DISCOVERING THE
ART OF MATHEMATICS

TRUTH, REASONING, CERTAINTY & PROOF

MATHEMATICAL INQUIRY IN THE LIBERAL ARTS

Julian F. Fleron and Philip K. Hotchkiss,
with Volker Ecke and Christine von Renesse

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Truth, Reasoning, Certainty, and Proof

by Julian F. Fleron and Philip K. Hotchkiss

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Preface: Notes to the Explorer

Yes, that's you - you're the explorer.

"Explorer?"

Yes, explorer. And these notes are for you.

We could have addressed you as "reader," but this is not a traditional book. Indeed, this book cannot be read in the traditional sense. For this book is really a guide. It is a map. It is a route of trail markers along a path through part of the world of mathematics. This book provides you, our explorer, our heroine or hero, with a unique opportunity to explore this path - to take a surprising, exciting, and beautiful journey along a meandering path through a mathematical continent named the infinite. And this is a vast continent, not just one fixed, singular locale.

"Surprising?" Yes, surprising. You will be surprised to be doing real mathematics. You will not be following rules or algorithms, nor will you be parroting what you have been dutifully shown in class or by the text. Unlike most mathematics textbooks, this book is not a transcribed lecture followed by dozens of exercises that closely mimic illustrative examples. Rather, after a brief introduction to the chapter, the majority of each chapter is made up of Investigations. These investigations are interwoven with brief surveys, narratives, or introductions for context. But the Investigations form the heart of this book, your journey. In the form of a Socratic dialogue, the Investigations ask you to explore. They ask you to discover the mathematics that is behind music and dance. This is not a sightseeing tour, you will be the active one here. You will see mathematics the only way it can be seen, with the eyes of the mind - your mind. You are the mathematician on this voyage.

"Exciting?" Yes, exciting. Mathematics is captivating, curious, and intellectually compelling if you are not forced to approach it in a mindless, stress-invoking, mechanical manner. In this journey you will find the mathematical world to be quite different from the static barren landscape most textbooks paint it to be. Mathematics is in the midst of a golden age - more mathematics is discovered each day than in any time in its long history. Each year there are 50,000 mathematical papers and books that are reviewed for *Mathematical Reviews*! *Fermat's Last Theorem*, which is considered in detail in <u>Discovering that Art of Mathematics - Number Theory</u>, was solved in 1993 after 350 years of intense struggle. The 1$ Million Poincaŕe conjecture, unanswered for over 100 years, was solved by **Grigori Perleman** (Russian mathematician; 1966 -). In the time period between when these words were written and when you read them it is quite likely that important new discoveries adjacent to the path laid out here have been made.

"Beautiful?" Yes, beautiful. Mathematics is beautiful. It is a shame, but most people finish high school after 10 - 12 years of mathematics *instruction* and have no idea that mathematics is beautiful. How can this happen? Well, they were busy learning mathematical skills, mathematical reasoning, and mathematical applications. Arithmetical and statistical skills are useful skills everybody should possess. Who could argue with learning to reason? And we are all aware, to some degree or another, how mathematics shapes our technological society. But there is something more to mathematics than its usefulness and utility. There is its beauty. And the beauty of mathematics is one of its driving forces. As the famous **Henri Poincaŕe** (French mathematician; 1854 - 1912) said:

> The mathematician does not study pure mathematics because it is useful; [s]he studies it because [s]he delights in it and [s]he delights in it because it is beautiful.

Mathematics plays a dual role as both a liberal art and as a science. As a powerful science, mathematics shapes our technological society and serves as an indispensable tool and language in many fields. But it is not our purpose to explore these roles of mathematics here. This has been done in many other fine, accessible books (e.g. [COM] and [TaAr]). Instead, our purpose here is to journey down a path that values mathematics from its long tradition as a cornerstone of the liberal arts.

Mathematics was the organizing principle of the *Pythagorean society* (ca. 500 B.C.). It was a central concern of the great Greek philosophers like **Plato** (Greek philosopher; 427 - 347 B.C.). During the Dark Ages, classical knowledge was rescued and preserved in monasteries. Knowledge was categorized into the classical liberal arts and mathematics made up several of the seven categories.[1] During the Renaissance and the Scientific Revolution the importance of mathematics as a science increased dramatically. Nonetheless, it also remained a central component of the liberal arts during these periods. Indeed, mathematics has never lost its place within the liberal arts - except in the contemporary classrooms and textbooks where the focus of attention has shifted solely to the training of qualified mathematical scientists. If you are a student of the liberal arts or if you simply want to study mathematics for its own sake, you should feel more at home on this exploration than in other mathematics classes.

"Surprise, excitement, and beauty? Liberal arts? In a mathematics textbook?" Yes. And more. In your exploration here you will see that mathematics is a human endeavor with its own rich history of human struggle and accomplishment. You will see many of the other arts in non-trivial roles: dance and music to name two. There is also a fair share of philosophy and history. Students in the humanities and social sciences, you should feel at home here too.

Mathematics is broad, dynamic, and connected to every area of study in one way or another. There are places in mathematics for those in all areas of interest.

The great **Betrand Russell** (English mathematician and philosopher; 1872 - 1970) eloquently observed:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.

It is my hope that your discoveries and explorations along this path through the infinite will help you glimpse some of this beauty. And I hope they will help you appreciate Russell's claim that:

> ... The true spirit of delight, the exaltation, the sense of being more than [hu]man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

Finally, it is my hope that these discoveries and explorations enable you to make mathematics a real part of your lifelong educational journey. For, in Russell's words once again:

> ... What is best in mathematics deserves not merely to be learned as a task but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement.

Bon voyage. May your journey be as fulfilling and enlightening as those that have served as beacons to people who have explored the continents of mathematics throughout history.

[1] These were divided into two components: the *quadrivium* (arithmetic, music, geometry, and astronomy) and the *trivium* (grammar, logic, and rhetoric); which were united into all of knowledge by philosophy.

Navigating This Book

Before you begin, it will be helpful for us to briefly describe the set-up and conventions that are used throughout this book.

As noted in the Preface, the fundamental part of this book is the Investigations. They are the sequence of problems that will help guide you on your active exploration of mathematics. In each chapter the investigations are numbered sequentially. You may work on these investigation cooperatively in groups, they may often be part of homework, selected investigations may be solved by your teacher for the purposes of illustration, or any of these and other combinations depending on how your teacher decides to structure your learning experiences.

If you are stuck on an investigation remember what **Frederick Douglass** (American slave, abolitionist, and writer; 1818 - 1895) told us: "If thee is no struggle, there is no progress." Keep thinking about it, talk to peers, or ask your teacher for help. If you want you can temporarily put it aside and move on to the next section of the chapter. The sections are often somewhat independent.

Investigation numbers are bolded to help you identify the relationship between them.

Independent investigations are so-called to point out that the task is more significant than the typical investigations. They may require more involved mathematical investigation, additional research outside of class, or a significant writing component. They may also signify an opportunity for class discussion or group reporting once work has reached a certain stage of completion.

The Connections sections are meant to provide illustrations of the important connections between mathematics and other fields - especially the liberal arts. Whether you complete a few of the connections of your choice, all of the connections in each section, or are asked to find your own connections is up to your teacher. But we hope that these connections will help you see how rich mathematics' connections are to the liberal arts, the fine arts, culture, and the human experience.

Further investigations, when included are meant to continue the investigations of the area in question to a higher level. Often the level of sophistication of these investigations will be higher. Additionally, our guidance will be more cursory.

Within each book in this series the chapters are chosen sequentially so there is a dominant theme and direction to the book. However, it is often the case that chapters can be used independently of one another - both within a given book and among books in the series. So you may find your teacher choosing chapters from a number of different books - and even including "chapters" of their own that they have created to craft a coherent course for you. More information on chapter dependence within single books is available online.

Certain conventions are quite important to note. Because of the central role of proof in mathematics, definitions are essential. But different contexts suggest different degrees of formality. In our text we use the following conventions regarding definitions:

- An *undefined term* is italicized the first time it is used. This signifies that the term is: a standard technical term which will not be defined and may be new to the reader; a term that will be defined a bit later; or an important non-technical term that may be new to the reader, suggesting a dictionary consultation may be helpful.

- An ***informal definition*** is italicized and bold faced the first time it is used. This signifies that an implicit, non-technical, and/or intuitive definition should be clear from context. Often this means that a formal definition at this point would take the discussion too far afield or be overly pedantic.

- A **formal definition** is bolded the first time it is used. This is a formal definition that suitably precise for logical, rigorous proofs to be developed from the definition.

In each chapter the first time a biographical name appears it is bolded and basic biographical information is included parenthetically to provide some historical, cultural, and human connections.

Misplaced Darling

Grey skies
Purple rainbows
Pigeons in the gutters
Black water
Pink sunshine
Children in the wars
Saturday morning
Misty mind set
Will there ever be a savior?
Who made this world so polluted?
Can I blame it on them?
Or you?
Or I?
Dragon flies
And long goodbyes
Where will this life take us?
Away in a dream
or so it may seem.
This couldn't possibly be reality.

Ingeri Nel Kristine Sheline Eaton (American Poet, Steelworker and Chocolatier; 1970 -).

Chapter 1

Doubt

The eyes are not responsible when the mind does the seeing.

Publilius Syrus (Syrian Author; ca. 100 B.C. - ca 0 B.C.)

The question is not what you look at, but what you see.

Henry David Thoreau (American Author, Philosopher and Naturalist; 1817 - 1862)

As described in the section Notes to the Explorer, this book is for *your* explanation of mathematics. So it is natural for it to start with an investigation where you get to first imprint yourself on this exploration.

1. Name, describe, or otherwise identify five things that you are certain of. In your list, aim for as much diversity and uniqueness as you can.

2. For each of the things that you listed, describe why/how you are certain of them.

3. Do you question your perception of reality? Explain.

This book is about truth, reasoning, certainty and proof. These words are used fairly often in everyday language. Unfortunately, they are used so loosely that they have lost some of their power to illuminate deep matters that have captivated the human intellect throughout its history. This includes the very nature of knowledge - what we know and how we know it.

Truth, certainty and proof are radical notions. Mathematics plays a fundamental role in our understanding of these things - and of the nature of knowledge.

To understand how radical truth, certainty and proof are, one needs something to contrast them against. It is for this reason the first chapter is entitled "Doubt". We hope your investigations here induce in you a healthy skepticism. That these explorations make you call into question some of your most cherished beliefs. That your experiences nurture your willingness to doubt and question. For only then can you fully appreciate the radical nature of truth, certainty, proof, and the role that mathematics plays in our understanding of these fundamental notions.

1.1 Disturbing Puzzles

We begin with some puzzles.

4. How many Leprechauns are there in the image shown in Figure 1.1? How certain are you of your answer? Explain.

5. In a moment you will cut a copy of the image (which is included in the appendix) into three pieces. If you rearrange the pieces so there is no overlapping, is there a way to have any other number of Leprechauns than what you found in Investigation **4**? Explain.

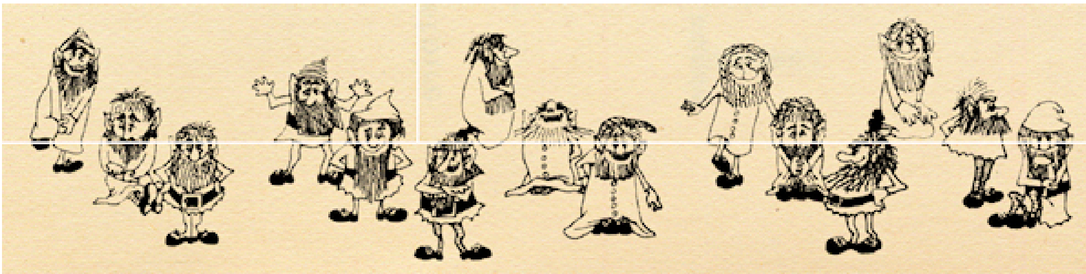


Figure 1.1: The Vanishing Leprechaun by Pat Lyons.

6. Cut the image into three pieces as indicated. Rearrange these pieces so they are not overlapping and so only whole Leprechauns are formed. How many Leprechauns are there now?

7. How is this possible?

Created as far back as the 1880's, ***vanishing puzzles*** can be quite vexing. "The Vanishing Leprechaun," the name of the vanishing puzzle you just completed, was invented by **Pat Patterson Lyons** (Canadian magician and illustrator; 1933 -) in 1968. Here are two more.

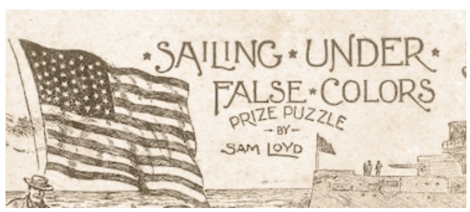


Figure 1.2: "Flying Under False Colors Puzzle" by **Sam Loyd** (American chess master and puzzle inventor; 1841 - 1911).

8. The name of the **Loyd's puzzle** in Figure 1.2 suggests the flag is false. What is wrong with the flag?

9. The goal of Loyd's puzzle is to cut the flag into exactly two pieces, rearranging the pieces to fit back together - like a jigsaw puzzle - so the resulting flag has 13 stripes. Either solve this puzzle or explain why you think it cannot be done. (An enlarged version appears in the appendix.)

10. **Stover's puzzle**, pictured in Figure 1.3 (a copy of which also appears in the appendix) is similar to the Vanishing Leprechauns. How many red pencils are there? Blue pencils?

11. Cut the puzzle as indicated and rearrange the pieces so whole pencils are formed again. How many red pencils are there now? Blue pencils?

12. How could the number of pencils of each color change? Explain.

13. Return to the Vanishing Leprechauns and explain how a Leprechaun appears/disappears as the pieces are arranged.

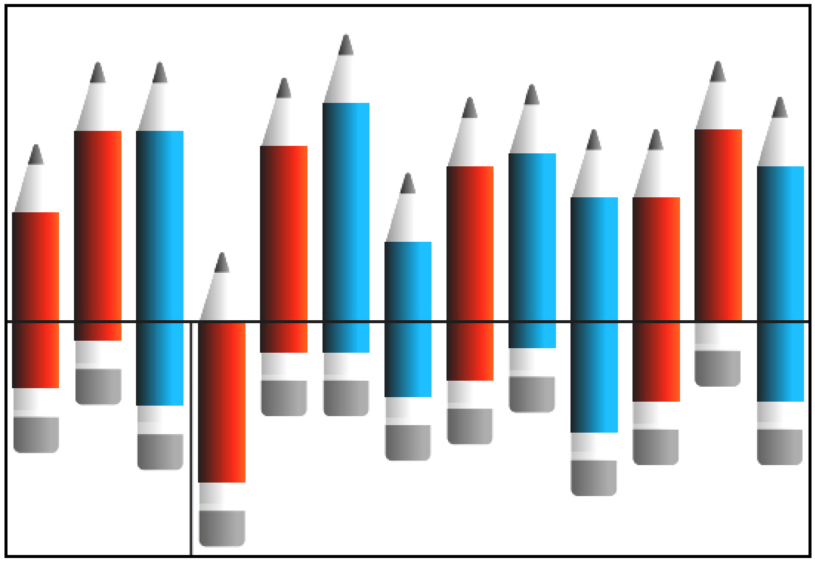


Figure 1.3: “Vanishing Pencils” by **Mel Stover** (Canadian puzzle inventor; 1912 - 1999).

The last puzzle is much older, attributed to **William Hooper** (; -) in 1794[1] by perhaps the greatest mathematical puzzler of all time, **Martin Gardner** (American mathematics and science writer; 1914 - 2010).

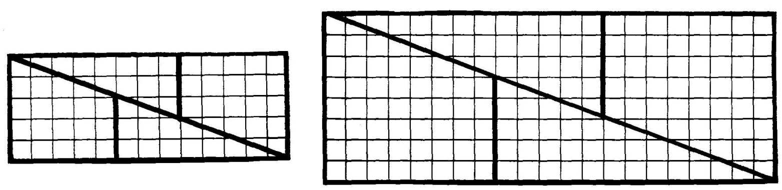


Figure 1.4: Magical rectangles, shape A on the left and B on the right, by William Hooper.

14. In each of the puzzles above you rearranged pieces in non-overlapping ways. What particular attributes changed to make the puzzles troubling?

15. When pieces are rearranged as those above were what particular mathematical aspects of the puzzles cannot/do not change? How do you know these aspects cannot be changed upon rearrangement of the pieces?

16. Determine the area of Shape A in Figure 1.4.

17. Determine the area of Shape B in Figure 1.4.

Copies of Shape A and Shape B are included in the appendix. Cut them out and then cut along the solid lines.

18. Rearrange the pieces that make up Shape A to form a square. What is the area of the square?

[1]See Mathematics, Magic and Mystery, Dover, 1956, p. 131

19. Rearrange the pieces that make up Shape B to form a square. What is the area of the square?

20. Rearrange the pieces that make up Shape A to form the shape of a T. What is the area of this shape?

21. Rearrange the pieces that make up Shape B to form the shape of a T. What is the area of this shape?

22. How do your observations about the areas of the rearranged shapes compare with the original areas? Compare and contrast this result with your observation in Investigation **15**. How can this contrast be resolved?

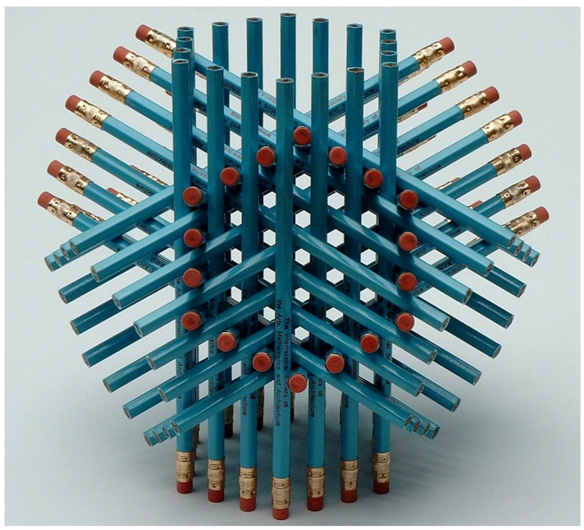


Figure 1.5: "72 Pencils" by **George Hart** (American mathematician, artist and computer scientist; 1955 -). Comissioned for The International Society of the Arts, Mathematics, and Architecture.

1.2 Optical Illusions

> This whole creation is essentially subjective, and the dream is the theater where the dreamer is at once scene, actor, prompter, stage manager, author, audience, and critic.
>
> **Carl Jung** (Swiss psychologist; 1875 - 1961)

23. Look at the image in Figure 1.6. Which of the two light-colored horizontal lines seem longer?

24. How do these two lines actually compare?

25. Perhaps you have seen optical illusions like this before. But suppose you knew nothing about such illusions and were given Ponzo's test. Would you be certain that your initial perception was correct?

26. Quickly look at the image in Figure 1.7. Do the lines that divide this "checkerboard" into "squares" appear to be straight?

27. Are these lines straight? Explain how you determined this.

28. Is there some way that you can visualize this image so you can see its geometric aspects correctly without your perception being tricked? Explain.

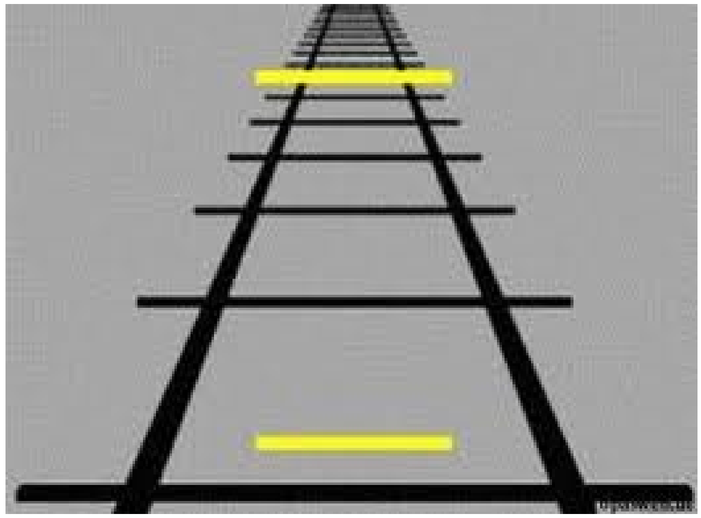


Figure 1.6: Perception experiment developed by **Mario Ponzo** (Italian psychologist; 1882 - 1960) in 1913.

> Perhaps much of the magic - the mystery - lies in this: a good photograph convinces us that it does indeed render everything; we would not know how to add to its completeness. And yet half a dozen photographs made in a given room at a given hour will each describe a different reality. None renders everything, and the best are those that describe without evasion or vagueness a single, coherent perception: not everything, but the precise intuition of a truth. Photography has enabled us to explore the richness of that swarming, shifting, four-dimensional continuum that we choose to call reality. It has shown us that what is "out there" is not a catalogue of discrete facts and Ptolemaic measurements but a sea of changing relationships.[2]
>
> **Lee Friedlander** (American photographer; 1934 -)

29. In his quotation above, describe what Friedlander is telling us about perception.

30. Can you describe ways in which optical illusions or varied perceptions can be used to manipulate people's beliefs?

Temple Grandin (American veterinary scientist; 1947 -) is one of the world's leading veterinary scientists and is the foremost designer of livestock handling and large animal veterinary facilities. She has autism. She attributes her unique ability to design humane environments for animals to "thinking in pictures" which is the name of her autobiography. At the age of 18 she made a ***squeeze machine*** to help her feel held since physical contact caused enormous stress. These machines are now widely used as therapeutic, stress-relieving devices.

A wonderful film biography of Temple's life, struggles with autism, and successes in veterinary sciences entitled Temple Grandin was released in 2010 and starred **Claire Danes** (American actress; 1979 -). Watch the video section about her re-discovery of the Ames room.[3]

[2]From E.J. Bellocq: Storyville Portraits, p. 12, Museum of Modern Art (New York), 1970

[3]This appears at minutes ?? - ?? in the video and can be found online at `http://www.youtube.com/watch?v=chxCNEsu3YU` .

