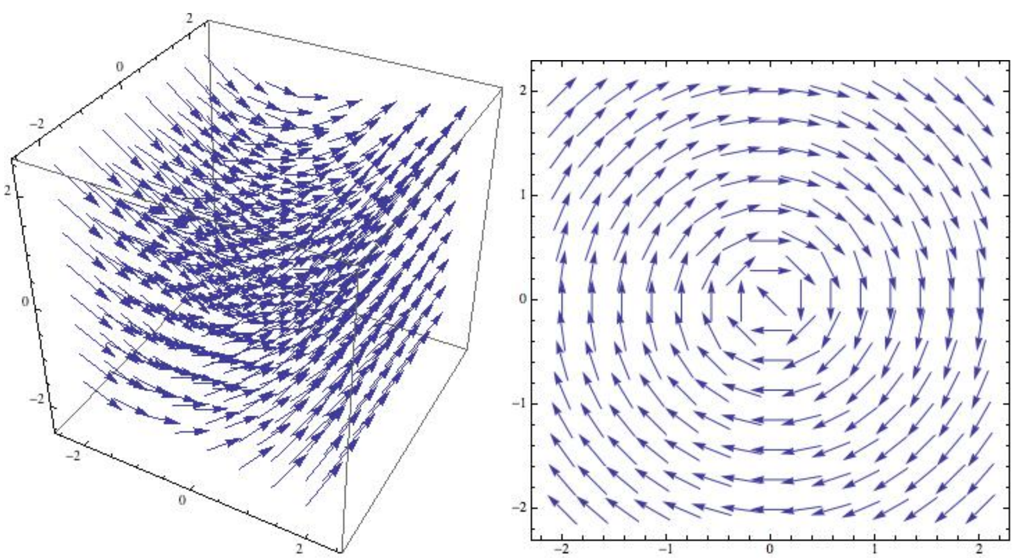

Figure 4.1: Vector Fields

In mathematics, we use a **vector field** to represent the velocity of the water at different points. Vector fields are also used to represent electric and magnetic fields, forces, and velocity of other fluids like air or the dust between stars in a galaxy. Intuitively, a vector field consists of an arrow at each point. We saw an example of a vector field when we computed the gradient.

The general formula for a vector field is $\overrightarrow{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$.

Problem 179. *Sketch the vector field* $\overrightarrow{F}(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}\hat{\imath} + \dfrac{-x}{\sqrt{x^2+y^2}}\hat{\jmath}$.

Now suppose we are studying a river. We have a net in part of the river, and want to write an integral to represent the amount of water flowing through the net.

Problem 180. *What units do we want our answer to be in?*

Problem 181. *Suppose the velocity field in the river is* $\overrightarrow{F}(x,y,z) = 3m/s\hat{\imath}$, *and the net can be described by the equation* $S(t,u) = (1,t,u)$ *for* $-2 \leq t \leq 2$ *and* $0 \leq u \leq 5$ *(all distances in meters).*

1. *Draw a sketch of the net and the river flow.*

2. *At what rate is water is flowing through the net? Explain your reasoning.*

Problem 182. *Now suppose that the velocity field of the river is the same as in the previous problem, but that the net now has equation $S(t,u) = (t,1,u)$ for $-2 \le t \le 2$ and $0 \le u \le 5$. At what rate is water flowing through the net? Explain your reasoning.*

Problem 183. *What happens if the net is at an angle to the current?*

Problem 184. *Use properties of the dot product to explain why, if $\overrightarrow{d}$ is a vector and $\hat{u}$ is a unit vector, $\overrightarrow{d} \cdot \hat{u}$ gives the component of $\overrightarrow{d}$ in the $\hat{u}$ direction.*

Definition 6. *The* ***flux*** *of $\overrightarrow{F}(x,y,z)$* ***across the surface*** *$S(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$ is defined by*

$$Flux = \iint_S \overrightarrow{F} \cdot \hat{n}\, dA,$$

where $\hat{n}$ is a field of unit vectors perpendicular ("normal") to the surface.