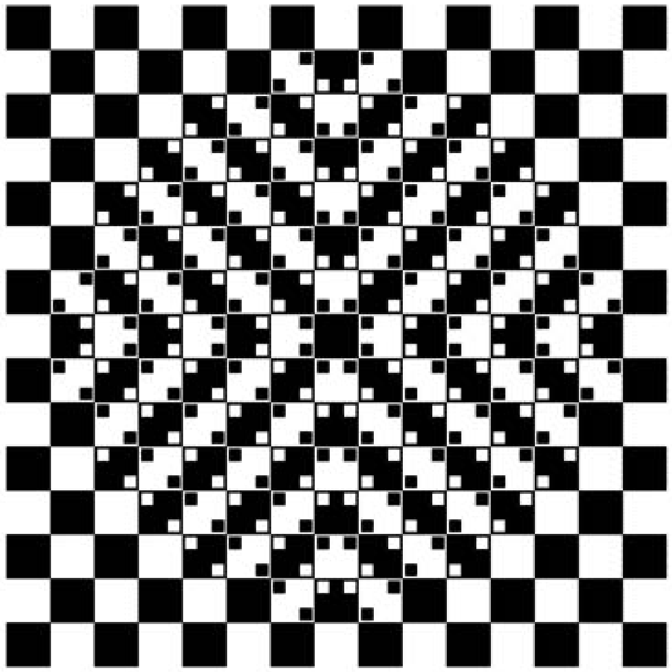


Figure 1.7: Distorted checkerboard.

31. Are there lessons about learning and alternative perspectives that this clip illustrate? Explain.

32. Describe the Ames room, what the illusion is and how it is created.

33. How powerful of an illusion is this? Explain.

34. Find several examples of optical illusions that you find compelling. Explain why they are of particularly interesting to you.

This section opens with a quotation from Jung and, indeed, there are many optical illusions that play important roles in psychology as well as other fields outside of mathematics. See the Connections section for more examples.

1.3 Biases in Perspective Illustrated by Literature, Fiction, Theater and Movies

The 2001 movie Shallow Hal[4] stars **Jack Black** (American actor; 1969 -) as shallow Hal Larson who is hypnotized to see everyone by their inner rather than their outer ward appearances. He falls madly in love with Rosemary Shanahan who he sees as the beautiful **Gwenyth Paltrow** (American actress; 1972 -) while the rest of the world sees the "real" Rosemary - a homely, obese woman. Hal's friend decides to "save" Hal from this "awful" fate:

Hal: You screwed me, man! l had a beautiful, caring, funny, intelligent woman, and you made her disappear!

[4] By Sean Moynihan, Peter Farrelly and Bobby Farrelly. Trailer in online at `http://www.youtube.com/watch?v=NMLZnY2nLcw` .

Mauricio: Oh, no, l didn't. l just made Rosemary appear. There's a difference. lt's called reality.

Hal: Hey, if you can see something and hear it and smell it, what keeps it from being real?

Mauricio: Third-party perspective. Other people agreeing that it's real.

Hal: [The all-time love of my life,] that's what I had with Rosemary! I saw a knockout! I don't care what anybody else saw!

So, what is real?

"Allegory of the Cave" is a parable from The Republic by **Plato** (Greek philosopher; circa 420 BC - circa 340 BC).

35. Read or read about "Allegory of the Cave." Describe what the two different perspectives are that form the central focus of this parable.

36. From their perspective, do the prisoners have *any* way to know that theirs is not the "true" reality? I.e. could we be prisoners ourselves?

Flatland by **Edwin Abbott Abbott** (English educator; 1838 - 1926) is a Victorian satire of race, class, and gender prejudices.

37. Read about or read the first several sections of Flatland. Describe the limitations faced by the citizens of Flatland.

38. What interesting mathematical issues arise as one considers how Flatlanders perceive their world?

39. Until Flatland is visited by Sphere, Flatlanders have no way to know that their perceptions are badly constrained and do not represent the "real" world. Might our perceptions be similarly constrained? How?

Connections to the higher dimensions are woven beautifully into the science fiction novel Factoring Humanity by **Robert J. Sawyer** (Canadian author; 1960 -). The ***Necker cube***, the two-dimensional representation of the cube that is the first three-dimensional object most people learn to draw, also plays a critical role in this novel.

40. Draw a Necker cube.

41. The square face that appears on the lower left, is it the front face or the back face?

42. In fact, by focusing your sight on different parts of the cube you should be able to see this face as *either* the front or the back. Can you get the Necker cube to form a "shift" like this?

43. Do you think that people that were once blind or people that were not educated in a traditional setting see the Necker cube as an appropriate representation of a cube? Do a little bit of research to see if this is indeed the case.

Of course, biases in our perspectives do not require fiction.

44. Read about Black Like Me by **John Howard Griffin** (American journalist and author; 1920 - 1980). Describe the central feature of this work of nonfiction.

45. Do you know similar books or similar situations where people have purposefully created situations so they could see the world from a very different perspective?

46. Find several other pieces of literature, movies, or theatrical pieces where there are major challenges to our perspective. For each, provide the title/setting and a brief description.

1.4 Challenges of Perceptions from Art

Optical illusions deceive by playing on weaknesses in our senses; our ability to appropriately perceive. There are many examples from the world of art where the artists deceive us in more purposeful, artistic ways. Several examples appear in Figure 1.8 - Figure 1.11.

Figure 1.8: Autumn Cycling by **Rob Gonsalves** (Candaian artist; 1959 -), 1994.

47. Find three additional examples of art in which the artist uses art to make us question our perception. Provide the name of the piece, the name of the artist and a brief explanation of why you find the piece particularly compelling.

1.5 Collective Misperceptions - Myths, Misperceptions and Paradigms

> The great enemy of the truth is very often not the lie, deliberate, contrived and dishonest, but the myth, persistent, persuasive and unrealistic.
>
> **John F. Kennedy** (American President; 1917 - 1963)

A classic example of a myth is the *myth* that you should not swim for 30 minutes after you have eaten to avoid cramps and potential drowning. Evidence suggests otherwise. Despite what is widely believed, Poinsettas are not lethal to ingest. And gum doesn't really stay in your stomach for 7 years when you swallow it, contrary to what your parents may have told you. You do in fact use more than 10% of your brain regularly. And so on.

If there weren't so many myths that needed to be debunked the show Mythbusters would not be such a popular, long-running show.

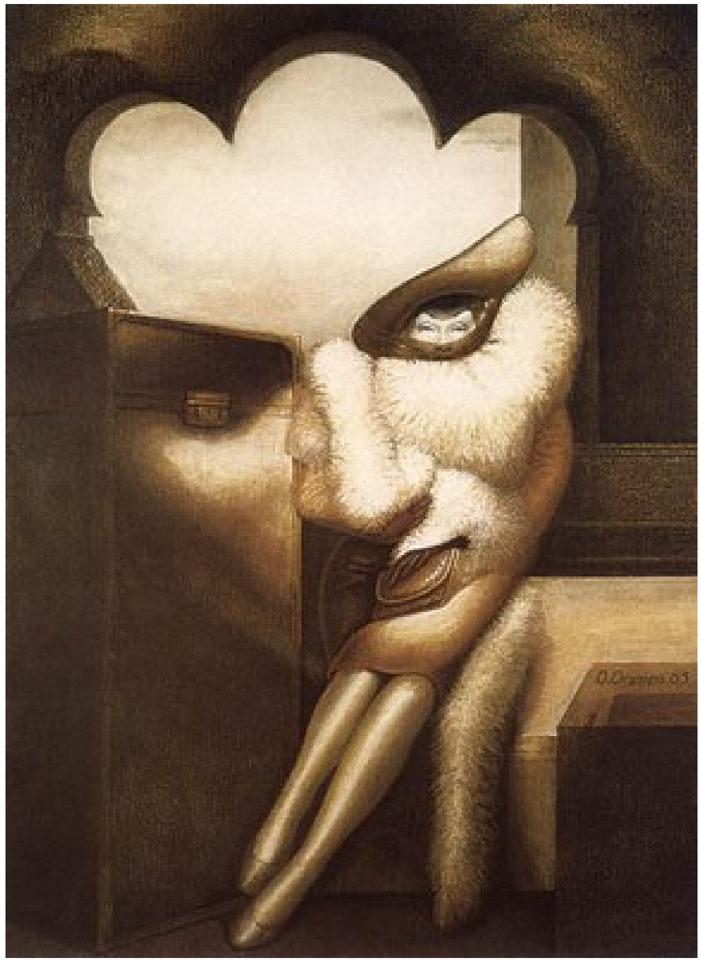


Figure 1.9: Marlene by **Octavio Ocampo** (Mexican artist; 1943 -), 1983.

48. Find several widely-held myths that have been debunked that you think are interesting.

Sometimes the exact genesis of a myth can be determined. Widespread folk-lore attributes the discovery of the spherical shape of the earth to **Christopher Columbus** (Italian Explorer; 1451 - 1506) - he sailed west to prove there was a route to the Far East. However, virtually all evidence suggests that the shape of the earth was known to be spherical to the majority of cultures from at least the time of the ancient Greeks. Indeed, as investigated in Discovering the Art of Mathematics - Geometry, **Eratosthenes of Cyrene** (Greek Mathematician and Geographer; circa 276 B.C. - 195 B.C.) calculated the circumference of the spherical earth with a remarkable degree of accuracy some 2,250 years ago. So how did the urban myth of Columbus that is so widely accepted arise? The myth arises largely from the widely popular work The Life and Voyages of Christopher Columbus which was written by **Washington Irving** (American Author; 1783 - 1859) in 1828. This fictionalized account was taken by many as historically accurate.

Everyone is entitled to his own opinion, but not his own facts.

Daniel Patrick Moynihan (American politician; 1927 - 2003)

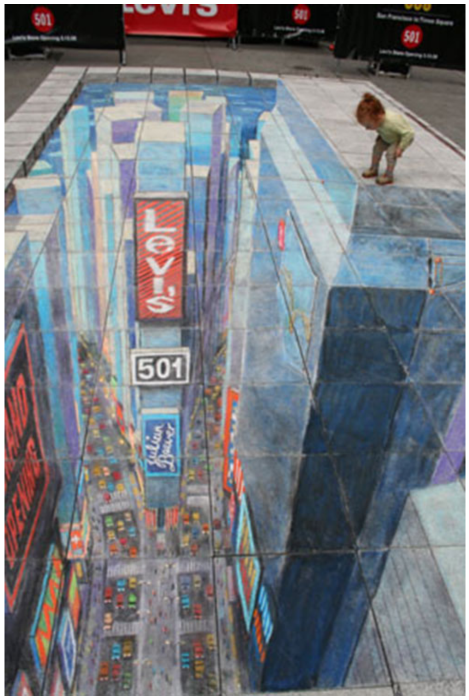


Figure 1.10: “Times Square in Times Square” by **Julian Beever** (English artist; 1959 -).

Generally myths are spread through communities and accepted facts. In contrast, problems with perceptions provide “direct evidence” which appear to support facts that are not well founded. And these *misperceptions* are so common that they effect important, real-world, day-to-day activities. Shopping and advertising, finances and economic decision-making[5], social relations, politics of all sorts, and even legal decision-making. In August, 2011 the state of New Jersey radically changed its rules on eyewitness identification in trials due to a “troubling lack of reliability.” Identifications are often plagued by misperceptions caused by optical illusions, pre-conceived views held by witnesses that impact their perception of the events they witness, stress, and/or coercion.

Challenges like these to perception are studied by people in psychology, marketing, religion, economics, law, and other fields. Misperceptions continue to have important consequences. They often affect large numbers of people.

It is often the case that these misperceptions are not seen as such by a group or community. Rather the results of these misperceptions are believed by the community; they are taken to be true; taken as facts about the world.

> ‘Believe’ is what we think we need to do when our existential understanding [of anything] has to be supplemented to fill in the spaces of our ignorance. As long as we remember that, it’s not so dangerous. Those who think that ‘believing’ IS knowing, or is a substitute for knowing (or learning), or takes priority over knowing, are dangerous.[6]

[5]Consider, for example, the following quote from **Ben Bernanke** (American economist; 1953 -), Chairman of the United States Federal Reserve, on the mortgage/housing collapse of the late 2000’s which precipitated that country’s recession: “Although the high rate of delinquency has a number of causes, it seems clear that unfair or deceptive acts and practices by lenders resulted in the extension of many loans, particularly high-cost loans, that were inappropriate for or misled the borrower.” Quoted in “Fed Sets Rules Meant to Stop Deceptive Lending Practices”, by Steven R. Weisman, *New York Times*, July 15, 2008.

[6]Personal communication, 3/8/2011.

Figure 1.11: "Beneath Every Street" by Julian Beever.

Lou Jean Fleron (American labor educator; 1940 -)

49. Find several examples where the "beliefs" of a community were accepted as facts and had outcomes that were dangerous, damaging or deplorable.

In order to reach the Truth, it is necessary, once in one's life, to put everything in doubt – so far as possible.

Rene Descartes (French mathematician, philosopher and writer; 1596 - 1650)

So, have you begun to put things in doubt a bit?

50. Describe something that is generally considered to be true but that have doubted at some point in your life. Strive to be very precise in your description of the "fact" that is in question. Briefly describe the evidence you generally hear/heard in support of this "fact". Then explain why you question/questioned this "fact".

51. Repeat Investigation **50** for an entirely different "fact". This new example should be quite far away from that in Investigation **50** in many critical respects (subject area, time developed, context, etc.).

For our purposes here, please avoid things that are entirely subjective (e.g. "anchovies are nasty"), things that relative (e.g. "abortion is a sin" which requires a notion of sin that is fundamentally dependent of one's religious views), or things that require extensive elaboration beyond what is included in the statement (e.g. "Obama is a bad President" which is often a value statement and only becomes a statement with some factual truth value if quantified in much greater detail).

52. Repeat Investigation **50** for a third time, striving to find something that is quite different from the examples in Investigation **50** and Investigation **51**.

53. Describe, precisely, something in mathematics that is generally considered to be true but that you no not necessarily believe. Explain why you question this "fact".

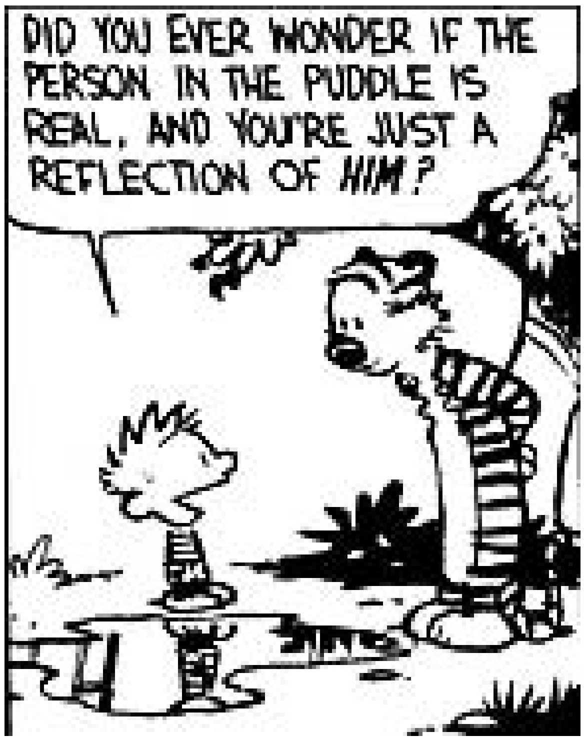


Figure 1.12: Calvin and Hobbes by **Bill Watterson** (American cartoonist; 1958 -).

You create your own universe as you go along.

Winston S. Churchill (British politician; 1874 - 1965)

On 30 October, 1938 CBS radio network broadcast a theatrical version of the Martian invasion science fiction novel War of the Worlds by **H.G. Wells** (; -). Presented as news bulletins coming out of West Windsor Township, New Jersey, many people believed that the broadcast was that of an actual invasion by Martians. There was widespread panic - the story of the panic was a front-page article in the next day's *New York Times*.

> Henry Ward Beecher used to tell of the Free Will Methodist preacher and the Predestinarian Presbyterian preacher who agreed to exchange pulpits. As they met on the road, each on the way to the other's pulpit, the Presbyterian said, "Brother, does it not give you pleasure and glory to God that before the earth itself was formed, we were destined in the mind of God to have this exchange this morning?" To which the Methodist replied, "Well, if that is so, then I ain't going," and he turned his horse around.[7]
>
> **Peter J. Gomes** (American Theologian and Preacher; - 2011)

[7]From The Good Book: Reading the bible with Mind and Heart, p. xi.

54. Explain the moral and implications of the story Gomes attributes to Beecher.

Knowledge is knowing that we cannot know.

Ralph Waldo Emerson (American poet, writer and philosopher; 1803 - 1882)

As a graduate student in theoretical physics **Thomas Kuhn** (American physicist and philosopher of science; 1922 - 1996) began a project which he later published a "full report" on. The report is The Structure of Scientific Revolutions, a critical contribution to the philosophy of science.

This book begins by "appropriating" the word *paradigm* from its Latin and extending a new meaning to it in the context of the philosophy of science, one approximately captured by a typical dictionary definition:

Paradigm - A set of assumptions, concepts, values, and practices that constitutes a way of viewing reality for the community that shares them, especially in an intellectual discipline.

A ***paradigm shift*** is a when a dominant paradigm is overthrown; rejected and replaced by a new paradigm. Paradigm shifts generally take place over very short periods of time and are precipitated by important events that shake our beliefs in the dominant paradigm and force us to assimilate a new perception of reality into our world view.

Paradigm shifts are generally revolutionary, but few revolutions are paradigm shifts. The Internet is a revolution in how humans communicate. But humans have been communicating with different means throughout their history; the Internet is a radical breakthrough in communication. But it is not a paradigm shift.

Figure 1.13: Frank and Ernest by **Bob Thaves** (American cartoonist; 1924 - 2006).

We don't see things the way they are, we see things the way we are.

Anais Nin (American author; 1903 - 1977)

55. Find and describe a paradigm shift in the physical sciences.

If the Columbus myth were true, that the earth was considered by most to be flat and that Columbus reached the East by sailing west, then this would have been a classic paradigm shift. Perceptions of the true nature of the earth would have been fundamentally changed by his voyages.

The word paradigm shift has made it into the popular lexicon. On Monday, 26 September, 2011 the lead story on SI.com was entitled "Paradigm Shift" and described the huge surprise among NFL fans that the Buffalo Bills and Detroit Lions, two of the most losing teams in professional football in the past decade, had begun the season undefeated at 3-0. As a tongue-in-cheek reference this is fine. But this is not a particularly meaningful use of the term. Kuhn helped us understand how the dominant paradigm can affect our world view. The use of the idea of paradigms in many social sciences and humanities similarly helps us understand how our beliefs can bias our perceptions of reality.

56. Find and describe a paradigm shift in the social sciences or the arts.

Einstein's theory of relativity was a historic paradigm shift. In 2011 the principal investigators of the OPERA experiment at the CERN - the world's largest particle physics laboratory - announced they had repeated generated beams of neutrinos which traveled at greater the speed of light, violating Einstein's theory of relativity as we know it. This would have precipitated a paradigm shift. It was not until a year later that detection errors causing these readings were discovered.

> Every man's world picture is and always remains a construct of his mind and cannot be proved to have any other existence.
>
> **Erwin Schrödinger** (Austrian physicist; 1887 - 1961)

57. Find and describe a paradigm shift in the biological sciences.

Examples of paradigms also arise as parts of fictional literature, providing morals about the impact of paradigms. In The Wizard of Oz our perceptions are constantly manipulated. In the movie version Oz seems more real than Kansas as it is in color when Kansas is black and white. But there are Munchkins, witches and wizards? It certainly was a paradigm shift for the Munchkins when they discovered the Wizard was a fraud. It was a paradigm shift for Dorothy when she realized that she could not discern the difference between dreams and reality - perhaps a tip of the hat to Descartes' famous cogito below? Wicked by **Gregory Maguire** (American author; 1954 -) takes things a step further. In his alternative novel (later made into a Broadway musical which won three Tony awards and is one of the longest-running Broadway shows in history) we learn the "Wicked Witch", whose name is Elphaba, is in fact a victim of racism, bullying, and false accusation at the hands of Glenda the "Good Witch". Instead of a menace, the green-skinned Elphaba is a crusader seeing to unveil both the Wizard and Glenda as the frauds and manipulators they are. While fictional, both speak to the importance of paradigm shifts.

58. Find and describe a paradigm shift that plays a fundamental role in a work of literature or a movie.

History has shown, repeatedly, that some of our most closely held "truths" are really prejudices that have biased much of what we "know". One of the signs of the ability to truly learn - what many would call "to be educated" - is to be able to look beyond your biases. We study paradigms to illustrate how important this is.

One of the great sources of doubt, and one of the great paradigm shifts, was from **Rene Descartes** (French mathematician, philosopher and writer; 1596 - 1650), in Latin:

> Cogito, ergo sum,

or in English:

> I think, therefore I am.

Certainly because you are thinking begin you must have some existence. This is not so revolutionary. However, Descartes implication was much deeper - namely that *all* else was in question. Your body, your surroundings, and all that you perceive as reality my not be as your senses perceive. Your senses may be misperceiving, the entire experience could be a dream/hallucination, or you may be being systematically deceived. With this level of doubt, how does one actually establish the existence of something as simple as a chair? Can you really prove that it is there? How?

59. Go back to the list of five things you were certain of that you made at the beginning of this chapter. Are you just as certain of these things, or are is there room for some doubt? Explain.

60. Are you more of less convinced of the "correctness" of your perception of reality than at the outset of this chapter? Explain.

61. Moving forward, will you be more inclined to question conventional wisdom, what you are told, and what you taught? Explain.

62. In the future, will you ask "Why?" more often, seek to understand more deeply, and seek greater justification for things that you choose to accept as certain? Explain.

1.6 Connections

> Your vision will become clear only when you look into your heart ... Who looks outside, dreams. Who looks inside, awakens.
>
> **Carl Jung** (Swiss psychologist; 1875 - 1961)

> How dreadful knowledge of the truth can be when there's no help in truth.[8]
>
> **Sophocles** (Greek author; circa 497 BC - circa 406 BC)

- Powers of Ten - This is described elsewhere. How to cross reference things like this? These are ALL pictures of the SAME thing! Look how different they all look!!
- The Necker cubes from different perspective points. How to record this somehow?
- Bodyworlds? What is the exact connection here? Just that we don't know all of these things so close to us really look like? How they operate? A mystery inside of us?
- Seeing Through Maps: The Power of Images to Shape Our World View by Ward L. Kaiser and Denis Wood
- Masters of Deception: Escher, Dali & the Artists of Optical Illusion, edited by Al Seckel.
- Pavement Chalk Artist: The Three-Dimensional Drawings of Julian Beever by Julian Beever.

[8] From Oedipus Rex.

Chapter 2

Existence of $\sqrt{-1}$: A Case Study

> [Complex numbers are] a fine and wonderful refuge of the divine spirit - almost an amphibian between being and non-being.[1]
>
> **G. W. Leibniz** (German mathematician and philosopher; 1646 - 1716)

> I have often been surprized, that Mathematics, the quintessence of Truth, should have found admirers so few and so languid. Frequent consideration and minute scrutiny have at length unraveled the cause, viz. that though Reason is feasted, Imagination is starved; whilst Reason is luxuriating in it's proper Paradise, Imagination is wearily traveling on a dreary desert.
>
> **Samuel Taylor Coleridge** (English poet and philosopher; 1772 - 1834)

With all of the talk of doubt, you may be getting the impression that mathematicians are a pessimistic, cynical, and closed-minded group. In fact, mathematicians are generally quite the opposite - optimistic, creative, and imaginative dreamers who invent all sorts of spectacular worlds. Before we undertake the larger investigation of types of reasoning and the ways in which truth is established, it seems appropriate to get a sense how this creative, imaginative dreaming can coexist with high standards of certainty.