Problem 185. *Remember that cross product we did back in Theorem 3? Which direction does it point? Is it a unit vector?*

Problem 186. *Use the result from the previous problem to explain why*

$$\iint_S \overrightarrow{F} \cdot \hat{n}\, dA = \iint_S \overrightarrow{F} \cdot (\overrightarrow{S}_u \times \overrightarrow{S}_v)\, du\, dv.$$

Problem 187. *Let $\overrightarrow{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by $\overrightarrow{F}(x,y,z) = (1,1,-1)$.*

1. *Compute the flux of $\overrightarrow{F}$ across the square $S(x,y) = (x,y,2)$ for $-2 \le x \le 2$ and $-2 \le y \le 2$.*

2. *Was your answer to the previous problem positive or negative? Is this what you would have expected?*

Problem 188. *Let $\overrightarrow{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by $\overrightarrow{F}(x,y,z) = (0,0,-z)$.*

1. *Find the flux of $\overrightarrow{F}$ down through the surface given by $S(u,v) = (u,v,u^2+v^2)$ where $u^2+v^2 \le 4$.*

2. *Find the flux of $\overrightarrow{F}$ up through the surface given by $S(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$ where $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.*

3. *Compare and contrast.*

Problem 189. *Compute the flux of the vector field $\overrightarrow{F}(x,y,z) = (x,y,z)$ out of the sphere $x^2+y^2+z^2 = 9$.*

4.1 Divergence and Gauss's Theorem

Definition 7. *The* ***divergence*** *of a vector field* $\overrightarrow{F}(x,y,z) = P(x,y,z)\ \hat{\imath} + Q(x,y,z)\ \hat{\jmath} + R(x,y,z)\ \hat{k}$ *is defined as*

$$\textbf{div}\ \overrightarrow{F} := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Problem 190. *Wow, that definition of divergence looks kind of like a dot product to me! If it were the dot product of something with* $\overrightarrow{F}$*, what would the something be?*

Problem 191. *Draw the vectorfield* $F(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ *and compute its divergence. Is the divergence positive or negative? Can you "see" the sign of the divergence in your picture?*

Problem 192. *Draw the vectorfield* $F(x,y,z) = \frac{x}{(x^2+y^2+z^2)^6}\hat{\imath} + \frac{y}{(x^2+y^2+z^2)^6}\hat{\jmath} + \frac{z}{(x^2+y^2+z^2)^6}\hat{k}$ *and compute its divergence. Is the divergence positive or negative? Can you "see" the sign of the divergence in your picture?*

Theorem 5 (Gauss's Divergence Theorem – first proven in 1826)**.** *If* $\overrightarrow{F}$ *is a vector field on the region surrounded by a closed surface, then the flux of* $\overrightarrow{F}$ *out through the surface is equal to the integral of the divergence of* $\overrightarrow{F}$ *over the enclosed region.*

Problem 193. *Write Gauss's Divergence Theorem as an equation with an integral on each side. Explain in your own words what each integrals stands for. Does the theorem makes sense to you?*

Problem 194. *Redo Problem 189 using Gauss's Divergence Theorem.*

The Divergence Theorem is an example of a higher-dimensional version of the Fundamental Theorem of Calculus. Both have the general form

$$\int_{\text{object}} \text{"derivative" of } f = \int_{\text{boundary of object}} f. \tag{4.1}$$

Problem 195. *Explain how each piece of Gauss's Divergence Theorem fits into the framework of Equation 4.1.*

Problem 196. *Write down the Fundamental Theorem of Calculus from Calculus 1 (which we use all of the time to evaluate integrals). Explain how each piece of this equation fits into the framework of Equation 4.1.*

4.2 Rowing and more Fundamental Theorems

Imagine you are rowing on a river along the curve $c : \mathbb{R} \to \mathbb{R}^2$. The river has a current $F(x,y)$ which may be with or again you while you travel along your chosen path. We want to compute how much the current has overall helped or hindered us.

Problem 197. *1. Make a fairly large and careful sketch of the curve $C(t) = (3\cos t, 3\sin t)$ for $0 \le t \le \pi$. This is the path you are rowing.*

2. Let $\overrightarrow{F}(x,y) = -\frac{x^2}{9}\hat{i}$. For each of $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$, sketch the two vectors $\overrightarrow{C}'(t)$ and $\overrightarrow{F}(C(t))$, both based at the point $C(t)$. How are these vectors related to your boat and the current?

3. For each of the t values in part 2, compute the component of $\overrightarrow{F}$ in the $\overrightarrow{C}'(t)$ direction and label that point on the graph with this value. What do you notice? Did the current help you or hinder you on your path?