So we shall build something. Something that is surprising, beautiful, and certain. We shall construct the *field of complex numbers.*

2.0.1 Introducing a New (Mathematical) Character

Most people are not worried about the certainty of standard arithmetic. They can combine numbers through addition, subtraction, multiplication and division. There are algorithms that can be followed to compute. There are rules and laws upon which this systems has been built (topics that we will consider shortly, calling them *axioms* or *postulates* when we do), such as the **commutative property of multiplication** which says that $a \times b = b \times a$ for any numbers a, b.

Into this setting we are simply going to add one mathematical object, one more mathematical character. We will require it to satisfy the defining rules of arithmetic upon which the system we are used to has been built. It is simply a new character which we have brought forward through the power of our imagination.

We have added this character to the mix of our everyday arithmetic simply to explore what might happen.

This is a work of imagination just as much as the creation of a character in literature, a compositional theme in a painting, etc. Samuel Taylor Coleridge coined the term "suspension of disbelief" which he thought was central to an audience's ability to imagine the often illusory settings of poetry,

[1] Quoted in "Thinking the Unthinkable: The Story of Complex Numbers (with a Moral)," by Israel Kleiner, *Mathematics Teacher*, Oct. 1988.

theater and literature. If you are troubled by our new character, try to suspend disbelief and give this new character a chance.

INTRODUCING the **imaginary unit**

$$i = \sqrt{-1}.$$

2.1 Basic Questions from Arithmetic

1. Can you explain why the product of a positive number and a negative number is a negative number? Are you confident in your explanation? Does this result make intuitive sense to you? Explain.

2. Can you explain why the product of two negative numbers is a positive number? Are you confident in your explanation? Does this result make intuitive sense to you? Explain.

3. Why are square roots of negative numbers supposedly nonexistent?

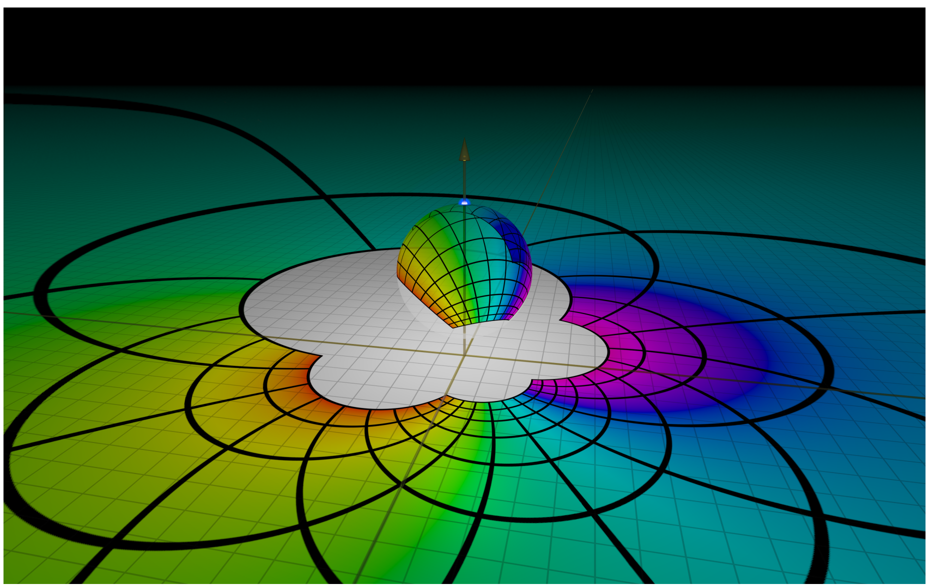


Figure 2.1: Möbius transformation from the beautiful video "Möbius Transformations Revealed" available at `https://www.youtube.com/watch?v=JX3VmDgiFnY`.

2.2 The Complex Number Field

> This is a major theme in mathematics: things are what you want them to be. You have endless choices; there is no reality to get in your way. On the other hand, once you have made your choices... then your new creations do what they do, whether you like it or not. This is the amazing thing about making imaginary patterns: they talk back![2]
>
> **Paul Lockhart** (American mathematician and teacher; -)

[2]From A Mathematician's Lament, p. 25.

We have made our choice, to denote our new character $\sqrt{-1}$ by the symbol i. We said we expected it to behave, by following the normal rules of arithmetic. We must be able to add it to itself, $i + i$. Shouldn't we just call this $2i$? Similarly, we can get $3i, 4i, \ldots$. We can also multiply any number by i. It is natural to write $2 \times i = 2i$ and $\pi \times i$ as πi. Having created numbers like bi for any real number b we can then add real numbers. There is no obvious way to combine the real number with our new characters, so the result of adding a to bi will simply be the number $a + bi$. Simply from the addition of one character we seem to have gotten an entire new cast:

Numbers of the form $a+ib = a+\sqrt{-1}b$, with a and b real numbers, are called **complex numbers**. In $a + ib$ we call a the **real part** of $a + ib$ and we call b the **imaginary part** of $a + ib$.

4. Explain why $i^2 = -1$.

5. Simplify $i^3, i^4, i^5, \ldots$ so each is written in the standard form of a complex number. (I.e. no power of i other than i^1.)

6. Use Investigation **5** to determine a simple rule that which can be used to simplify i^n.

In the Flatland the Movie, the main character Arthur Square asks:

> Does your sense of wonder go only as far as your eyes can see? ...Friends, do not fear what you cannot see. Mathematics, reason and imagination will reveal the truth.

2.2.1 Complex Arithmetic

In each of the problems below, use the standard rules of arithmetic together with your rules for powers of i to compute the indicated product, simplifying so each product is expressed in the standard form of a complex number:

7. $(2 + i) \cdot (1 + 3i)$.

8. $(3 + 4i) \cdot (1 + i)$.

9. $(2 + 3i) \cdot (-2 + 2i)$.

10. $(-2 - 2i) \cdot (-1 + 3i)$.

11. $(-1 - 3i) \cdot (2 - 2i)$.

12. $(-3 + i) \cdot (-2 - 2i)$.

13. $(-2 - 2i) \cdot (-4 - i)$.

14. $(4 - i) \cdot (-2 + 2i)$.

Complex numbers can be plotted on an **Argand plane** where the horizontal axis is the real coordinate and the vertical axis is the imaginary coordinate. (Note: Here and below it is important to pick a single scale and stay with that scale. We suggest using centimeter graph paper where all scales have axes that are the same scales and centimeter rulers.)

15. Plot each of the factors as well as the product in Investigation **7** on an Argand plane constructed on graph paper.

16. Repeat Investigation **15** for the data in Investigation **8** on a separate Argand plane.

17. Repeat Investigation **15** for the data in Investigation **9** on a separate Argand plane.

18. Repeat Investigation **15** for the data in Investigation **10** on a separate Argand plane.

19. Repeat Investigation **15** for the data in Investigation **11** on a separate Argand plane.

20. Repeat Investigation **15** for the data in Investigation **12** on a separate Argand plane.

21. Repeat Investigation **15** for the data in Investigation **13** on a separate Argand plane.

22. Repeat Investigation **15** for the data in Investigation **14** on a separate Argand plane.

There is nothing mysterious about points in the plane, is there? Does the existence of $(2, 1)$ concern you?

The number $2 + i$ is really no more mysterious than the point $(2, 1)$ in the *Cartesian plane*, only we have equipped the complex numbers with a multiplication for which the points in the plane generally lack. Then why does $2 + i$ concern you?

In fact, **John Stillwell** (Australian mathematician; 1942 -) describes algebraic results of **Diophantus** (Greek mathematician; circa 205 - circa 289) which give a picture that is "extraordinarily close to what we now regard as the 'right' way to interpret complex numbers.[3]" This was nearly 2,000 years ago!

Ironically, Rennaisance mathematicians did not rediscover this natural way to model the complex numbers. The first known to have suggested this in its full generality is **Caspar Wessel** (Norwegian surveyor; 1745 - 1818). Nahin[4] tells us:

> The problem [of taming the complex numbers geometrically] was suddenly and quite dramatically solved by... Wessel. This is both remarkable and, ironically, understandable, when you consider that Wessel was not a professional mathematicians but a surveyor. Wessel's breakthrough on a problem that had stumped a lot of brilliant minds was, in fact, motivated by the practical problems he faced every day in making maps, i.e. by the survey data he regularly encountered in the form of plane and spherical polygons. There was no family tradition in mathematics to guide him, as both his father and hist father's father were men of the cloth. It was Wessel's *work* that inspired him to success where all others had failed before.

So why don't we call it the Wessel plane? Because the presentation of his paper in 1797 and its publication in 1799 happened in relative obscure forms and the small audience did not carry the ideas far enough for them to germinate largely.

There are many cases in mathematics where the original discoverer/inventor of important ideas is not given the appropriate credit. In this case the credit went to **Jean-Robert Argand** (Swiss bookkeeper; 1768 - 1822) of whom little is known. The story of his (re-)discovery of the natural way to represent complex numbers geometrically is also somewhat miraculous. Not himself an academic, he thought highly enough of his work to have a pamphlet containing his work self-published. His name did not even appear on the title page. The prominent **Adrien-Marie Legendre** (French mathematician; 1752 - 1833) received a copy of this pamphlet began to share it with other mathematicians. The ideas were subsequently published in 1813 with a plea in the article for the discoverer/inventor to come forward. Argand saw this plea, came forward and was declared the namesake for this representation of the complex numbers in the very next issue of the journal.[5]

2.2.2 Polar Form

Translating the complex numbers into a different representation will provide a first sense of their utility. Just as the number $1\frac{1}{2}$ can be equally well expressed as a single fraction $\frac{3}{2}$ or a decimal

[3] Yearning for the Impossible: The Surprising Truths of Mathematics, pp. 34-6. Paul Nahim has a slightly different view in the Introduction to An Imaginary Tale: The Story of $\sqrt{-1}$.

[4] p. 48 of An Imaginary Tale: The Story of $\sqrt{-1}$

[5] Ibid, pp. 73-4.

1.5, complex numbers can be written in different forms. One useful form is the **polar form** $r \cdot e^{i\theta}$ where $o \leq r$, $0 \leq \theta < 360$ and $e = 2.718\ldots$ is the *base of the natural logarithm* named in honor of **Leonhard Euler** (Swiss mathematician; 1707 - 1783). r is called the **magnitude** or **modulus** of the complex number and is simply its distance from the origin. θ is called the **argument** of the complex number and is simply the angle between the positive real axis and the line from the origin to the complex number in question, measured in the counter-clockwise sense. The argument is generally measured in *radians*, but for our purposes here the translation will be successful if we use angle measures in degrees.

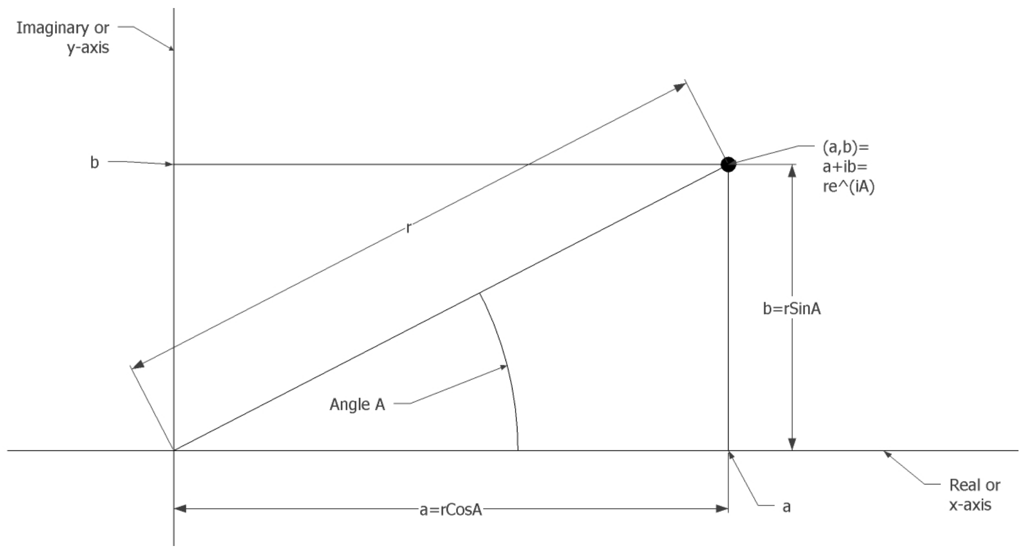


Figure 2.2: A point on the (Argand) plane.

Example 1 i is one cm. from the origin and it's argument is 90°. So we write $i = 1 \cdot e^{i \cdot 90^\circ}$.
Example 2 $3 - 4i$ is 5 cm. from the origin and its argument is about 307°. So we write $3 - 4i \approx 5 \cdot e^{i \cdot 307^\circ}$.

Please notice we have *not* said what the precise role of the symbols e^i, for now they are simply part of the notational format of the way we express the polar form of the complex numbers.

23. Using a protractor and cm. ruler, approximate the polar form of each of the complex numbers in Investigation **7** - Investigation **14** and record the results in a table like the one below:

$a+ib$	Mag	Arg	Polar Form	$c+id$	Mag	Arg	Polar Form	$(a+ib)\cdot(c+id)$	Mag	Arg	Polar Form
$2+i$				$1+3i$				$-1+7i$			

24. How is the argument of the product related to the arguments of the factors?

25. How is the magnitude of the product related to the magnitudes of the factors?

26. Investigations Investigation **24** and Investigation **25** should give you a very geometric characterization of the multiplication of two complex numbers. Describe it completely.

27. Using this geometric conceptualization of multiplication, explain why the product of two positive real numbers is positive.

28. Now use this geometric view to explain why the product of a negative real number and a positive real number is negative.

29. Now use this geometric view to explain why the product of two negative real numbers is positive.

30. In the novella (resp. movie) Flatland by Edwin Abbott Abbott (resp. Flatland the Movie) the characters are flat geometric figures confined to a flat world, entirely unaware of a third dimension. In a dream sequence A Square (resp. Arthur Square) visits Lineland where the inhabitants are short line segments whose movement is restricted to an infinite line. The King of Lineland is only aware of his two neighbors (resp. Queens) which adjacent to him on his number-line world. This is all he knows. So he is badly startled when, from out of nowhere, his visitor appears in his world. Does this storyline resonate with your previous view of real multiplication, considered at the outset of this chapter, and the view that you now have? Explain.

> It was a wild thought, in the judgment of many; and I too was for a long time of the same opinion. The whole matter seemed to rest on sophistry rather than on truth. Yet I sought so long, until I actually proved this to be the case.[6]
>
> **Rafael Bombelli** (Italian mathematician; 1526 - 1572)

> Because all conceivable numbers are either greater than zero, less than zero or equal to zero, then it is clear that the square root of negative numbers cannot be included among the possible numbers And this circumstance leads us to the concept of such numbers, which by their nature are impossible and ordinarily are called imaginary or fancied numbers, because they exist only in the imagination.[7]
>
> **Leonhard Euler** (Swiss mathematician; 1707 - 1783)

31. If $\sqrt{-1}$ "exists in our imaginations" as Euler claims, then does it really exist? Is this existence the same or different that numbers like $0, -1, \sqrt{2}$ or π? Explain.

32. Return to the quote by Cooleridge that opens this chapter. Why does it appear in this chapter? Do you think that the mathematics you have investigated here may refute Cooleridge's claim?

2.3 Expanding Our Notion of Numbers via Solving Equations

No justification was given for the introduction of imaginary and complex numbers above. Here we trace a bit of the evolution of number through the solution of equations - a historically important time line.

In the early sixteenth century, mathematical problem solving competitions were common. Scholarly reputations were largely based on these contests because "not only could an immediate monetary prize be gained by proposing problems beyond the reach of one's rival, but the outcomes of these challenges strongly influenced academic appointments."[8] One of mathematics' great disputes arose out of these competitions.

The players in the dispute were three. **Antonio Maria Del Fiore** (Italian mathematician; -) was the pupil of **Scipione del Ferro** (Italian mathematician; 1465 - 1526) who was one of the greatest of the early competitors. **Nicolo Tartaglia** (Italian mathematician; 1500 - 1557) saw his

[6] Quoted in "Thinking the Unthinkable: The Story of Complex Numbers (with a Moral)" by Israel Kleiner, Mathematics Teacher, October 1988, pp. 583-92.

[7] Quoted in "Thinking the Unthinkable: The Story of Complex Numbers (with a Moral)" by Israel Kleiner, Mathematics Teacher, October 1988, pp. 583-92.

[8] From The History of Mathematics: An Introduction by David M. Burton, p. 305.

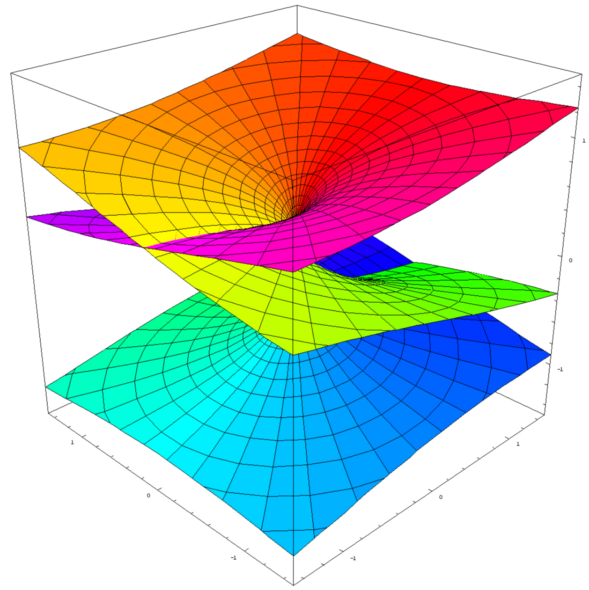


Figure 2.3: Riemann surface for the complex valued cube root function.

father killed and had his face nearly destroyed in the sack of Brescia as a 12 year old. Despite living in poverty, Tartaglia "was determined to educate himself.[9] Throughout his life he was known as the "stammerer" because of the difficulties to his speaking that his childhood injuries caused. **Girolamo Cardano** (Italian mathematician; 1501 - 1576), aka Cardan, had a life that was "deplorable." He "divided his time between intensive study and extensive debauchery.[10]"

Cardano horned in on the ongoing competitions between Fiore and Tartaglia, pretending to befriend Tartaglia. Once he gained his confidence Cardano managed to have Tartaglia share his secret to solving an important class of cubic equations. Cardano then published the results in his famous book Ars Magna in 1545, beginning "one of the bitterest feuds in the history of science, carried on with name-calling and mudslinging of the lowest order.[11]"

In 1572 Bombelli used a leap of imagination to compute with $\sqrt{-1}$. He did this when considering the equation $x^3 = 15x + 4$ for which Cardano's cubic equation predicts a root of $\sqrt[3]{2 + 11 \cdot \sqrt{-1}} + \sqrt[3]{2 - 11 \cdot \sqrt{-1}}$.

33. Show that $\sqrt[3]{2 + 11 \cdot \sqrt{-1}} = 2 + i$.

34. Show that $\sqrt[3]{2 - 11 \cdot \sqrt{-1}} = 2 - i$.

35. Use these results to find the root predicted by Cardano's formula.

36. Show that this (*real*) number is a root of the cubic.

While this root could have been found in other ways, we see Cardano's formula is that much more powerful with the use of complex numbers as its use is now not limited. The same can be said for the solution of quadratic equations. You need not worry about the dreaded ***discriminant*** $b^2 - 4ac$ being nonnegative. So we have the first known illustration of a famous maxim of **Jacques Hadamard** (French mathematician; 1865 - 1963):

[9]Ibid, p. 306.
[10]Ibid, p. 307.
[11]Ibid, p. 308.

> The shortest and best way between two truths of the real domain often passes through the imaginary one.[12]

37. Solve the equation $2x - 6 = 0$.

38. Solve the equation $6x - 2 = 0$.

39. If you did not know anything about fractions, what would you say about the solubility of the equation in Investigation **38**?

40. There is nothing particularly problematic about the solution to the equation in Investigation **38**, unless... Express the solution as a decimal. Is there anything troubling or potentially problematic about this number?

41. Solve the equation $2x + 6 = 0$.

42. Find out when negative numbers first came into regular use. Without negative numbers, what would you say about the solubility of the equation in Investigation **41**?

43. Your solution in Investigation **41**, can you show me this quantity concretely? I.e. does it exist in our physical reality? Where?

44. Solve the equation $x^2 - 2 = 0$.

45. Can you describe your solutions to the equation in Investigation **44** exactly? I.e. what is their exact numerical value?

46. Like $\sqrt{9} = 3$, some square roots are simple. However, whenever n is a positive integer and $\sqrt{n}$ is not a whole number then the decimal expansion of $\sqrt{n}$ is an infinite decimal that *never* repeats. So, would you say that your solution to Investigation **44** is an exact, concrete entity which exists in our physical world?

> The idea of the continuum seems simple to us. We have somehow lost sight of the difficulties it implies...We are told such a number as square root of 2 worried Pythagoras and his school almost to exhaustion. Being used to such queer numbers from early childhood, we must be careful not to form a low idea of the mathematical intuition of these ancient sages, their worry was highly credible.
>
> **Erwin Schrödinger** (Austrian physicist; 1887 - 1961)

Let us consider the equation $y = x^2 + 1$ where x and y are real numbers.

47. Graph this equation in the ***Cartesian plane***, the standard $x - y$ plane that you were likely introduced to in middle or high school.

48. As we have used them, how does the Argand plane differ from the Cartesian plane?

49. What does this graph suggest about real solutions to the equation $x^2 + 1 = 0$?

50. Suppose now that we considered the equation $w = z^2 + 1$ where z and w are complex numbers. In how many dimension would the graph of this "simple" quadratic function live? Explain.

51. Find the two distinct complex solutions to the equation $z^2 + 1 = 0$.

52. Based on what you have seen above, do you think that these numbers are any less legitimate than the solutions to the other equations considered above? Explain.

[12] From "An Essay on the Psychology of Invention in the Mathematical Field."

What about higher order equations, can we find new solutions there using complex numbers? Let's begin by investigating higher powers of complex numbers.

53. Using graph paper, draw an Argand plane. Choose a point z with magnitude greater than one and relatively small argument. Graph z on your plane.

54. Graph the points $z^2, z^3, z^4, ..., z^{10}$ on your plane as well.

55. With line segments or a curve of your choice, connect z to z^2, z^2 to z^3, z^3 to z^4, etc. What shape do you see?

56. Will something similar happen no matter what point z you start with? (Hint: Think about different categories of magnitudes and arguments you might choose.)

57. Find a dozen spirals in nature.

58. The world "natural" has come up many times in our discussion of the complex numbers. Do you find them more natural now?

59. Using your experience with spirals above, find the three distinct complex solutions to the equation $z^3 - 1 = 0$.

60. Find the three distinct complex solutions to the equation $z^3 + 8 = 0$.

61. For each of the equations whose solutions are considered in this chapter, record the ***degree*** (the highest exponent of the variable x), the number of real solutions and the number of complex solutions (which includes the real solutions as well). Notice something?

62. The **Fundamental Theorem of Arithmetic** says that every polynomial of degree n (i.e. which has the form $a_n x^n + a_{n-1}x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0$ has exactly n complex roots regardless of whether the coefficients a_j are complex are real. What does this suggest about the most natural arena to solve polynomial equations?

Here's one last example if you still question whether the complex numbers are very natural.

Above we measured the arguments of complex numbers in degrees. Typically ***radian*** measure for angles should be used. In radians $2\pi = 360°$, so $\frac{\pi}{2} = 90°$, etc.

63. What is the degree equivalent of the radian measure π?

64. Determine what complex number $e^{i\cdot\pi}$ represents.

65. Rewrite the equation above so all nonzero terms are on the left.

66. What important numbers does your equation include? What important operations? Is it remarkable that one valid equation contains so many fundamental mathematical objects and operations?

2.4 Conclusion - The Values of $\sqrt{-1}$

This chapter has been included as a case study to provide a concrete example of how mathematicians build new worlds - something that will provide valuable context for the upcoming consideration of *deductive reasoning* which sets mathematics apart from other areas of human knowledge. Built on the foundation provided by the real number system, an entirely new world has been created.

At the outset of the chapter the motivation was light as the desire involve you in a somewhat playful exploration of this new world. There is nothing light about the importance of the invention/discovery of the complex number field. As you have seen, they play a fundamental role in our understanding of certain properties of arithmetic, of the solution of algebraic equations, and the underlying structure of algebraic equations. In the Further Investigations section you also see critical perspectives that the complex numbers bring to trigonometry.

Based on an imaginary unit, the importance of the complex numbers extends into many real world applications. The maxim[13] of **Keith Devlin** (English mathematician and science writer; 1947 -)

> There can be very little of present-day science and technology that is not dependent on complex numbers in one way or another

is a powerful reminder that we should not be too quick to judge the power of imagination. Imagine the mockery that a financial request to study imaginary numbers would have had to endure at the hands of the media and politicians if it was a new field of inquiry. And yet, it is one of the most important chapters in the history of mathematics - one that continues to pay enormous real-world benefits to this day.

2.5 Further Investigations

2.5.1 Motivation for the Polar Form

Above the polar representation was introduced simply as a notational aid. Who would pick such awkward notation, especially when it includes already used mathematical objects and operations, the base e of the natural logarithm and the process of exponentiation, that may not even make sense here? In fact, some mathematician made the remarkable connection between complex numbers, the exponential function and the trigonometric functions that makes this a powerful tool.

This mathematician was **Leonard Euler** (Swiss mathematician; 1707 - 1783) - one of history's greatest mathematicians - who wrote to share the usefulness of his newly defined mathematical object with many different mathematicians through the 1740's.[14] This object played an important role in seminal work Introductio in Analysis Infinitorum published in 1748.

Define the **complex exponential** by $e^{i \cdot y} = cosy + i \cdot siny$ for all real numbers y. While this is simply a definition, it is of such great importance, and to mathematicians revealing such a fundamental truth, that it is referred to as **Euler's formula**.[15]

67. For a dozen of the complex numbers in Investigation **23** use the definition of $e^{i \cdot y}$ to convert the number from polar form back into its standard complex form. Show that within measurement error, the polar form agrees with the standard form.

68. Prove that for any complex number $x + iy$ the polar form agrees with the standard complex form.

69. Suppose you have two complex numbers in polar form, $re^{i \cdot y}$ and $se^{i \cdot t}$. Simplify the product $\left(re^{i \cdot y}\right)\left(se^{i \cdot t}\right)$ until it clearly illustrates the geometric characterization of complex multiplication in Investigation **26**.

70. Does the definition of polar form now seem more reasonable?

[13] From Mathematics: The New Golden Age

[14] See An Imaginary Tale: The Story of $\sqrt{-1}$ by Paul J. Nahin, pp. 143-4.

[15] While Euler's work on the complex exponential was the definitive work that gave it it's central place in mathematics, other mathematicians had discovered it without recognizing its importance earlier, including **Roger Cotes** (English mathematician; 1682 - 1716) and **Abraham de Moivre** (French mathematician; 1667 - 1754). (See An Imaginary Tale: The Story of $\sqrt{-1}$ by Paul J. Nahin, pp. 162 - 166 and e: The Story of a Number by Eli Maor, p. 160.)

71. Using the rules for exponents, provide a definition for e^{x+iy} where $x+iy$ is any complex number. This provides the complete definition of the complex exponential.

72. What is the magnitude of a complex number of the form e^{x+iy}? Prove your result.

73. You should now be able to translate freely among the three different representations for complex numbers: $a+ib$, $re^{i\cdot y}$ and $e^{x+i\cdot y}$. Explain how.

2.5.2 Trigonometric Identities

In his wonderful book Visual Complex Analysis, **Tristan Needham** (English mathematician; -) tells us:

> All trigonometric identities may be viewed as arising from the rule for complex multiplication... Every complex equation says two things at once.

Is this really the case that the entire zoo of trigonometric identities are encoded in complex multiplication? Let's give the following trigonometric identity a try:

$$cos(\theta+\phi) = cos\theta cos\phi - sin\theta sin\phi.$$

74. Write $e^{i\cdot(\theta+\phi)}$ in the standard form of a complex number.

75. Explain why $e^{i\cdot(\theta+\phi)} = e^{i\theta}\cdot e^{i\phi}$.

76. Write both of the polar forms in Investigation **75** as complex numbers in the standard form.

77. Compute the product to express the right hand side of the equation in Investigation **75** in standard form.

78. Explain why the real parts of the expressions in Investigation **75** and Investigation **74** must be equal. Equate them to derive the desired formula.

79. What result do you obtain when you equate the imaginary parts?

80. How hard was it to derive these formulas this way?

2.5.3 Legitimizing the Formula $e^{x+iy} = e^x(cosy + isiny)$

The power series approach below can be used to fully justify the definition of the complex exponential e^{x+iy}. As the rules for exponents have already enabled you to break this exponential into a real part and a complex part, we will just concentrate on the complex part e^{iy} here.

81. Use a graphing calculator or online grapher to graph the functions $siny$ and $y-\frac{y^3}{3\cdot 2}$ on the same graph.

82. Repeat Investigation **81** for the functions $siny$ and $y-\frac{y^3}{3\cdot 2}+\frac{y^5}{5\cdot 4\cdot 3\cdot 2}$.

83. Repeat Investigation **81** for the functions $siny$ and $y-\frac{y^3}{3\cdot 2}+\frac{y^5}{5\cdot 4\cdot 3\cdot 2}-\frac{y^7}{7!}$.

84. You should see a pattern in the polynomial functions you are being asked to graph together with the sin function. Find the next three such polynomials and graph them together with the sin function.

85. How do the graphs of the polynomials compare to the graph of the sin function?

86. As you continue to use the higher degree polynomials that are generated by this pattern (these are called the ***Taylor polynomials*** for sin) what do you think will happen?

In fact, the Taylor polynomials *converge* to the *sin* function for any value of y. I.e. $\sin y = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}$.

87. The ***Taylor series*** for *cosy* is $\cos y = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!}$. Graph several of the Taylor polynomials together with the *cos* function to convince yourself that they too look to converge.

88. The Taylor series for e^t is $\sum_{n=0}^{\infty} \frac{t^n}{n!}$ Graph several of the Taylor polynomials together with the exponential function to convince yourself that they too look to converge.

89. Substitute iy in as the variable to find a Taylor series representation for $e^{i \cdot y}$.

90. Reduce all of the powers of i in the Taylor series so no higher power than the first occurs.

91. Group the real terms together. What do you notice?

92. Group the imaginary terms together. What do you notice?

93. So what does this all you to conclude about e^{iy}?

There are a few technical details that have to be cleaned up to have a rigorous proof.[16] But you have what Euler and the mathematicians prior to the late nineteenth century would certainly consider a proof of the legitimacy of this important equation.

2.6 Connections

> There can be very little of present-day science and technology that is not dependent on complex numbers in one way or another.[17]
>
> **Keith Devlin** (English mathematician and science writer; 1947 -)

Here we provide references where interested readers can find out about some of the important, "real-world" applications of complex numbers:

- Fluid Flow (hence much of aeronautical engineering) - Chapter 2, Section 6 of Complex Variables by Norman Levinson and Raymond M. Redheffer.
- Electrical Engineering - Chapter 5, Section 3 of An Imaginary Tale: The Story of $\sqrt{-1}$ by Paul J. Nahin.
- Kepler's Laws and Satelite Orbits - Chapter 5, Section 3 of An Imaginary Tale: The Story of $\sqrt{-1}$ by Paul J. Nahin.

To get a sense of the central importance of complex numbers to mathematics it is interesting to view the history of the prime number theorem. This theorem describes the distribution of the prime numbers, the building blocks for all the whole numbers under multiplication, by approximating the density of primes. Specifically, the **prime number theorem** says that as N grows without bound the density of primes in the range $1, 2, 3, \ldots N$ is approximately $\frac{1}{ln(N)}$. Symbolically, $\frac{\pi(N)}{N} \approx \frac{1}{ln(N)}$ where the function π counts the number of primes. The great **Carl Friedrich Gauss** (German mathematician and scientist; 1777 - 1855) experimented empirically and claims to have known this

[16] For example, can you believe that it is not always ok to rearrange the order of terms in an infinite series? See Discovering the Art of Mathematics - The Infinite for discussion.

[17] From Mathematics: The New Golden Age

result at the age of 15 or 16! **Adrien-Marie Legendre** (French mathematician; 1752 - 1833) was aware of this result about the same time, the mid 1790's. However, almost 100 years would pass before this result was proven, independently, in 1896 by **Jacques Hadamard** (French mathematician; 1865 - 1963) and **Charles Jean de la Vallée-Poussin** (Belgian mathematician; 1866 - 1962)! Both of these proofs involved complex numbers and complex function theory. In fact, all proofs through 1948 did! The most important result about the whole numbers and a "'real variable' proof of the prime number theorem, that is to say a proof not involving explicitly or implicitly the notion of an analytic function of a complex variable, has never been discovered, and we can now understand why this should be so…[18]" It was not until 1949, over 150 years since its formulation, that the Prime Number Theorem was proven without the use of complex numbers and complex functions! These proofs were due to **Atle Selberg** (Norwegian mathematician; 1917 - 2007) and, perhaps not independently, **Paul Erdös** (Hungarian mathematician; 1913 - 1996).

[18]From the 1932 text The Distribution of Prime Numbers by Albert Ingham; cited on p. 125 of Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics by John Derbyshire.

Chapter 3

Establishing Truth - Certainty and Burdens of Proof

In this chapter we would like to consider the ways in which we establish truth. We believe that by considering the ways in which we seek to establish truth in several different real-world settings will help illuminate aspects of mathematics that are often not understood. In later chapters we will see that when we proceed to analyze the foundations of mathematics we will be repaid with far reaching implications to the nature of truth and certainty.

3.1 The U.S. Courts System - A Case Study

O.J. Simpson (American Athlete and Actor; 1947 -) was the first professional football (American) player to rush for more than 2,000 yards in a single season. In addition to stellar collegiate and professional athletic careers he was a successful actor and television salesperson. His personal life was much less successful. On 12 June, 1994 Simpson's ex-wife **Nicole Brown** (American Housewife; 1959 - 1994) and her friend **Ronald Graham** (American Model and Waiter; 1968 - 1994) were brutally murdered. Simpson became a suspect and was to be charged before he attempted to flee in a slow-speed chase televised live on 17 June, 1994. This began a highly publicized, sensationalized, divisive, and racially charged series of trials.

The outcome of the trials were that Simpson was *acquitted* of double murder charges while he was found responsible for the *wrongful deaths* of Brown and Graham. How someone can be tried twice with opposite results is confusing to some. The reason we describe this case here is that it can help us understand different standards of proof.

	Civil Court	Criminal Court
Decision	Whether one individual has harmed another	Whether an individual has broken a law
Parties	A *plaintiff* brings the case against a *defendant*	A government *prosecutor* brings the case against a *defendant*
Questions	Was there *damage*? Who is *responsible* for the damage?	Was a *crime* committed? Who *committed* the crime?
Finding/Conclusion	*Responsible*, or not, for damage	*Guilty* or *innocent* of crime
Outcome of Positive Finding	The *remedy* is *damages* which are often monetary	The *penalty* is incarceration or other deterrent
Burden of Proof	*Perponderence of evidence*	*Beyond a reasonable doubt*

Notice all of the technical legal terms that have been used without being given formal definitions. The definitions of these terms are determined by statute, by legal precedence, and by court interpretation. These terms, and those like them that make up the legal vernacular, have very specific, precise definitions despite the fact that these definitions evolve over time.

So what is the truth? Did Simpson brutally murder Brown and Goldman? Only a few people actually know. But our court system has been developed so we have a structure, however fallible, that enables us to try to find relief for damages and crimes.

3.2 Science and Religion

> You're right, evolution is just a theory. Kind of like gravity.
>
> **Bumpersticker - author anonymous** (; -)

The ***Scopes Monkey Trial*** was a 1925 case in which charges were brought against **John Scopes** (American Teacher and Geologist; 1900 - 1970) for teaching the theory of evolution in a Tennessee school. This case is one of the more well-known cases in American legal history. It was widely followed as it happened, providing a huge stage for the opposing lawyers **Clarence Darrow** (American Lawyer; 1857 - 1938) and **William Jennings Bryant** (American Politician; 1860 - 1925) to joust over the political, legal, and religious implications. Scopes was convicted and his conviction was later overturned on appeal.

The debate over the teaching of evolution in the schools continues to this day.

While this is not a place to address this topic in general, it does bear relevance to our effort to understand how we establish truth.

The implicit suggestion of the bumper sticker above is that if we are to question evolution because it is "just a theory", we might as well question gravitation as well. It certainly is just a theory. But, like evolution, it is a profoundly useful theory that has been justified by hundreds of thousands of experiments over centuries. We consider the "theory of gravitation" to be true without giving it much though, but continue to question the "theory of evolution." It is of interest that in modern physics, gravity is no longer considered a fundamental force. Einstein's Theory of Relativity (yes, another theory) purports to explain what we have thought of as gravity is actually travel along *geodesics* in *spacetime* whose "lines" have been curved by the presence of matter. In this context, gravity appears part of the exotic geometries considered in Discovering the Art of Geometry rather than its own independent truth of nature.

More generally, the ongoing debate between religious and scientific views is based largely on the different paradigms that one accepts/believes when one searches for truth. The paradigms of religion are based on faith. The paradigms of science are based on experience and reason. This debate continues to rage in popular books, including the following by noted scientists and religious leaders:

Anarchy Evolution: Faith, Science, and Bad Religion in a World Without God by Greg Graffin and Steve Olson.[1]

The God Delusion by Richard Dawkins[2]

The End of Faith: Religion, Terror, and the Future of Reason by Sam Harris

> The net of science covers the empirical universe: what is it made of (fact) and why does it work this way (theory). The net of religion extends over questions of moral meaning and value. These two magisteria do not overlap, nor do they encompass all inquiry (consider, for starters, the magisterium of art and the meaning of beauty). To cite the arch cliches, we get the age of rocks, and religion retains the rock of ages; we study how the heavens go, and they determine how to go to heaven.

[1] **Greg Graffin** (American musician and biologist; 1964 -) is the lead singer of the influential and highly successful punk band Bad Religion. In addition to his success as a songwriter and musician, Graffin earned a Ph.D. in Evolutionary Biology from Cornell University. He regularly teaches courses at both Cornell and UCLA. The book referenced here is closely related to his Ph.D. dissertation.

[2] Dawkins also wrote the widely regarded The Greatest Show on Earth: The Evidence for Evolution.

Stephen Jay Gould (American biologist and historian of science; 1941 - 2002)

3.3 Who Are You Going to Believe?

The number π is perhaps the most fundamental constant in mathematics. Approximated to six decimals places as $\pi \approx 3.1415926$, the number π is what mathematicians call a *transcendental irrational*. Its decimal representation goes on indefinitely without any repeating pattern ever appearing. Π is not the solution to any algebraic equation with *rational number* coefficients. Despite these difficulties, π is central to mathematics. Most basically, it is the constant the relates ***linear dimensions*** (e.g. the radii, diameters, and heights) of circles, spheres, cylinders, cones and other objects which fundamentally involve circles to the areas, surface areas, and volumes of these objects. These relationships were known to the ancients even though their knowledge of numerical approximations to the actual value of π progressed *very* slowly over the millennia.

In the American King James version of the Bible, 1 Kings 7 describes *Solomon's temple*. Solomon fetches Hiram of Tyre who forges a hemispherical brass tub. In 1 Kings 7:23 the size of this hemisphere is described as follows:

> And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about.

1. Draw a hemisphere and carefully label all of the dimensions that are given in the passage from 1 Kings.

2. What part of the hemisphere is being measured as "thirty cubics did compass it round about"? What is the precise mathematical term that describes the measurement of this part?

3. Find a formula which expresses the measurement in Investigation **2** as a function of linear dimensions of a hemisphere.

4. Pretend that you did not know the value of the mathematical constant π. Use the equation in Investigation **3** and the quantities given in 1 Kings to find a **Biblical Value of** π. Does this bother you? Explain.

Bill #246 of the 1897 Indiana General Assembly was:

> A Bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the State of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the Legislature of 1897.

The bill was brought for legislative action by **Taylor I. Record** (American Indiana State Representative; -) on behalf of **Edwin J. Goodwin** (American physician; ca. 1825 - 1902). In the bill Goodwin claims to have found solutions to the *Three Greek Problems of Antiquity* which we will later see had *already* been *proven* to be impossible to solve.

5. One of the "facts" that would have been legislated by the bill is the legitimacy of a circle with radius 5 and circumference of 32. Find a formula which expresses the relationship between the circumference and radius of a circle.

6. Pretend that you did not know the value of the mathematical constant π. Use the equation and data in Investigation **5** to determine the numerical value of π that would have resulted if this legislative act was passed.

The Indiana House Committee on Education recommended that the bill be passed. Subsequently, it was approved the the Indiana House of Representatives unanimously 67 to 0! Luckily **Clarence Abiathar Waldo** (American mathematician and educator; 1852 - 1926), then Chair of the Purdue University Department of Mathematics, was in attendance when the bill was taken up by the Indiana Senate. At the General Assembly to lobby for legislative support for the University he reverted to his role as teacher, educating a sufficient number of Senators that the bill was indefinitely postponed.

7. Classroom Discussion: The title of this chapter is "Who are you going to believe?" Have these examples changed and/or informed the way that you think of the fiats of "authorities"? Can you think of other, similar examples?

Of course, these examples have a certain definiteness because they involve fairly clear statements about a well-established mathematical constant. But the implications are far broader. For example, the distinguished **Peter J. Gomes** (American Preacher and Theologian; 1942 - 2011), who served as Minister and Plummer Professor of Christian Morals at Harvard from 1974 until his death, spoke eloquently about "reading the Bible with mind and heart." He wrote:

> The last thing the faithful wish for is to be disturbed. Thus it is easy to favor the Bible over the gospel... Could it be that we spend so much time trying to make sense of the Bible, or making it conform to our set of social expectations, that we have failed to take to heart the essential content of the preaching and teaching of Jesus? ...If we are sincere in wanting to know what Jesus would do, we must risk the courage to ask what he says, what he asks, and what he demands. Only if we do so will we be able to move, however cautiously and imperfectly, from the Bible to the gospel.[3]

8. Classroom Discussion: From whom does the authority of the Bible come from? Does the answer depend on whether we approach the question from historical, political, religious, or personal perspectives? In Part 2, "The Uses and Abuses of the Bible," of The Good Book: Reading the Bible with Mind and Heart Rev. Gomes discusses in great length what the Bible tells us about race, what it tells us about Semites, what it tells us about women, and what it tells us about homosexuality. What is the difference between the Bible and the gospel that Gomes seeks so carefully to distinguish? What impacts might these distinctions have on what we choose to believe is moral?

9. Classroom Discussion: The Pulitzer- and Tony-Award winning play Proof was the first full-length work by playwright **David Auburn** (American playwright; 1969 -). Lead character Catherine is the daughter of the crazy mathematician Robert, Hal the diligent graduate student. The passage below is fundamental to the dénouement:

> Catherine: You think you've figured something out? You run over here so pleased with yourself because you changed your mind. Now you're certain. You're so... *sloppy*. You don't know anything. The book, the math, the dates, the writing, all that stuff you decided with your buddies, it's just evidence. It doesn't finish the job. It doesn't prove anything.
>
> Hal: Okay, what would?
>
> Catherine: *Nothing.*
>
> You should have trusted me.

How do burdens of proof differ in real life? In love, medicine, religion, friendships, media, politics,...?

3.4 The Scientific Revolution

This is not a book about faith. We hope to illustrate different standards of proof. In doing so, we need to consider the context of the *Scientific Method* that is central to scientific reasoning.

[3]From The Scandalous Gospel of Jesus: What's So Good About the Good News?, pp. 19,23.

The terms *Renaissance*, *Age of Enlightenment*, and *Scientific Revolution* are terms invented by contemporary historians to describe what they see as important periods in European culture.

10. In your own words, briefly describe the Renaissance.

11. Cite a few mathematical and/or scientific advances that played important roles in the Renaissance.

12. In your own words, briefly describe the Age of Enlightenment.

13. Cite a few mathematical and/or scientific advances that played important roles in the Age of Enlightenment.

14. In your own words, briefly describe the Scientific Revolution.

15. Cite a few mathematical and/or scientific advances that played important roles in the Scientific Revolution.

16. Which of these periods is subsumed by the others?

Fundamental to these periods was the development of the *Scientific Method*.

17. In your own words, describe the Scientific Method.

18. Illustrate, using a few different examples of paradigms that you considered above, how the scientific method was employed to result in important paradigm shifts.