Our example used only two dimensions but you can imagine that a submarine in the ocean could ask the same question about ocean currents and would need three dimensions. This motivates the following definition:

Definition 8. *The **line integral of the vector field** $\overrightarrow{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$ over the curve C given by equation $r(t) = (x(t), y(t), z(t))$ is defined by*

$$\int_C \overrightarrow{F} \cdot d\mathbf{r} := \int_a^b \overrightarrow{F}(r(t)) \cdot r'(t)\, dt$$

Problem 198. *Compute the line integral of the vector field and curve given in Problem 197 for $0 < t < \pi$. How does this relate to the values you computed before? Did the current help or hinder?*

Problem 199. *Compute the line integral of the vector field and curve given in Problem 197 for $0 < t < 2\pi$. How does this relate to Problems 197 and 198? Did the current help or hinder?*

Problem 200. *Now imagine taking a different path through the river (but starting and ending at the same points). Would this change the value of your line integral? Make a conjecture and choose a few paths to test it (using the above current vector field).*

Problem 201. *Compute the line integral $\int_C \overrightarrow{F} \cdot d\mathbf{r}$ where the path of your boat C is given by $c(t) = (t^2, t^3)$, $0 < t < 1$ and the current $\overrightarrow{F}$ is given by $\overrightarrow{F}(x,y,z) = (x+y, y)$. Test you abovfe conjecture by choosing a different path for the line integral. What do you notice?*

Problem 202. *Test your conjecture again by computing the line integral $\int_C \overrightarrow{F} \cdot d\mathbf{r}$ where $\overrightarrow{F}$ is given by $\overrightarrow{F}(x,y) = (1, 2y)$ and C is the line segment from $(0,0)$ to $(5,5)$.*

It is difficult to see what needs to be true for a vector field F to allow the line integral to be path independent:

Theorem 6. ***The Fundamental Theorem for Line Integrals*** *Suppose that C is a curve starting at point P and ending at point Q. If the vector field $\overrightarrow{F}$ is the gradient of a function g, and everything is continuous, then*

$$\int_C \overrightarrow{F} \cdot d\mathbf{r} = g(Q) - g(P).$$

Problem 203. *Explain how Theorem 6 fits into the general framework of Equation 4.1.*

Problem 204. *Explain in your own words what Theorem 6 states and then see if you can use it to simplify your solutions of Problems 198, 199, 201 and 202.*

Problem 205 (Extension Question)**.** *Use the chainrule to prove Theorem 6.*

If we want to use Theorem 6 to compute a line integral $\int_C \overrightarrow{F} \cdot d\mathbf{r}$, we need to find an "anti-derivative" g of $\overrightarrow{F}$: a function whose gradient is $\overrightarrow{F}$. But as you nocticed above, not every vector field is the gradient of a function.

Definition 9. *A vector field is said to be* ***conservative*** *if it is the gradient of some function. If a vector field $\overrightarrow{F}$ is the gradient of a function g, g is called its* ***potential function****.*

Problem 206. *Is $\overrightarrow{F}(x,y) = (x^2 + y^2, 2xy)$ conservative? If so, find its potential function.*

Problem 207. *Is $\overrightarrow{F}(x,y) = (xy, x - y)$ conservative? If so, find its potential function.*

Problem 208. *Why are conservative vector fields useful? Think about gravitational fields, electric fields, etc.*

There are two more fundamental theorems you should at least have heard about:

Theorem 7. ***Green's Theorem*** *Suppose C is a piecewise smooth, simple, closed curve that is the boundary of a region R in the plane and oriented so that the region is on the left as we move around the curve. Suppose $F = P\hat{i} + Q\hat{j}$ is a smooth vector field on a open region containing R and C. Then*

$$\int_C \overrightarrow{F} \cdot d\mathbf{r} = \int_R (Q_x - P_y)\ \mathrm{dxdy}$$

Problem 209. *Explain how Theorem 7 fits into the general framework of Equation 4.1 on page 32.*

Problem 210. *Remember ∇ from Problem 190? What would the cross product $\nabla \times \overrightarrow{F}$ be? This is called the* ***curl*** *of $\overrightarrow{F}$. The curl measures how much a vectorfield "rotates". Imagine sticking a paddle into the current where the blade in front can spin around. The direction in which it spins the most is the direction of the curl.*

Theorem 8. ***Stokes' Theorem*** *If S is a surface and C is the boundary curve of that surface, and $\overrightarrow{F}$ is a vector field which is differentiable on S, then*

$$\int_C \overrightarrow{F} \cdot d\mathbf{r} = \int_S (\mathbf{curl}\,\overrightarrow{F}) \cdot \hat{n}\ dA.$$

Problem 211. *Explain how Theorem 8 fits into the general framework of Equation 4.1 on page 32.*