3.5 Mathematics and Deductive Reasoning

> Faith is different from proof; the latter is human, the former is a Gift from God.
>
> **Blaise Pascal** (; -)

The Scientific Revolution is considered one of the most important intellectual developments in history. The resulting world-view has dominated science, and the *social sciences* who have sought to mimic its methods in many ways, for centuries.

Much less well-known, but equally important, is the revolution that formed the *epistemological* basis for mathematics. This revolution took place approximately two millenia before the Scientific Revolution. Here we consider this revolution from a general, theoretical perspective. In the next chapter it is explored in more practical, concrete ways.

Aristotle (Greek philosopher; 384 BC - 322 BC) was the first we know to undertake a detailed study of what we now call *logic*. Central to our purpose was his effort to illustrate what we now call deductive reasoning. In Prior Analytics he writes, "...certain things being supposed, something different from the things supposed results of necessity because these things are so." **Deductive reasoning** is the method of reasoning which links *premises* and axioms to *conclusions* - the truth of the premises and axioms logically necessitating the truth of the conclusion.

If conclusions are built on premises, one may fairly wonder how the premises were derived. For concreteness, let us consider one of the earliest, large-scale use of deductive reasoning: Euclid's Elements. In 13 books, Euclid deductively establishes 468 propositions. Upon what do they rest? Euclid begins with 23 definitions. These are *primitive terms* which are taken to need no further articulation. For example, Definition 5 states: A **straight line** is a line which lies evenly with the points on itself. He has 5 *common notions*. Among them, Common Notion 1 states: Things which equal the same thing also equal one another. Finally, he has five ***axioms*** or ***postulates***, which are the unquestioned assumptions or building blocks for the system of Euclidean geometry he is setting out to create. Axiom 1 states: To draw a straight line from any point to any point. This enables

the use of a straightedge in this geometry. Axiom 3 states: To describe a circle with any center and radius. This enables the use of a compass in this geometry. Euclid is now free to build a world where conclusions follow by logical necessity from the axioms and common notions. His first result is:
Proposition I.1 An equilateral triangle can be constructed from a given straight line.

19. Draw a straight line segment.

20. Proposition 1 says that an equilateral triangle, a triangle with all sides congruent, can be constructed from this straight line. Use the tools given by Axioms 1 and 3 - a straightedge and compass - to construct an equilateral triangle.

21. Prove that your triangle is equilateral. (Question: How does Common Notion 1 play into your proof?)

It is essential to note that it is impossible to refute this proposition if one starts from the same axioms and common notions. The truth, built from first principles, with necessity required at each stage of building, is eternal within the system that Euclid set out. This is essential for it has enabled mathematics to be "more successful than other branches of human knowledge in its endeavor to erect a reliable and lasting structure of thought."[4]
It is lasting because:

> In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation adds a new story to the old structure.
>
> **Herman Henkel** (German mathematician; 1839 - 1873)

One may *only* question results of deductive system by questioning the foundational postulates of the system. The remainder of the structure has been built as logical consequences from this foundation. It is for this reason that "each generation adds a new story to the old structure."

What happens when one questions or disputes any part of the foundation of the system? While much of the growth of mathematics builds vertically on established systems, important advances have come from beginning anew. By removing just one of Euclid's postulates - the 5^{th} postulate which is often called the *parallel postulate* - geometer's of the nineteenth century created entirely new, important geometries. **János Bolyai** (Hungarian mathematician; 1802 - 1860) said of these geometries:

> I have discovered such wonderful things that I was amazed... Out of nothing I have created a strange new universe.

These new geometries did not discredit or invalidate Euclidean geometry. They exist as alternatives to Euclidean geometry.

But shouldn't many geometries pose a problem? Which is the real geometry? In the introductory case study - civil versus criminal trials in the United States court systems - we saw that there were different systems for different purposes. The same is true with the different geometries. Carpenters and engineers work in Euclidean geometry as our existence on small scales is essentially flat. Airline pilots and sea captains work in spherical geometry, following routes along the curved surface of the earth. Bolyai's geometry, *hyperbolic geometry*, a model of which is illustrated in the Escher print in Figure 3.1, is the geometry that is the basis for Einstein's theory of relativity. Taxicab geometry, which you will meet below, is the geometry of taxicab drivers. Each of these geometries is quite useful, although quite different.

---

[4] In the words of the historian of mathematics **Morris Kline** (American mathematician and educator; 1908 - 1992).

Unlike the U.S. court system, where the burdens of proof are different, the burdens of proof in each of these geometries is equally high. They are built deductively and each is equally consistent. Each is true - what matters is which is useful in a given context.

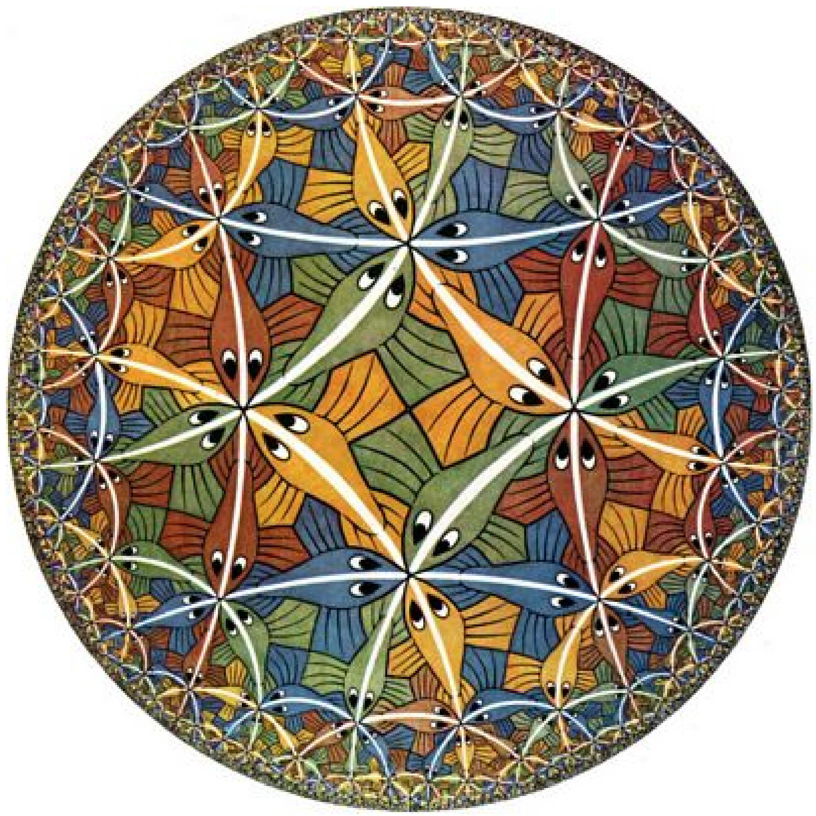


Figure 3.1: Circle Limit III by **Maurits Cornelis Escher** (Dutch graphic artist; 1898 - 1972).

Our introduction of $\sqrt{-1}$ in the previous chapter is a nice illustration of how mathematics grows/evolves deductively. There are definitions and axioms upon which the *field of real numbers* is constructed.[5] Once defined it is a simple matter to extend it to the form the field of complex numbers by the inclusion of the imaginary unit $i = \sqrt{-1}$. All that needs to be done is insure that all of the axioms for real numbers are well defined and valid for the complex numbers. This is not a hard task.

From one, deductively established, world - the field of real numbers - we have created another.

3.6 Differential and Integral Calculus - A Brief Case Study

Calculus is one of the most important areas of mathematics, a universal tool to study continuous change, rates and aggregates. It is used throughout engineering, all of the sciences, economics, and any area that studies continuously varying quantities in mathematical ways. Its roots date back to **Archimedes** (; -) and his exploration of circles and spheres, which is revisited later in this chapter and also in Discovering the Art of Mathematics - Calculus. A number of seventeenth

[5] It is important to note, a logical development of this system is quite sophisticated. It is a structure that needs to logically support all of the calculus, including *limits* and *completeness*.

century mathematicians made important contributions. Nonetheless, it was over a very short period that a complete theory of calculus was developed. It was developed independently by two men, **Isaac Newton** (English mathematician and physicist; 1642 - 1727) and **G.W. Leibniz** (German mathematician and philosopher; 1646 - 1716). The year 1666 is known as Newton's annus mirabilis, or year of miracles. In his "prime of age for invention, and minded mathematics and philosophy more than at any time since"[6] he left Cambridge to avoid the plague that had spread across many of England's urban areas to return to his family's country home in Woolsthorpe. In this year of isolation he developed calculus, developed the universal law of gravitation and made fundamental advances to what we know about optics.

Despite its remarkable success, calculus was *not* build on a rigorous axiomatic foundation. In studying instantaneous change, it rested on the notion of ***infinitesimals*** - infinitely small yet non-zero numbers. While most mathematicians continued blithely along buoyed by the success of the calculus, serious objections to there cavalier attitude about their subject were levied. Addressed to "infidel mathematicians", **Bishop George Berkeley** (Irish philosopher; 1685 - 1753) wrote The Analyst which was a scathing attack on the lack of rigor in the the foundations of calculus. The passage below is particularly interesting as it makes clear note of the logical relationships between different parts of the calculus, all pointing clearly back to the lack of substance in the nature of infinitesimals:

> It must, indeed, be acknowledged, that [Newton] used Fluxions, like the Scaffold of a building, as things to be laid aside or got rid of, as soon as finite Lines were found proportional to them. But then these finite Exponents are found by the help of Fluxions. Whatever therefore is got by such Exponents and Proportions is to be ascribed to Fluxions: which must therefore be previously understood. And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?

In his wonderful novel-like book A Tour of the Calculus, **David Berlinski** (; -) says:

> If the calculus is much like a cathedral, its construction the work of centuries, it remained until the nineteenth century a cathedral suspiciously suspended in midair, the thing simply hanging there, with no one absolutely convinced that one day the gorgeous and elaborate structure would not come crashing down and fracture in a thousand pieces.

Eventually gaps in the foundation arose that could not be ignored:

> The crisis struck four days before Christmas 1807. The edifice of calculus was shaken to its foundations...The nineteenth century would see ever expanding investigations into the assumptions of the calculus, an inspection and refitting of the structure from the footings to the pinnacle, so thorough a reconstruction that the calculus would be given a new name: *analysis*. Few of those who witnessed the incident of 1807 would have recognized mathematics as it stood 100 years later.[7]
>
> **David Bressoud** (; -)

Our understanding of the real number line, what constitutes a *function*, and how to rigorously work with *limits* were essential to this refitting. For those with interest in learning more, Berlinsky's A Tour of the Calculus is wonderful and its level is compatible with the level of the book you are currently working through. Bressoud's A Radical Approach to Real Analysis expects a level of mathematical experience including two-semesters of calculus, but it has a beautiful historical trajectory that can be appreciated on its own.

[6] Isaac Newton.

[7] From A Radical Approach to Real Analysis.

3.7 The United States Declaration of Independence and Constitution - A Case Study

The United States Declaration of Independence begins:

> We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty and the pursuit of Happiness.

The authors of the Declaration of Independence and the framers of the United States Constitution after them were influenced by political philosophers of the 17th century, like **John Locke** (English philosopher; 1632 - 1704), who had been profoundly influenced by the scientific revolution. The notion of self-evident truths are the axioms or postulates for a new political system that is being established.

The United States Constitution is the supreme law of the United States. It is a system based of law based on seven Articles, a Bill of Rights (the first 10 Amendments), and 17 subsequent Amendments. The framers wished to set up a system where future decisions could be based on rational thought and logic. It seems very similar to the system set out by Euclid for geometry, doesn't it?

Yet there are fundamental differences.

22. The Declaration of Independence says "*all* men are created equal." Did it really mean all men? Describe several ways in which the term "men" has been reinterpreted since the time of the original writing.

23. Write a brief summary of the 1896 United States Supreme Court case Plessy vs. Ferguson.

24. Write a brief summary of the 1954 United States Supreme Court case Brown vs. Board of Education.

The cases Plessy vs. Ferguson and Brown vs. Board of Education both involved the Equal Protection Clause of the Fourteenth Amendment. While the Amendment itself had not changed, the Supreme Court ruling was exactly the opposite in 1954 as it was in 1896!

Over time the ways in which we interpret and apply the Constitution change. The Supreme Court adjudicates the interpretations and applications. Some jurists are considered ***originalists*** who seek to interpret the Constitution strictly in terms of the framers original intent. Even this strictness cannot begin to compare with that of mathematics. In mathematics logic alone is all that is needed to adjudicate the validity of conclusions.

3.8 π - A Final Case Study

> We may have brought these creatures into existence (and that is a serious philosophical question in itself) but now they are running amok and doing things we never intended. This is the Frankenstein aspect of mathematics - we have the authority to *define* our creations, to instill in them whatever features or properties we choose, but we have no say in what behaviors may then ensue as a *consequence* of our choices.[8]
>
> **Paul Lockhart** (American mathematician and teacher; -)

Below we will formally define the number π. It is essential to note that once it is defined, its "behaviors" are no longer under our control. It's behaviors are under control of the rules of logic within the system in which it was defined. Biblical will, the will of legislators, none of these matter.

[8] From A Mathematician's Lament, p. 108.

There is nothing that gets to be decided once a definition is agreed upon, π is what it is and it is impervious to our will. Luckily, it provides the opportunity for wonderful discoveries that can be made about it. As **David Chudnovsky** (American mathematician; 1947 -) says, "Exploring π is like exploring the universe."

3.8.1 The Mystique of π

In 2004 **Daniel Tammet** (English author; 1979 -), who is a savant born with high functioning autism which he has written about in several beautiful and important books,[9] recited correctly from memory the first 22,514 digits of π. For over 5 hours, he rattled off digits of π one after another.[10]

25. What do you know about the number π? Please be as complete and specific as possible. You should include what you know about the number itself, what you know about its properties, where it arises in mathematics, what role it may play in culture, etc.

26. What you know about π, how have you learned it?

27. Are there things about π that you are curious about or interested in learning more about? If there are specific things, please describe them. If your interest is more general, please describe the nature of this interest. If you are not interesting, please indicate why you are not.

The meme below was recently posted on the Facebook account of Star Trek's **George Takei** (American actor; 1947 -):

> Pi is an infinite, nonrepeating decimal - meaning that every possible number combination exists somewhere in pi. Converted into ASCII text, somewhere in that infinite string of digits is the name of every person you will ever love, the date, time, and manner of your death, and the answers to all the great questions of the universe. Converted into a bitmap, somewhere in that infinite string of digits is a pixel-perfect representation of the first thing you saw on this earth, the last thing you will see before your life leaves you, and all the moments, momentous and mundane, that will occur between those two points. All information that has ever existed or will ever exist, the DNA of every being in the universe, EVERYTHING: all contained in the ratio of a circumference and a diameter.

Are these things really true about π? How could anyone possibly be certain of things like this? There is no way to check all of the digits. No way to search for the unlimited list of different strings that might appear. π is, to many, one of the great mysteries of mathematics.

The fascination with π is nothing new. The table below gives important moments in humanities search for ever better approximations of the value of π. Notice that these efforts span most of the worlds cultures over the past four milennia.

Notable Events in the Search for πs Decimals

[9] Born on a Blue Day, Embracing the Wide Sky and Thinking in Numbers

[10] See `https://www.youtube.com/watch?v=AbASOcqc1Ss` for a selection of a documentary on Daniel which contains footage of this feat.

Mathematician/Culture	**Year**	**Nationality**	**Correct Digits**
Babylonians	c. 2000 BC		3.1
Egyptians	c. 2000 BC	Egypt	3.1
Bible			3
Plato	c. 380 BC	Greek	3.14
Archimedes	c. 250 BC	Greek	3.14
Hon Han Shu	c. 130		3.1
Ptolemy	150	Greek	3.141
Wang Fau	c. 250		3.1
Liu Hui	263	China	3.14159
Tsu Chhung-Chih	c. 480		3.1415926
Aryabhata	499	Indian	3.1415
Brahmagupta	c. 640	Indian	3.1
Al-Khowarizmi	c. 800	Persian	3.141
Fibonacci	1220	Italian	3.141
Al-Kashi	1429	Persian	14 digits
Nilakantha	c. 1501	Indian	9 digits
Viete	1579	French	9 digits
Romanas	1593		15 digits
Ludolph Van Ceulen	1596	Dutch	20 digits
Ludolph Van Ceulen	1615	Dutch	35 digits
Isaac Newton	1665	English	15 digits
Sharp	1699		71 digits
Machin	1706	English	100 digits
De Lagny	1719	French	111 digits
Matsunaga	1739		50 digits
Vega	1794		136 digits
Rutherford	1824		152 digits
Dase	1844		200 digits
Clausen	1847		248 digits
Lehmann	1853		261 digits
Shanks	1853		530 digits
Lindeman			π transcendental
Ferguson	1945		530 digits
Ferguson	1947		710 digits
Reitwiesner	1949		2,037 digits
Felton	1958		10,020 digits
Shanks and Wrench	1961		100,265 digits
Guilloud and Dichampt	1967		500,000 digits
Guilloud and Boyer	1973		1,001,250 digits
Tamura	1982		16,777,206 digits
Kanada and Tamura	1988		201,326,551 digits
Chudnovskys	1989		1,011,196,691 digits
Takahashi and Kanada	1997		17,179,869,142 digits
Rabinowitz and Wagon	1995		Spigot algorithm discovered
Bailey, Borwein and Plouffe	1995		Hexadecimal digit extraction algorithm discovered

Many of these events are rediscoveries, successes of one culture not readily shared with others - all looking for more digits. Archimedes' method is the method employed in the majority of the events noted through the turn of the seventeenth century. When the number of correct digits does not increase from one line to the next, this is because the approach is new or newly discovered within a culture. Nilakantha was the first to use an approach based on infinite series, using Mādhava's series (which is usually known as Gregory's series as it was much better known through its European rediscover). Once this approach was reintroduced in Europe by Newton *all* of the subsequent results were based on that approach - adding terms from a series representation for π. This accounts for the rapid increase in the number of digits found. Ferguson's first result was the last advance done by hand, his second result done with the aid of a calculator. All subsequent advances were results based on infinite series and computed with the help of electronic computers. In fact, correctly calculating digits of π has long been a way to test computer hardware and software.

Yes, but how in the world does one know that these are really the correct digits of π? For questions like this, we need to begin to prove things about π.

3.8.2 Defining π

To satisfy the rigors of deductive reasoning we must make careful, unambiguous definitions upon which our mathematical structure will build. The standard definition of the mathematical constant **pi**, denoted by the Greek letter π, is that it is the unique real number satisfying

$$C = 2\pi r \quad \text{or } C = \pi d$$

for every Euclidean circle where C is the circumference and r is the radius, or $d = 2r$ is the diameter, of the circle. Equivalently, this means

$$\pi = \frac{C}{2r} \text{ or } \pi = \frac{C}{d}.$$

28. Can you explain why the ratio of the circumference to the diameter should be the same for *every* circle?

This formula for π is one of the most widely taught formulas in school mathematics. It is considered with such repetition that it takes on a nearly self-evident veneer. This is badly misleading.

Could you explain why the ratio of the circumference to the diameter is the same for every circle? If not, then the definition for π is entirely nonsensical. This definition *presupposes* that the ratio of the circumference to the diameter is the same for every circle. For the definition to be logically appropriate one would already have to establish this fundamental result about circles; that is one must *prove* the *theorem* that for every Euclidean circle the ratio $\frac{C}{d}$ is the same.

Euclid's Elements, so historically important in the development of deductive reasoning and the basis of most high school geometry curricula, is *silent* on this matter. Nowhere in the 468 propositions that are proven over the course of its 13 books does Euclid consider the circumference of the circle! Ancient mathematicians from many cultures tried to find values for π, clearly suggesting they thought the ratio was constant for all circles. But a definitive understanding of the discovery of proof of this fundamental result has yet to be found.

So how do we make sense of $C = \pi d$? We can begin empirically, as the ancients likely did.

29. Find eight circular objects of significantly different sizes. For each carefully measure the diameter and the circumference. Explain how you determined the diameter.

30. For each of your circles, compute the ratio $\frac{C}{d}$.

31. How close are your ratios to π? If there are any ratios that are dramatically different than π, remeasure them.

32. Construct a graph for your data points with the horizontal axes starting at $d = 0$ and including the entire range of diameters and the vertical axes the vertical axes starting at $C = 0$ and including the entire range of circumferences. Once you have created your graph, plot your data points. What do you notice about your data?

Figure 3.2: Pizza Pi.

33. Draw the line $C = \pi d$ on your graph. If you measurements were perfect your data would fall exactly on this line if $\frac{C}{d} = \pi$ for all circles. Is your data within an acceptable range to give you some faith that $\frac{C}{d} = \pi$ for all circles? Explain.

3.8.3 Other Shape's πs

Mathematics is *not* a science. No collection of empirical data is considered sufficient to establish a result. What we require is certainty - a proof. One modern way to approach this problem is to look at it more generally:

Big Question: As we scale other planar shapes does the ratio of their perimeter to some diameter remain constant?

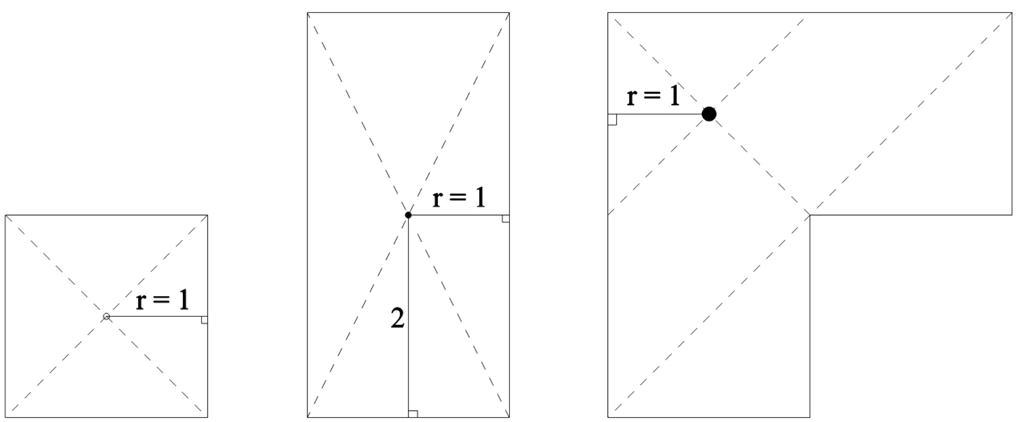


Figure 3.3: Square, rectangle and right-angled gnomon.

34. Choose one of the shapes in Figure 3.3 to investigate. Find the perimeter of your chosen shape, justifying your work.

35. Uniformly scale/magnify your chosen shape so that the radius is scaled from 1 to 2. Draw the new shape and find its perimeter, justifying your work.

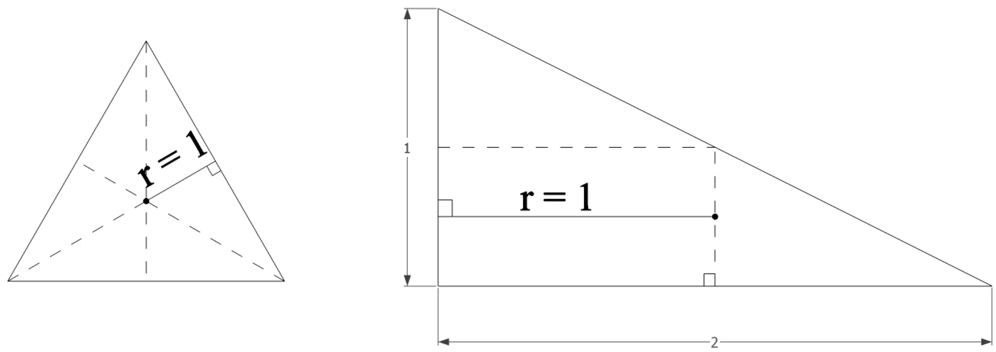


Figure 3.4: Equilateral and right triangle.

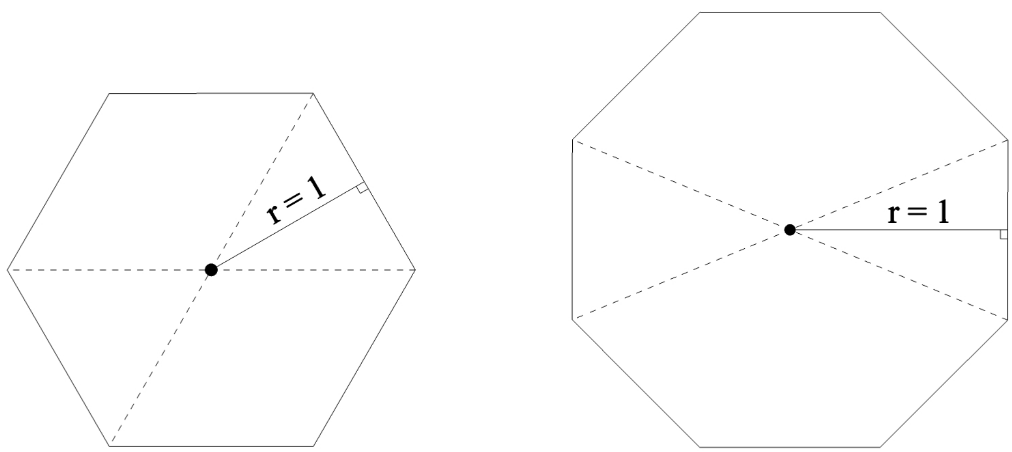


Figure 3.5: Regular hexagon and regular octagon.

36. Repeat Investigation **35**, this time scaling your originally chosen shape so the radius is scaled from 1 to 3.

37. Repeat Investigation **35**, this time scaling your originally chosen shape so the radius is scaled from 1 to 4.

38. Repeat Investigation **35**, this time scaling your originally chosen shape so the radius is scaled by a factor of your choice.

39. Determine the ratio $\frac{\text{Perimeter}}{2\text{radius}}$ for the shapes in each of the Investigation **34** - Investigation **38**. Surprised?

40. Scale the shape so the indicated length is scaled from 1 to r where $r > 0$ is any scaling factor. Determine the perimeter, P, and the ratio $\frac{P}{2r}$.

41. You now have *proven* a formula for the perimeter of your chosen shape as it is scaled to any size. Excellent! Isn't it nice to understand why this works and where it came from rather than being asked to mindlessly plug numbers into an arbitrary formula you were given?

42. Compare and contrast this formula with the formulae $C = \pi d$ and/or $C = 2\pi r$ for circles. Are you surprised that you have found a π-like constant for your chosen shape? Explain.

43. Repeat Investigation **34** - Investigation **41** for one of the shapes in Figure 3.4

44. Repeat Investigation **34** - Investigation **41** for one of the shapes in Figure 3.5.

Group Discussion - With other groups, share your results. What have you found? What are the implications for the existence of "other shapes πs" as a way of determining their perimeters?

In fact, it is generally the case that any well-behaved shape will have its own π-like constant ratio that relates the perimeter to the scale r of the shape. This is a fundamental consequence of the notion of *dimension*. The perimeter measures the length of the *boundary* of the shape, the boundary being one-dimensional. Similarly, the length indicated by r is also one-dimensional. As the shape is scaled, both the boundary and the indicated length increase proportionally. Specifically, if we scale by a factor of m the new perimeter is mP and the new indicated length is mr so the ratio of the two is:

$$\frac{mP}{m(2r)} = \frac{P}{2r}.$$

This ratio is unchanged!

Does this prove that $\frac{C}{d}$ is a constant for all circles? Only if one has carefully applied the modern machinery of dimension theory in Euclidean n-spaces. One can use calculus as well.[11] But *none* of these approaches were available to the ancient Greeks. Most teachers and mathematicians, and their books and websites and lectures and class materials, sweep all of this under the rug. Euclid's Elements was a paradigm shift, seeking to make mathematical truth eternal by specifying an exact foundation and building deductively upon it. It considered all major areas of mathematical knowledge of the day. Yet there is *NO* mention of $\frac{C}{d}$ being constant. The ancients believed this ratio was constant empirically, but the absence of proof of this fact from the historical record makes clear the difficulty of obtaining a geometric proof in the spirit of Euclid.

It is dishonest for this issue to be ignored or distorted in this way. In reference to the number $\sqrt{2}$ and the continuum of real numbers the great **Erwin Schrödinger** (Austrian physicist; 1887 - 1961) said:

> We have somehow lost sight of the difficulties it implies...We are told such a number as $\sqrt{2}$ worried Pythagoras and his school almost to exhaustion. Being used to such queer numbers from early childhood, we must be careful not to form a low idea of the mathematical intuition of these ancient sages, their worry was highly credible.

As you will see below, $\sqrt{2}$ caused a paradigm shift in mathematics, and a drowning of the discoverer of the example that triggered it! Yet $\sqrt{2}$ arises simply as the length of the diagonal of a unit square. In contrast, the geometrical appearance of the number $\pi = \frac{C}{d}$ is more complicated and this constant is dramatically more complicated as a number.[12]

Instead, significant attention should be placed on the absence of a deductive demonstration in Euclid's Elements and the other works of the time that have been handed down to us.

3.8.4 Area πs

π also arises in the area formula circle. Why is that so? We could send you out to measure the areas of some circles, but measuring the area of circles is quite hard. Archimedes was a major contributor to our understanding of circular areas and perimeters. The interested reader is referred

[11] See e.g. p. 543 of Geometry (1982) by Moise and Downs.

[12] For more see Chapter 21.2 of Elementary Geometry from an Advanced Standpoint by E.E. Moise and the paper "Circular reasoning: who first proved that C/d is a constant?" by David Richeson available at `http://arxiv.org/abs/1303.0904`.

to the chapter "Areas" in Discovering the Art of Mathematics - Calculus for more on data collection and the beautiful methods of Archimedes.

Instead, here we'll return to general geometric shapes.

45. Why does the area formula for a circle, $A = \pi r^2$, involve r^2 and not simply r?

46. Do you think the area formulas for our families of shapes should all involve r^2 even though the perimeters involved only r? Explain.

47. For each of the shapes in Investigation **34** - Investigation **38** determine the area of the shape.

48. Determine the ratio $\frac{\text{Area}}{r^2}$ for each shape. Surprised?

49. Scale the original shape so the radius is scaled from 1 to r where $r > 0$ is any scaling factor. Determine the area, A, and the ratio $\frac{A}{r^2}$.

50. You now have *proven* a formula for the area of your chosen shape as it is scaled to any size. Compare and contrast this formula with the formulae $A = \pi r^2$ for circles. Does it make sense to say that you have found a π-like area constant for your chosen shape? Explain.

51. Repeat Investigation **49** and Investigation **50** for your shape in Investigation **43**.

52. Repeat Investigation **49** and Investigation **50** for your shape in Investigation **44**.

As with perimeters, this behavior is perfectly natural from the perspective of dimension. The area is a measure of the *interior* of the shape, the interior being two-dimensional. The length indicated by r is one-dimensional. As the shape is scaled by a factor of m the new indicated length is mr, the area of the new interior is $m^2 A$ and the ratio of the new area to the square of the new length is:

$$\frac{m^2 A}{(mr)^2} = \frac{A}{r^2}.$$

As with perimeters, this ratio is unchanged!

At this stage in the discussion with perimeters we noted that there was no clear evidence in the historic record that the ancient Greeks were able to prove that this ratio is constant. In contrast, the constancy of the ratio for areas of circles is proven as Proposition II of Book XII of Euclid's Elements. This despite the fact that measuring, approximately, circumference is simple while it is remarkably hard to measure area.

In terms of rigorous foundations it would be more reasonable to define π as the ratio $\frac{A}{r^2}$ which we can prove using basic geometry!

This brings us to a very interesting question: If $\pi \approx 3.14159$ is the constant for the circumference (if we measure via $C = \pi d$, otherwise the constant is 2π since $C = 2\pi r$), why does this constant have *anything* to do with the area constant?

53. For each of the three shapes you have chosen to investigate, compare their π-like perimeter constant to their π-like area constant. Are any of these constants the same? Do you see any relationships between the two constants?

Returning to the case of circle, the miracle that the constant in question appears to be π in both cases is something that needs to be proven so we can have certainty.

Enlarged copies of the sectored circles in Figure 3.6 appears in the appendix.

54. Cut out the sectored circle on the left. Try to rearrange the pieces in a way that they create a shape that resembles a shape whose area is easy to determine.

55. Repeat Investigation **54** with the sectored circle in the middle of Figure 3.6. Try to make an arrangement that is similar for each of these circles.

56. Repeat Investigation **54** with the sectored circle on the right of Figure 3.6. Try to make an arrangement that is similar for each of these circles.

57. Suppose you kept sectoring the circle into more and more congruent sectors. Could you continue to arrange the pieces as you did above?

58. If you continued to do this indefinitely, in the *limit* what would be the resulting shape?

59. Determine the area of this shape in terms of the original dimensions of the circle.

60. Explain how this establishes a direct link between the perimeter constant and the area constant for a circle.

61. Return to the shapes in Figure 3.5. Can you adapt your strategy to give an alternative explanation why the perimeter and area π-like constants for these shapes are equal?

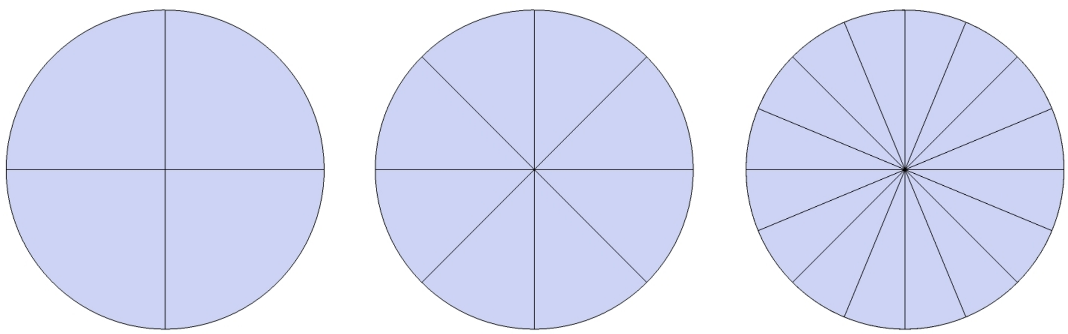


Figure 3.6: Sectored circles.

This argument has appeared many times historically, including a seventeenth century Japanese text and the works of **Leonardo da Vinci** (I talian painter, sculptor, architect, musician, mathematician, engineer, inventor, anatomist, geologist, cartographer, botanist, and writer; 1452 - 1519).[13] It was likely known much earlier than this. It is hard to believe that Archimedes was unaware of it.

We now have a real link which unites the circumference constant for circles back to the area constant. This approach is *not* typical of the type of geometry practiced by Euclid. Rather, it is typical of that practiced by Archimedes, in ways that foreshadowed the development of calculus almost two milennia later. We now, finally, have proof that the perimeter constant is the same as the area constant - our friend π.

3.8.5 Taxicab π

We now consider an alternative geometry. In this new geometry, called ***Taxicab geometry***, distances are measured the way taxicabs drive city whose streets are laid out so they form a regular, square grid. If you wished to travel by cab from one corner to the opposite corner on the same block a cab would have to drive up one block and then over one block. Hence, the distance between these two points is 2. This is a different way to measure distances than in Euclidean geometry. In

[13]See pp. 17-19 of A History of Pi by Petr Beckmann.

Euclidean geometry the distance would be, calculated via the Pythagorean theorem, $\sqrt{2}$. This is the distance the proverbial crow flies. But a taxicab cannot drive diagonally through the center of a block.

62. Give a precise definition of a circle.

63. On square grid graph paper choose and clearly mark an origin. Find and mark a point whose taxicab distance from the origin is 3 blocks.

64. Find and mark another point whose taxicab distance from the origin is 3 blocks.

65. Repeat Investigation **64**.

66. Continue to repeat Investigation **64** until you have found all points whose taxicab distance from the origin is 3 blocks. How many such points have you found? In what shape are these points located?

The taxicab geometry we would like to consider is ***continuous taxicab geometry***, where one can follow the streets which make up our grid but can also move along "alleyways" that occur *anywhere* between the main streets as long as they run parallel to the streets.

67. Utilizing alleyways, find several more points whose taxicab distance from the origin is 3.

68. If you continued using such alleyways, draw the figure which represents all points that are a distance of 3 blocks from the origin.

69. What shape is formed by all of these points?

70. Your shape is formed by all points that are a fixed distance from the origin. Return to Investigation **62**. What do we call such a shape?

71. Determine the circumference of your circle of radius 3. Explain carefully how you have determined the circumference.

72. Now draw a circle of radius 2.

73. What is the circumference of this circle?

74. Now draw a circle of radius 1.

75. What is the circumference of this circle?

76. Based on these examples, make a conjecture about the circumference of a circle of radius r in taxicab geometry for positive integers $r > 0$.

77. Prove your conjecture.

78. Having proved the formula for the circumference of circles, determine the value of *pi* in Taxicab geometry.

3.8.6 Spherical π

You may object that $\pi = 4$ is disingenuous as we are measuring distances so differently in Taxicab geometry. Taxicab geometry *is* a legitimate geometry. Let us then consider a more natural geometry - the geometry of the surface of the earth. We are, after all, creatures of the earth.

As noted earlier, the relinquishing of Euclid's parallel postulate caused a revolution in geometry in the nineteenth century. Ancient mariners and astronomers did not wait that long to study spherical geometry - they were adept at it close to two milennia earlier.

Find a large spherical object. Lenart spheres are wonderful manipulatives for exploring spherical geometry. If you do not have access, a basketball or other similarly sized spherical object will work fine.

79. On a flat surface, how can you use a piece of string and writing instrument to draw a circle?

80. How do you measure the radius of your circle?

81. You should be able to repeat this same process on your sphere. Draw a circle on the sphere in this way. Does it look like a legitimate circle?

82. Is the string the shortest distance *along the surface of the sphere* to get from your origin to the circle? If so, does it make sense to call this the radius?

83. Measure your radius and measure the circumference of your circle.

84. Draw another circle on the sphere. Measure its radius and its circumference.

85. Repeat Investigation **84** drawing a circle whose size is much different than those already drawn.

86. Repeat Investigation **84**, again trying to draw a circle of a significantly different size.

87. If you let your radius become so large that the circle is in the hemisphere opposite to the "center" where the string is anchored, what happens to the circumference of the circle as the radius gets larger.

88. For each of your circles, compute the ratio $\frac{C}{2r}$ of the circumference to twice the radius. This should be our spherical π. Is it? Explain.

3.8.7 But Why $3.14159\ldots$?

So in *some* geometries, shapes have their own πs, often a different π for perimeter than for area. In some they don't.

So the Euclidean circles we are used to have their own π - one you have discovered is the circumference *and* area constant. But why is this π such a mysterious, complex number? After all, circles are the most symmetric, most perfect of shapes. Why is Euclidean circles' π so esoteric?

The reason is because our way of measuring length is not compatible with the nature of circles. We measure length with straight rulers and square units of area. This is fine for squares and triangles. But our entire measurement apparatus is contrary to the nature of circles. We have a different paradigm, a different perspective than the circle. The operation of the circle must be translated back into our language of lengths and areas. The translation is complex. It is π.

It should not be any other way, should it? π is the crowning jewel of the beauty of the circle - perfectly appropriate.

Our race's efforts to understand π is one of the greatest stories of exploration our history holds. Each culture in each age has worked toward understanding its secretes. And their efforts illustrate the evolution of our ways of knowing. What we have found in each case is that π transcends any finite method of measurement.

Each of the formulas below is a formula for π. Each is listed under the name(s) of the mathematician who we believe is the first to discover it.

Nilakantha Somayaji (Indian mathematician and astronomer; 1444 - 1544)

$$\frac{\pi}{8} = \frac{1}{2} - \frac{1}{4^2-1} - \frac{1}{8^2-1} - \frac{1}{12^2-1} - \frac{1}{16^2-1} - \dots$$

François Viéte (French mathematician; 1540 - 1603)

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \times \dots$$

John Wallis (English mathematician; 1616 - 1703)

$$\frac{\pi}{2} = \frac{2\times 2}{1\times 3} \times \frac{4\times 4}{3\times 5} \times \frac{6\times 6}{5\times 7} \times \cdots$$

Isaac Newton (English mathematician and physicist; 1642 - 1727)

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3\times 2^3}\right) + \frac{1\times 3}{2\times 4}\left(\frac{1}{5\times 2^5}\right) + \frac{1\times 3\times 5}{2\times 4\times 6}\left(\frac{1}{7\times 2^7}\right) + \dots$$

Mādhava (Indian mathematician; circa 1380 - circa 1420), **James Gregory** (Scottish mathematician; 1638 - 1675) and **G.W. Leibniz** (German mathematician and philosopher; 1646 - 1716)

$$frac\pi 4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

William Brouncker (English mathematician; 1620 - 1684)

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \dots}}}}$$

Abraham Sharp (English mathematician; 1653 - 1742)

$$\frac{\pi}{6} = \sqrt{\frac{1}{3}} \times \left[1 - \left(\frac{1}{3\times 3}\right) + \left(\frac{1}{3^2\times 5}\right) - \left(\frac{1}{3^3\times 7}\right) + \dots\right]$$

Leonhard Euler (Swiss mathematician; 1707 - 1783)

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Srinivas Ramanujan (Indian mathematician; 1887 - 1920)

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \left[\frac{(4n)!}{(n!)} \times \frac{[1103 + 26390n]}{(4 \times 99)^{4n}} \right]$$

David Chudnovsky (American mathematician; 1947 -) and **Gregory Chudnovsky** (American mathematician; 1952 -)

$$\frac{1}{\pi} = 12 \times \sum_{n=0}^{\infty} \left[(-1)^n \times \frac{(6n)!}{(n!)^3 (3n)!} \times \frac{13591409 + 545140134n}{640320^{3n+\frac{3}{2}}} \right]$$

David H. Bailey (American mathematician and computer scientist; 1948 -), **Peter Borwein** (Canadian mathematician; 1953 -) and **Simon Plouffe** (Canadian mathematician; 1956 -)

$$\pi = \sum_{n=0}^{\infty} \left[\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right]$$

Continued Fraction whose coefficients follow no pattern, like the digits of π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}}}}}$$

Each of these methods involves the infinite.

3.8.8 Current Status of π

Using continued fractions, in 1761 **Johann Heinrich Lambert** (Swiss mathematician and physicist; 1728 - 1777) was able to prove that π is **irrational** - it cannot be written as fraction using whole numbers. It is not hard to show[14] that the decimal expansion of any irrational number is non-repeating. Hence, the decimal expansion of π does not repeat.

Euler and others had long expected that π was even more esoteric, that it was **transcendental** - not the solution to any *polynomial equation* with *rational coefficients*. π resisted for quite some time until **Carl Louis Ferdinand von Lindemann** (German mathematician; 1852 - 1939) proved that π was transcendental in 1882. This was seen as a remarkable achievement.

In 1995 two remarkable results were obtained, both ***spigot algorithms*** for computing the digits of π. Up to that point all efforts to compute digits of π relied on infinite expressions and sophisticated floating point computer arithmetic. The first, by **Stanley Rabinowitz** (; -) and **Stan Wagon** (; -), was a method for computing the digits of π one at a time without any reference to previous digits or need for high precision, floating point arithmetic. All that need be specified in advance was how many digits were desired. The second, by **David H. Bailey** (American mathematician and computer scientist; 1948 -), **Peter Borwein** (Canadian mathematician; 1953 -) and **Simon**

[14] See Discovering the Art of Mathematics - The Infinite.

Plouffe (Canadian mathematician; 1956 -) allowed *any* single digit of π to be determined directly without any knowledge of previous digits or much significant calculation. The drawback? The digits are computed not as *base ten digits* but as *base sixteen digits*. Why does π submit more readily to base sixteen computations? We do not know.

And what about the claim in Takei's meme that the digits of π contain every possible finite string of digits with the expected frequency? Such numbers are called ***normal***. It is unknown whether π is normal. This remains a great mystery. It is a mystery we may never know the answer to.

While the open questions in mathematics far outnumber the questions that have been answered, this is another situation where there is a remarkable silver lining. Mathematicians have proven that the numbers that are normal outnumber those that are not by any measure. Picked randomly, a number is overwhelmingly likely to be normal - to have the remarkable properties claimed in the meme.

3.8.9 You and π

At the outset of this section you were asked some questions about your relationship with π. So how has your relationship with π changed through these investigations?

89. Write a brief essay of one- to two-pages which describes how your relationship with π has changed through the course of these investigations. Some possible topics to include in your essay are:

- What did you learn about π through these investigations?
- What was most surprising to you?
- Were some of the questions you had at the outset answered by your investigations?
- Are you more or less curious about π having completed these investigations?
- Does π make more or less sense to you having completed these investigations? Is this good or bad?

3.9 Further Investigations

3.9.1 When Perimeter Constant is Twice Area Constant

For some shapes the perimeter constant, $\frac{P}{r}$, is simply twice the area constant, $\frac{A}{r^2}$, like for the circle where we have $C = 2\pi r$ and $A = \pi r^2$. If you have some experience with calculus, there is a beautiful connection that explains this.

F1. Use the rules for derivatives to take the derivative of $A = \pi r^2$ with respect to the variable r.

F2. Use the rules for derivatives to take the derivative of the area formula for the other shapes you considered. For which shapes is the derivative of the area equal to the perimeter? Do you see anything these shapes share in common?

So *why* does this happen? And why does it only work for certain shapes? The definition of the derivative is:

$$\lim_{\Delta r \to 0} \frac{A(r + \Delta r) - A(r)}{\Delta r}.$$

Instead of analyzing this definition algebraically or numerically, let us analyze it geometrically.

F3. For the circle, draw a picture which represents $A(r + \Delta r) - A(r)$ geometrically.

F4. For each of the other shapes for which $A'(r) = P(r)$, draw a picture which represents $A(r + \Delta r) - A(r)$ geometrically.

F5. Each of these shapes is approximately a long, thin, distorted rectangle - whose width and length are easily approximated. Approximate the area in terms of quantities we are already working with: A, P, r and Δr.

F6. Use your approximation in the previous problem to simplify $\frac{A(r+\Delta r)-A(r)}{\Delta r}$. Beautiful, no?

It is now natural to wonder, "Why is the derivative of area equal to the perimeter?"

F7. Remember the derivative of area measures the rate of growth. Use your geometric pictures above to explain why, with this interpretation of the derivative, the derivative of area is the perimeter.

F8. Return to the shapes for which the derivative of area is *not* the perimeter. Use the interpretation of the derivative as rate of change and geometric pictures to describe why the derivative of area is not exactly equal to the perimeter.

This result is not limited to areas and perimeters, but extends throughout the dimensional ladder.

F9. Find the formulas for the surface area and volume of a sphere as a function of the radius r.

F10. Take the derivative of the volume. What do you notice?

F11. Can you extend your reasoning in Investigation **8** to describe why we would intuitively expect this result here?

For more about these topics, see:

- "Differentiating Area and Volume", J.I. Miller, *The Two-Year College Mathematics Journal*, Vol. 9, No. 1, Jan. 1978, pp. 47-8.
- "Solids in R^n Whose Area is the Derivative of the Volume", M. Dorff and L. Hall, *The College Mathematics Journal*, Vol. 34, No. 5, Nov. 2003, pp. 350-8.
- "Area and Perimeter, Volume and Surface Area", J. Tong, *The College Mathematics Journal*, Vol. 28, No. 1, Jan 1997, p. 57.

3.9.2 Taxicab Area

It is natural to wonder about the area of circles in Taxicab geometry. Since the city blocks in our model are square, each block naturally has an area of one square unit. This enables us to easily calculate areas of circles.

F12. Find the area of each of the circles in taxicab geometry whose circumferences you found above.

F13. Based on these examples, make a conjecture about the area of a circle of radius r for positive integers $r > 0$.

F14. Prove your conjecture.

Chapter 4

Types of Reasoning

> Science is the attempt to make the chaotic diversity of our sense-experiments correspond to a logically uniform system of thought.
>
> **Albert Einstein** (German physicist and author; 1879 - 1955)

Here we will investigate two critical types of reasoning - inductive and deductive. We shall see that each deals with patterns, relationships, connections, causes, prediction, control, explanation, and implications, but in very different ways.

What's My World is a game in which one person, who we'll call the **creator**, creates an imaginary world from a set of rules. The creator provides clues about their imagined world while the other players attempt to guess the rules that define this imaginary world. Each clue is given in the form "In my world there are $\boxed{}$ but no $\boxed{}$." To help find the rules, players can ask the Creator whether certain things are in the world and the clue returned by the Creator will be given in the same form. Sample clues for a game are:

"In my world there are birds but no cats."
"In my world there are gnats but no children."
"Is there chocolate in your world?" "In my world there are geese but no chocolate."

1. Guess what the rules are that defines this world. Are you confident?

2. In fact, these clues have been set up so there are several different possible answers. Find a few more.

3. What are some questions you could ask to help distinguish which of your guesses may be correct and which are not?

4. Other than asking directly, is there any way you can be absolutely certain that you have discovered the defining rule?

5. Choose a Creator and play What's My World. Describe the clues, how the game progressed and whether the rules were guessed.

6. Is there any way, other than asking the Creator, that you could be certain that you had determined the rules of this world? Explain.

> This universe, I conceive, like to a great game begin played out, and we poor mortals are allowed to take a hand. By great good fortune the wiser among us have made out some few of the rules of the game, as at present played. We call them "Laws of Nature," and honour them because we find that if we obey them we win something for our pains. The cards are our theories and hypotheses, the tricks our experimental verifications. But what

sane man would endeavor to solve this problem? The problem of the metaphysicians is to my mind no saner.[1]

Thomas Henry Huxley (English biologist; 1825 - 1895)

7. Use the quote by Thomas Huxley to explain how What's My World is related to the scientific method. Give several examples from your own world where this process of trying to find rules has lead you to false conclusions.

8. Consider the sequence William Henry Harrison, Abraham Lincoln, James A. Garfield, and William McKinley. Why are these four people on these list together? Can you think of a reason that these particular people were chosen? Explain.[2]

9. Suppose we add Warren G. Harding to the end of this list. Does your answer to the previous question still hold? If not, can you posit a new principle which governs our list?

10. Franklin D. Roosevelt would be next on the list. And then John F. Kennedy. Can you now guess what the list is?

11. There seems to be a pattern. Some principle in action. So perhaps it would be appropriate to make predictions based on this pattern? What predictions might be made? Did they come to fruition?

$$\begin{aligned} 6 &= 3+3 \\ 8 &= 3+5 \\ 10 &= 5+5 \\ 12 &= 5+7 \\ \vdots & \quad \vdots \end{aligned}$$

12. The vertical ***ellipsis*** $\vdots$ appear below the four equations above because they suggest that the equations form a pattern that continues. What do you think the next five equations are in this pattern?

13. Your answer to Investigation **12** may "fit" the data, but the typical answer does not "match" the pattern we were looking for. Here's a clue - the number 9 is never allowed. Can you find a different pattern that "fits" this new clue about the pattern?

14. These investigations are a bit like What's My World. Here are some more clues about the sequence of equations we have in mind - in addition to not using 9 anywhere, you cannot use any even number on the right. Nor can you use 15s. Can you find a pattern that "fits" these new clues?

15. The "rule" that governs the sequence of equations that is intended above gives rise to a *conjecture*. Use your observations above to complete the following:

Conjecture 1. *(Goldbach's) Every even number* ≥ 6 *can be written as the* $\boxed{}$ *of* $\boxed{}$ $\boxed{}$

16. See if you can check this conjecture for the first 25 examples.

[1] Quoted on p. 513 of The Colossal Book of Mathematics by Martin Gardner.
[2] This example is from Instructor's Guide to Mathematics: A Human Endeavor, 3^{rd} edition, by Harold R. Jacobs.

We call this ***Goldbach's conjecture*** because it was first posed by **Christian Goldbach** (German lawyer and mathematician; 1690 - 1764), an otherwise little-remembered mathematician, in a letter to the great **Leonard Euler** (Swiss mathematician; 1707 - 1783) in a letter dated 7 June, 1742.

The first examples above came from a game. The sequence of dead Presidents is pretty strange, but there couldn't be any underlying cause for this could there? But now with Goldbach's conjecture perhaps you have some sense that there must be some underlying cause, rationale, reason, effect, or agent at work.

In fact, the scientific method relies very much on the observation of patterns, relationships, connections, causes, prediction, control, explanation, and implications. The scientist hopes to find a theory that "fits" the data.

Inductive reasoning is the process of drawing general conclusions from limited (usually empirical) evidence. If the conclusions "fit" the data well enough they become known as scientific theories.

4.1 Challenges to Inductive Reasoning

In the next several investigations we would like to explore how many prime numbers there are.

17. Complete each of the following computations and determine if the resulting number is prime or not:

$$2 + 1 =$$

$$2 \cdot 3 + 1 =$$

$$2 \cdot 3 \cdot 5 + 1 =$$

$$2 \cdot 3 \cdot 5 \cdot 7 + 1 =$$

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 =$$

18. Do you think that the pattern in Investigation **17** will continue indefinitely? If so what type of reasoning are you using? If not, why not?

19. If the pattern in Investigation **17** did go on for ever, what could you conclude about the number of primes?

20. Complete the calculation

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1,$$

and then determine if the resulting number is prime. If it is not prime, completely factor the number into *prime factors*.

21. Repeat Investigation **20** for the number $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 + 1$.

22. Repeat Investigation **20** for the number $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 + 1$. (Hint: 347.)

23. What do these last results tell you about inductive reasoning?

24. In the preceding Investigations, if the number formed was not prime then what can you say about its prime factors? Do you think this pattern will continue? Why?

We will return to this pattern in the next Chapter and will see there that we can use it as the basis of a beautiful proof, due to **Euclid** (Greek mathematician; flourished circa 300 BC -), that there are infinitely many prime numbers. There is also a new, very nice alternative proof in the chapter "Proof."

Put a section right here about Euler's formula for polyhedra, Proofs and Refutations, and the ongoing debate. Have students do some by making GSU figures, folding nets, going and collecting polyhedral things around them (dice, books, etc.)

In a recent paper[3], **Lenny Jones** (; -), introduced a very interesting sequence of numbers:

$$\begin{aligned} s_0 &= 12 \\ s_1 &= 121 \\ s_2 &= 1,211 \\ s_3 &= 12,111 \\ s_4 &= 121,111 \\ &\vdots \end{aligned}$$

25. Find all of the prime factors of each of the numbers s_0, s_1, s_2 and s_3.

s_4 has a *prime factorization* of $s_4 = 281 \times 431$.

26. Determine which, if any, of the numbers $s_5, \ldots s_9$ are prime. For those that are not, keep track of any factors that you find in the table below. (Finding all prime factors may be difficult without suitable computer software.)

n	Term s_n	Factors
0	12	
1	121	
2	1211	
3	12111	
4	121111	281, 431
5	1211111	
6	12,111,111	
7	121,111,111	
8	1,211,111,111	
9	12,111,111,111	
10	121,111,111,111	61, 1985428051
11	1,211,111,111,111	
12	12,111,111,111,111	
13	121,111,111,111,111	
14	1,211,111,111,111,111	
15	12,111,111,111,111,111	
16	121,111,111,111,111,111	683, 177322271026517
17	1,211,111,111,111,111,111	

Notice that factors for a few of the terms in the sequence have been filled in.

27. See what factors you can find for the remaining entries in the table above.

A positive integer that is not *prime* is called **composite**. When we check to see if a number (resp. collection of numbers) is (resp. are) prime we are checking their ***primality***.

[3] "When does appending the same digit repeatedly on the right side of a positive integer generate a sequence of composite integers?" *American Mathematical Monthly*, vol. 118, no. 2, February 2011, pp. 153-60

28. Do you feel comfortable making a conjecture about the primality of *all* numbers in the sequence $\{s_n\}$? Explain.

29. Prove that all of the numbers $s_1, s_3, s_5, \ldots$ are *all* composite.

30. Prove that all of the numbers $s_0, s_3, s_6, \ldots$ are *all* composite.

31. Find another infinite ***subsequence*** of numbers from the sequence $\{s_n\}$ that you think will be composite. Give your reasons for making this conjecture.

32. You have now shown that a very large proportion of the numbers in the sequence $\{s_n\}$ - *infinitely many* of them - are composite. Would you like to amend your answer to Investigation **28**? Explain.

33. If you still had concern that maybe some terms in the sequence are prime, identify a specific term you would like to know about. Explain why you chose this term.

34. Each of the numbers s_0 - s_{100} are in fact composite. So in addition to your three infinite subsequences that are composite, all of those terms in the first hundred that might be worrisome are in fact composite as well. Would you now like to amend your answer to Investigation **28**? Explain.

35. Do you have any information that pertains to s_{136}? Explain.

36. Write out s_{136} in its entirety.

37. How does one prove that a number is prime? How long do you think it would take to check that s_{136} is prime if, in fact, it is? Explain.

38. Can you guess what the punch-line is going to be?[4]

39. What does this suggest about inductive reasoning?

4.2 Deductive Reasoning

Having seen many examples of the limitations of inductive reasoning, we return to deductive reasoning. A theoretical description of deductive reasoning was given in the previous chapter. Remember, deductive reasoning is a type of reasoning where the truth of conclusions follow as logical necessity from previously established premises and the axioms that serve as the foundation of the system.

A useful way to help understand the distinction between inductive and deductive reasoning is to compare corresponding terms in the different areas:

Inductive Reasoning		Deductive Reasoning
Empirical Evidence	$\leftrightarrow$	Proof
Fits data	$\leftrightarrow$	Matches Cause
Hypothesis	$\leftrightarrow$	Fact
Conjecture	$\leftrightarrow$	Theorem
Expectation	$\leftrightarrow$	Logical Conclusion
Reasonable Certainty	$\leftrightarrow$	Absolute Certainty

In this section we will explore some systems which are developed from simple sets of axioms so we can see how deductive systems germinate a rigorous foundation.

[4] Yes, s_{136} is the first term in the sequence that is prime.

4.2.1 Basic Example of Deduction: Introduction to Smullyan Island

Later, in the chapter "Knights and Knaves", you will study in some detail the fictional **Smullyan island**. It is an island whose cultural code consist of three axioms:

Smullyan Island - Axiom 1 All citizens are either knights or knaves.

Smullyan Island - Axiom 2 Knights always tell the truth.

Smullyan Island - Axiom 3 Knaves always lie.

You find yourself on Smullyan Island, talking to two natives - Don and Sancho. Don says, "We are both knaves."

40. Can Don be a knight? Explain.

41. What can you conclude about the validity of Don's statement? Explain.

42. Deductively establish whether Sancho is a knight or a knave.

What is important to note is that you solved the puzzle of Don and Sancho's types using only logic and the axioms of Smullyan Island's cultural code. Your conclusions are certain as long as the axioms hold.

4.2.2 Basic Example of Deduction: Sudoku Puzzles

Sudoku puzzles are popular puzzles where the goal is to fill in a 9 by 9 grid, which contains some initial clues, with numbers while satisfying each of the following rules:

Sudoku - Rule 1 Each grid square must contain one of the digits 1 through 9.

Sudoku - Rule 2 Every **row** must contain each of the digits 1 through 9 exactly once.

Sudoku - Rule 3 Every **column** must contain each of the digits 1 through 9 exactly once.

Sudoku - Rule 4 Every 3 by 3 sub-square must contain each of the digits 1 through 9 exactly once.

Consider the Sudoku puzzle in Figure 4.1.

43. What digit *must* go in the first row and fourth column? Prove that this is the only possibly digit for this location, explicitly describing what axioms you have used.

44. Where must the 5 in the first row be placed? Prove your result.

45. Where must the 8 in the first row be placed? Prove your result.

46. Where must the 6 in the first row be placed? Prove your result.

Notice that you have, using only the rules of Sudoku - the game's axioms, deductively established the identity of the entire first row for the puzzle to be solved.

47. Where must the 2 go in the lower, right 3 by 3 sub-square? Prove your result.

48. Where must the 6 go in the lower, right 3 by 3 sub-square? Prove your result.

49. Complete the remainder of the lower, right 3 by 3 sub-square, proving your result.

50. Where must the 8 in the eighth row be placed? Prove your result.

3	2	4			9		7	
		7			2			8
8		6		5		1	2	
9	3		5	1				
			8		4			
				9	3		5	1
	4	3		7		8		9
6			9			4		
	8		6			5	3	7

Figure 4.1: A Sudoku puzzle.

51. Where must the 4 in the eighth row be placed? Prove your result.

52. Complete the eighth row, proving that your result is correct.

Notice that at each stage you are proving that your entry *must* be correct. You are logically employing the rules of the game to make deductive conclusions which definitively establish the identity of certain entries. Notice also that some of your later deductions (moves) rely on previously established results (moves). The proof of each result (move) can be thought of as a theorem. You are building up a larger and larger system with more tools. Indeed, this is precisely one of the ways in which mathematics grows.

53. Solve the Sudoku puzzle, *only* adding entries when you can deductively establish that this *must* be the correct entry for this space.

You have now established, for a deductive, permanent fact, that your solution is the one and only solution to this Sudoku puzzle. You have established a Sudoku theorem. The results of mathematics are as final, permanent and definitive as yours. The difference is that mathematics is not constrained by the squares of a finite board, it continues to grow upward and outward in an unlimited number of possible directions.

4.2.3 Finite Geometries

On Smullyan Island there were axioms for the behavior of Knights and Knaves. These are the laws that govern the system; the postulates on which the system is based. Sudoku puzzles had rules that govern the puzzles. These are axioms for the puzzles. Here we will consider geometries based on simple axioms. Points, lines, and on are the undefined terms of this system.

A **three point geometry** is defined by the following axioms:

3PtGeo - Axiom 1 There exist exactly three distinct points.

3PtGeo - Axiom 2 Each two distinct points are on exactly one line.

3PtGeo - Axiom 3 Not all the points are on the same line.

3PtGeo - Axiom 4 Each two distinct lines are on at least one point.

54. Does a *consistent* three point geometry exist? If so, provide a model - a graphical or physical representation. If not, prove why not.

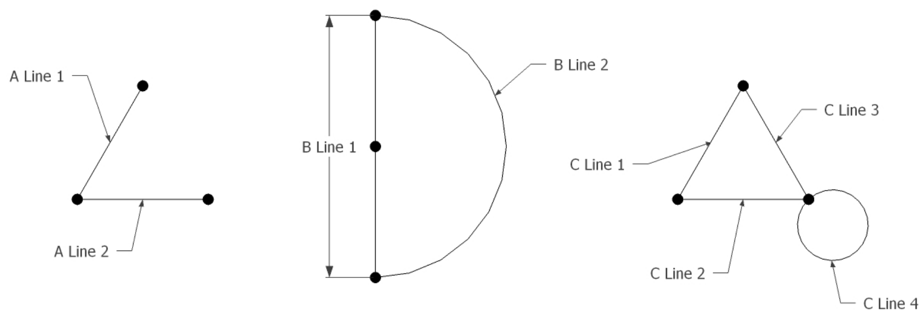


Figure 4.2: Candidates for three point geometries.

55. In Figure 4.2 are three potential candidates, A, B, and C, for a model for three point geometry. For each, determine whether it is a valid model for three point geometry, identifying the axioms that are violated if it is not a model.

56. Prove the following:

Theorem 1. *Each two distinct lines in three point geometry are on exactly one point.*

57. Explain why a direct result of this theorem is:

Theorem 2. *There are no parallel lines in three point geometry.*

58. Prove the following:

Theorem 3. *In three point geometry there exist exactly three lines.*

A **four point geometry** is defined by the following axioms:

4PtGeo - Axiom 1 There exist exactly four points.

4PtGeo - Axiom 2 Each two distinct points have exactly one line that contains both of them.

4PtGeo - Axiom 3 Each line is on exactly two points.

59. Does a *consistent* three point geometry exist? If so, provide a model - a graphical or physical representation. If not, prove why not.

60. Prove the following:

Theorem 4. *Any four point geometry has exactly 6 lines.*

61. Prove the following:

Theorem 5. *Every point of four point geometry has exactly 3 lines on it.*

62. Are there any parallel lines in four point geometry? Explain.

Developed in 1892 by **Gino Fano** (Italian mathematician; 1871 - 1952), a **Fano geometry** is defined by the following axioms:

Fano - Axiom 1 There exist at least one line.

Fano - Axiom 2 Every line of the geometry has exactly three points on it.

Fano - Axiom 3 Not all the points of the geometry are on the same line.

Fano - Axiom 4 For two distinct points, there exists exactly one line on both of them.

Fano - Axiom 5 Each two lines have at least one point on both of them

63. Make a model of Fano's geometry. Check carefully to insure that your model satisfies each of the axioms.

In the earlier axioms you were told the number of points in the geometry. Here you are not. While you have a model, might there be another model with more points?

64. The shaded box in Figure 4.3 represents your Fano geometry from Investigation **63** and a new point that would like to be added to your geometry. Prove that this point cannot be added without the axioms being violated.

65. State and prove a theorem about the number of points in any Fano geometry.

66. Prove the following:

Theorem 6. *In Fano's geometry each two distinct lines have exactly one point in common.*

67. State and prove a theorem about the number of lines any Fano geometry.

68. Does a *consistent* Fano geometry exist? If so, provide a model - a graphical or physical representation. If not, prove why not.

At the outset we said that the terms point and line were undefined. You might, rightly, wonder - who then will define the definition? The answer comes from **Henri Poincare** (French mathematician, physicist and philosopher; 1854 - 1912):

> Mathematicians do not study objects, but relations among objects; they are indifferent to the replacement of objects by others as long the relations don't change. Matter is not important, only form interests them.

69. Suppose that we name the "points" in Fano's geometry with the names Grigori, Maria, Fan, Alberto, Ingrid, Enrico, and Goro and suppose the names of the lines were Committee 1, Committee 2, . . . What then would Fano's geometry describe?

70. Explain how this relates to the Poincare quote.

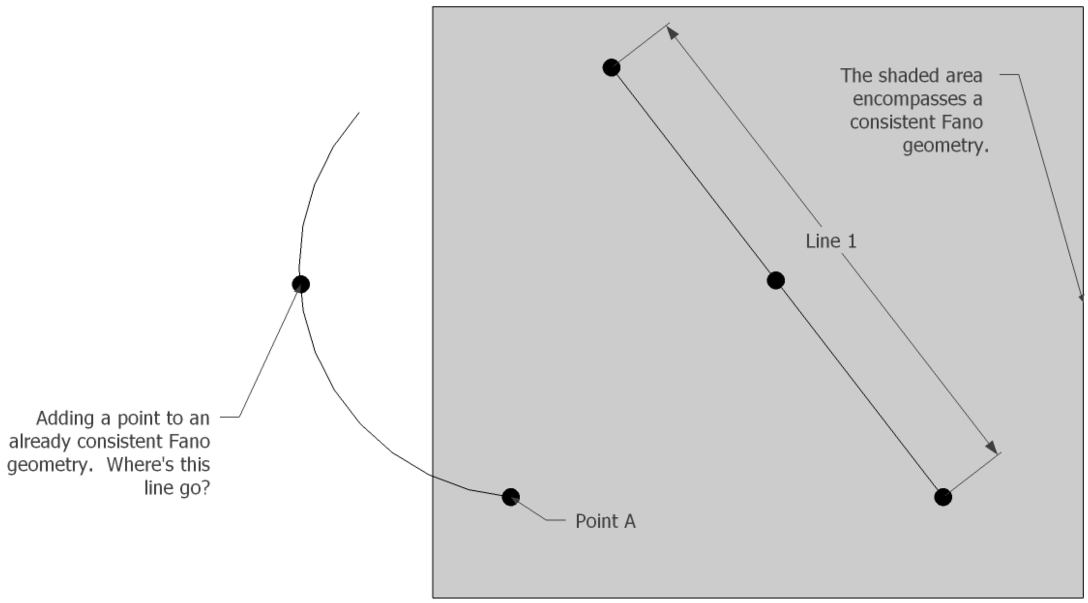


Figure 4.3: Can you expand a Fano geometry?

4.3 The Foundations of Mathematics

In chapter 3, "Establishing Truth - Certainty and Burdens of Proof", Herman Henkel noted, "In mathematics alone each generation adds a new story to the old structure." We see now why - when it rests on solid axiomatic foundation and its growth proceeds deductively, this tree of mathematics is upheld by the rules of logic.

The axiomatic foundations are long in coming. Euclid's Elements was certainly built on what was learned through the centuries-long mathematical work of surveyors and artisans. The universal axioms for the positive whole numbers are the *Peano axioms* developed by **Giuseppe Peano** (Italian mathematician; 1858 - 1932). While these axioms provide a rigorous foundation for the whole numbers, the lack of foundation did not seem to bother people who had been counting and doing arithmetic since there were people to count. Just like the shaky foundation of calculus did not keep people from using it.

The question becomes how much attention we wish to pay to the foundation. Einstein recounts his time with Euclid's Elements as:

> In your schooldays most of you who read this book made acquaintance with the noble building of Euclid's geometry, and you remember – perhaps with more respect than love – the magnificent structure, on the lofty staircase of which you were chased about for uncounted hours by conscientious teachers.

Our choice here is not to chase or herd you up such a staircase. We do not want you to spend years trying to ascend the rigid, enormous trunk that is the foundation of mathematics.

But we do want you to know that it is there. We want you to understand the essential point that one *can* go all the way back to the foundation and build deductively. In times of question, or turmoil, or historical interest, or philosophical interest, or scientific need, we can return to the roots of each branch of the tree of mathematics. We can investigate its formative axioms, much like we return to the Constitution for guidance. We can change them to begin new branches. But we do this without destroying anything. And in this way the tree of mathematics grows.

4.4 Further Investigations

F1. Find another subsequence of composite numbers in $\{s_n\}$ which is *not* a subsequence of one subsequences already found. (You will almost certainly need computer software - which is available readily online - to help you factor large numbers.)

Chapter 5

Proof

Proof is an idol before which the mathematician tortures himself.

Sir Arthur Eddington (British astrophysicist; 1882 - 1944)

A elegantly executed proof is a poem in all but the form in which it is written.

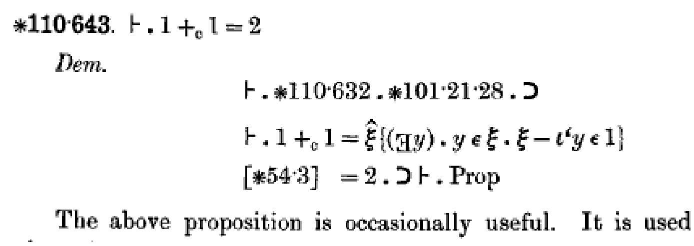
Morris Kline (American mathematician and educator; 1908 - 1992)

A good proof is one that makes us wiser.

Yuri I. Manin (Russian mathematician; 1937 -)

In the previous chapter you investigated how several different systems were built up from axioms, providing a solid foundation for deductive conclusions. Beginning with the foundation often necessitates a long, steep climb.

One of the most important contributions to the *foundations of mathematics* is the massive, three-volume <u>Principia Mathematica</u> by **Bertrand Russell** (British mathematician, philosopher, and author; 1872 - 1970) and **Alfred North Whitehead** (English mathematician and philosopher; 1861 - 1947) which sought to formalize mathematics from a single set of axioms and deduce results using the rules of symbolic logic. One of the results is the first fully *formalized* proof that $1+1=2$. Using several earlier results from the 680 page Volume I, this proof is completed on p. 68 of Volume II!



∗110·643. $\vdash . 1 +_c 1 = 2$

Dem.

$\vdash . *110{\cdot}632 . *101{\cdot}21{\cdot}28 . \supset$

$\vdash . 1 +_c 1 = \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota{}'y \in 1\}$

$[*54{\cdot}3] \quad = 2 . \supset \vdash . \text{Prop}$

The above proposition is occasionally useful. It is used

Figure 5.1: Last step in Russell and Whitehead's formal proof that $1+1=2$.

Although significant work on the foundations of mathematics continues, the day to day work of most mathematicians relies on proofs based on well accepted structures. For example, a well-known principles like the *distributive law* for integers is used regularly without concern.

In other words, up in the tree of mathematics, high above the foundations, mathematicians work building off of the results of others. As these previous results have been established deductively, mathematicians know that if they proceed from these established results their work will be valid as well. The role of deductive reasoning remains a sine qua non - essential ingredient - of the mathematical process. Only here, to signify that we are moving from known results to new results established logically, the words *proof* and *prove* are typically used to describe logical argument and logical process.

5.1 Number Theoretic Proofs

1. Explain what even and odd numbers are.

2. We often explain things intuitively. If you had to give a rigorous definition of even numbers, what would it be? What about odd numbers? Explain.

The typical definition of an **even number** is:
A positive integer is even if it can be written as $2n$ where n is some non-negative integer.
The definition of an **odd number** is analogous:
A positive integer is odd if it can be written as $2n+1$ where n is some non-negative integer.

3. Are your definitions in 2 equivalent to those just given? If so, prove your result. If not, provide an example which illustrates the difference.

4. Take several pairs of odd counting numbers and multiply each pair together. What do you notice about the products of these pairs of odd counting numbers?

5. Using the pattern you have observed in 4, state a conjecture that characterizes the product of any two odd counting numbers.

Here we demonstrate how this result can be proven deductively:

Proof. Denote the two counting numbers by a and b. By assumption, both a and b are odd. By definition this means that there are positive integers n and m so that $a = 2n+1$ and $b = 2m+1$. Then the product $a \cdot b$ is given by:

$$\begin{aligned} a \cdot b &= (2n+1)\cdot(2m+1) && (5.1)\\ &= 4nm + 2n + 2m + 1 && \text{(by the distributive law)}\\ &= 2\,(2nm + n + m)) + 1 && (5.2) \end{aligned}$$

$2nm + n + m$ is a positive integer and so, by definition, $a \cdot b$ is odd. □

6. Take several pairs of even counting numbers and add each pair together. What do you notice about the sums of these pairs of even counting numbers?

7. Using the pattern you have observed in 10 state a conjecture that characterizes the sum of any two even counting numbers.

8. Using the definition of even numbers, prove your conjecture about the sum of two even numbers.

9. Do you see a way to prove your conjecture using your definition in 2? Explain.

10. Take several pairs of odd counting numbers and add each pair together. What do you notice about the sums of these pairs of odd counting numbers?

11. Using the pattern you have observed in 10 state a conjecture that characterizes the sum of any two odd counting numbers.

12. Prove your conjecture about the sum of two odd numbers.

13. Take several pairs of counting numbers, one even and one odd, and multiply each pair together. What do you notice about the products of these pairs, one even and one odd, of counting numbers?

14. Using the pattern you have observed in 13 state a conjecture that characterizes the product of any two, one even and one odd, counting numbers.

15. Prove your conjecture about the product of an even and an odd number.

5.2 Proofs Without Words

Mathematical folklore holds that the great **Carl Freidrich Gauss** (German mathematician and scientist; 1777 - 1855) was once, as a very young child, scolded by being sent to the coat closet with a slate to determine the sum of the first hundred numbers: $1 + 2 + 3 + \ldots + 99 + 100$. The legend holds that he returned within a minute with the correct answer.

Figure 5.2 illustrates Gauss's method as it can be represented with blocks to determine the sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$.

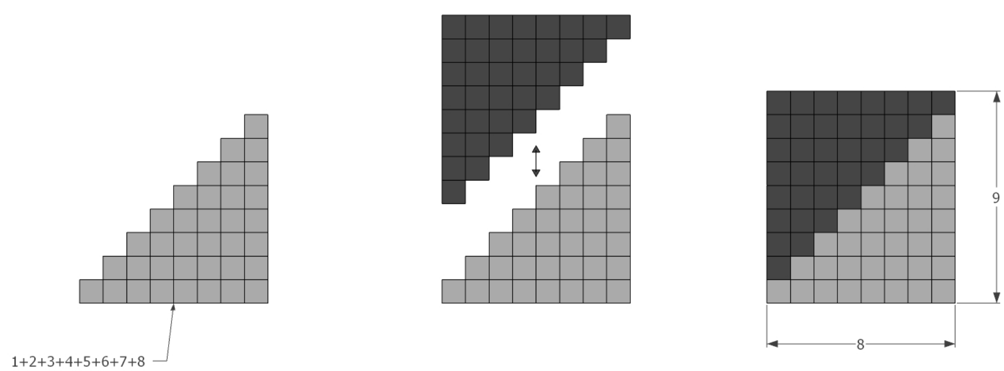


Figure 5.2: Determining the sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$.

16. Use Gauss's method to determine the sum Gauss was required to compute.

17. Use Gauss's method to determine the sum $1 + 2 + 3 + \ldots + 1,000,000,000,000$.

18. Suppose that n is a positive integer. Find an algebraic expression for the value of the sum $1 + 2 + 3 + \ldots (n - 2) + (n - 1) + n$.

19. Check that your result agrees with your answers to the two explicit problems computed previously.

20. Explain how Figure 5.3 provides a *proof without words* which proves the general result in 19

21. Determine the value of the following sums:

- $1 + 2 + 1$
- $1 + 2 + 3 + 2 + 1$
- $1 + 2 + 3 + 4 + 3 + 2 + 1$

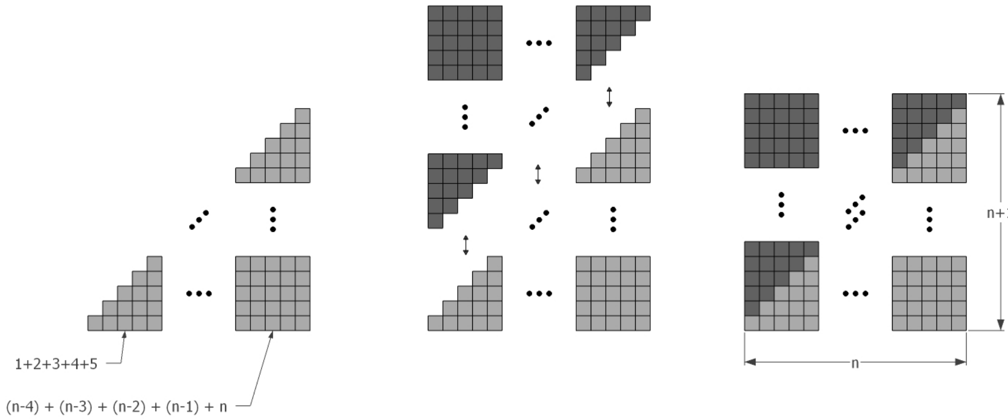


Figure 5.3: Determining the sum $1 + 2 + 3 + \ldots (n-2) + (n-1) + n$.

- $1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1$

22. What pattern do you see? Describe this pattern using the language of an algebraic equation.

23. Find a proof without words for this result.

24. Determine the value of the following sums:

- $1 + 3$
- $1 + 3 + 5$
- $1 + 3 + 5 + 7$
- $1 + 3 + 5 + 7 + 9$

25. What pattern do you see? Describe this pattern using the language of an algebraic equation.

26. Find a proof without words for this result.

5.3 Mathemagical Tricks

27. Choose two single-digit numbers. Then perform the following computations in order:

- Multiply the first number by 2
- Add 3 to the result
- Multiply the sum by 5
- Add the second number to the product
- Multiply the sum by 10.

Show these computations step by step and write down the end result.

A mathemagician - human in the form of your teacher or a peer, will now divine the identity of your two numbers from the result of your computation.

28. Do you think this is a compelling trick? Explain.

29. In an effort to understand how this trick worked, compile a list of beginning numbers and the final computations.

30. From this list in Investigation **29**, can you determine how it was that the mathemagician divined the two numbers in question? Explain.

31. It wouldn't be much of a trick if it only worked sometimes. Use algebra to prove that this trick will work for any pair of beginning numbers.

Here's another trick.

32. Choose a secret number. Then perform the following computations:

- Add 1 to the number chosen
- Multiply the sum by 3
- Add the square of the original number to this product
- Multiply the sum by 4
- Subtract 3 from the product
- Take the square root of the difference.

Show these computations step by step and write down the end result.

A mathemagician - human in the form of your teacher or a peer will now divine the identity of your two numbers from the result of your computation.

33. Do you think this is a compelling trick? Explain.

34. In an effort to understand how this trick worked, compile a list of beginning numbers and the final computations.

35. From this list in Investigation **34**, can you determine how it was that the mathemagician divined the two numbers in question? Explain.

36. It wouldn't be much of a trick if it only worked sometimes. Use algebra to prove that this trick will work for any beginning number.

5.4 Triangle Proofs

One of the fundamental results of Euclidean geometry, the typical high school geometry, concerns the sum of the interior angles in a triangle.

37. Define, in your own words, what an interior angle of a triangle is.

38. What is the sum of the interior angles in a triangle? State this result as a theorem. We will refer to this theorem as the **Triangle Sum Theorem**.

39. Do you have an intuitive understanding of why this result holds? If so, explain in detail. If not, is it surprising to you that every triangle has the same interior angle sum? Explain.

Our goal below will be to understand/prove why this result holds.

5.4.1 Measurement

A ***protractor*** is the typical tool used to measure angles.

40. Carefully draw a triangle.

41. Measure each of interior angles in your triangle. What is their sum?

42. Compare your sum with those of your peers. Do your answers agree? If not, try to understand why they do not agree. If so, does this prove the Triangle Sum Theorem? Explain.

43. Cut out your triangle. Now tear off the corners of your triangle. Is there a way you can arrange these three pieces to determine the sum of the angles in the original triangle? Explain.

44. Compare your sum with those of your peers. Do your answers agree? If not, try to understand why they do not agree. If so, does this prove the Triangle Sum Theorem? Explain.

45. Carefully draw another triangle, using a significant portion of a piece of paper. Carefully cut out your triangle. Make a single fold in your triangle so i) one corner of your triangle is folded over so it lies directly on the opposite side, and, ii) the fold line is parallel to the side the corner has been folded on to in i).

46. Can you find two more folds that enable you to determine the sum of the angles in your triangle? Draw a picture which illustrates your construction.

47. How does your origami-like construction compare to those of your peers? Do your results prove the Triangle Sum Theorem?

5.4.2 Myopia

48. Carefully draw another triangle.

49. At one vertex of the triangle continue one of the edges to form an infinite *ray* that extends beyond the triangle. You have created the ***exterior angle*** of the triangle at that vertex. Highlight and label this angle.

50. Now draw infinite rays that extend each of the other edges to show the other exterior angles of your triangle. (Note: At each vertex you have a choice of which line to extend. Extend the edge so that your figure looks like a pinwheel when complete; i.e. all of the exterior angles are measured in the same direction, either clockwise of counter-clockwise.)

51. At a given vertex how is the interior angle related to the exterior angle?

52. Carefully redraw your triangle and its extended edges as if you were viewing it from farther away.

53. Repeat Investigation **52** if you viewed your figure from even farther away.

54. Repeat Investigation **52** if you viewed your figure from even farther away.

55. Carefully redraw your figure as if you were standing infinitely far away.

56. Do these images suggest a value for the sum of the *exterior* angles of your triangle?

57. Use your result in Investigation **56** to prove the Triangle Sum Theorem.

5.4.3 Turtles

Seymour Papert (South African Educator; 1928 -) is one of the inventors of the computer programming language ***Logo*** which many generations of children have used to acquire:

> a sense of mastery over a piece of the most modern and powerful technology and. . . [establish] an intimate contact with some of the deepest ideas from science, from mathematics, and from the art of intellectual model building.[1]

You and some of your peers may remember working with the "Turtles" that drew graphics on a screen or physically on a floor. Papert grew to be disappointed in many of the ways the computer was used in education.[2] Nonetheless, Logo continues to be widely used by both children and adults[3] and Papert's role in predicting some of the impact of computers is remarkable. Well before personal computers, iPods, and the general availability of any type of wireless phone, Papert wrote:

> Computers can be carriers of powerful ideas and of the seeds of cultural change... They can help people form new relationships with knowledge that cut across the traditional lines separating humanities from sciences and knowledge of the self from both of these.[4]

Figure 5.4: Children programming the Logo Turtle.

Logo also provides a *bodily kinesthetic* vehicle for learning that is particularly valuable to some learners. In this section we will ask you to mimic what Logo would do by being a Turtle yourself.

58. With a few peers, create a triangle. Carefully measure the length of its sides and its angles.

59. You are now to write instructions to tell the Logo Turtle how do draw the triangle. You can tell it to turn right or left any angle. You can tell it to move forward or back any distance. As it moves, it will draw a line. You can assume that the Turtle starts at any vertex you like and is already facing a specific direction of your choice.

[1]From Mindstorms: Children, Computers, and Powerful Ideas, p. 5.

[2]See e.g. The Children's Machine: Rethinking School in the Age of the Computer.

[3]See e.g. StarLogo which is a powerful tool in modeling simulations which involve thousands of independent entities and the wonderful book Turtle Geometry: The Computer as a Medium for Exploring Mathematics which has key insights into *fractals*, computer graphics, and *artificial intelligence*.

[4]From Mindstorms: Children, Computers, and Powerful Ideas, p. 4.

60. Now test to see if your computer program actually draws the correct triangle. (Warning: You REALLY are supposed to physically try this - most people get this wrong because they think it is “too easy”. Use Logo if you want, there are free applets online, such as `http://www.mathsnet.net/logo/turtlelogo/index.html`. It's more fun to actually do it. Make a larger version of your triangle (perhaps by changing inches to feet) whose you tape on the floor using masking tape. Now blindfold a peer and give her/him the instructions from your program. See if your instructions really are correct.) Describe what happened in your test.

61. When your Turtle returns to its starting spot, there is no need to have it turn, the triangle is complete. But suppose that you include an instruction so the Turtle is turned to its original orientation as the last step of your instructions. What would this instruction be?

62. With this final instruction, what is the total number of degrees the Turtle has turned in making the Triangle?

63. Does the sum of the degrees turned by the Turtle depend on the specific triangle that it draws? Explain.

64. Use this result to prove the Triangle Sum Theorem.

5.4.4 Tessellations

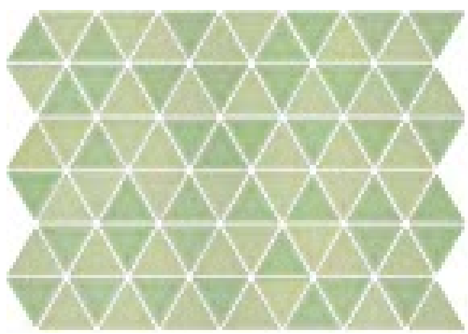


Figure 5.5: A ceramic tile floor - a tessellation by equilateral triangles.

Figure 5.5 shows a tiled floor. Because the floor is entirely covered by the tiles we call this arrangement a ***tessellation of the plane***.

65. Draw an *isosceles triangle*. Show how you can tessellate the plane with your triangle.

Seeing this might make you wonder if you can tessellate with an arbitrary triangle.

66. Draw a non-isosceles triangle. Now make a template of this triangle out of cardstock or cardboard.

67. On a large sheet of paper or chalkboard, use the template to draw a copy of your triangle. Now see what ways you can find to put another copy of your triangle together with the first to begin to create a tessellation.

68. Now add several more triangle copies as you try to tessellate. You may have to try several different arrangements - perhaps even very early in your arrangement.

69. Eventually you should be able to find a way to tessellate, regardless of which triangle you started with. Do so. Describe how the orientation of adjacent triangles in your tessellation are related. How might your tessellation be colored to illustrate this pattern?

70. Look at any vertex of your tessellation. Which angles surround this vertex? Is this true at every vertex?

71. Use this to prove the Triangle Sum Theorem for your triangle.

72. The applet available at `www.westfield.ma.edu/ComingSoon` allows you to dynamically change the shape of the initial triangle and see what happens to your tessellation. Do these tessellations provide a proof of the Triangle Sum Theorem? Why or why not?

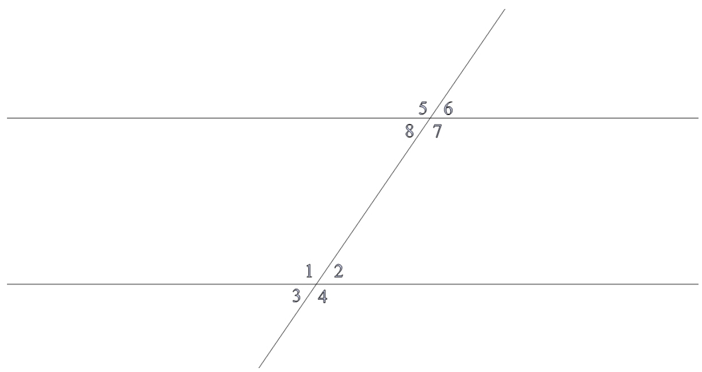


Figure 5.6: Parallel lines cut by a transversal.

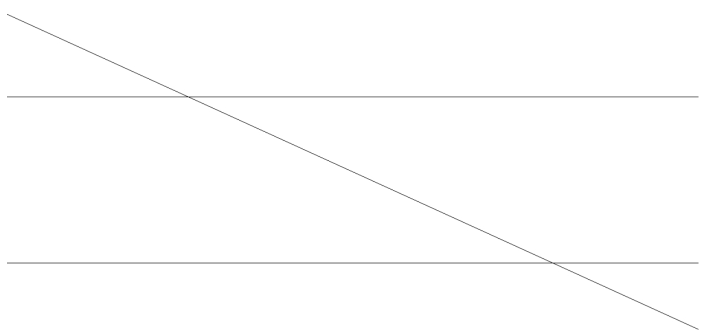


Figure 5.7: Parallel lines cut by a different transversal.

One of the earliest proofs of the Triangle Sum Theorem is reminiscent of your construction above. It appears as Proposition 32 in Book I of Euclid's Elements.

Euclid's proof relies on a fundamental result about parallel lines that he proves as Proposition 29 in Book I.

73. Pictured in Figure 5.6 is a pair of parallel lines cut by another line, usually called a ***transversal***. How are the angles labeled 1 - 8 related to one another?

74. Pictured in Figure 5.7 is the same set of parallel lines cut by a different transversal. How are the angles in this figure related to one another?

75. Is there a general result that holds for every transversal that intersects these two lines? If so, state it as a theorem and explain - at least intuitively - how you know this result is true. If not, illustrate the limitation with an example.

76. Figure 5.8 shows both transversals cutting the pair of parallel lines. Use your results above to *prove* the Triangle Sum Theorem for this one triangle.

77. If you are given a different triangle can this process be repeated? Explain. Does this give you a proof of the general result? Explain.

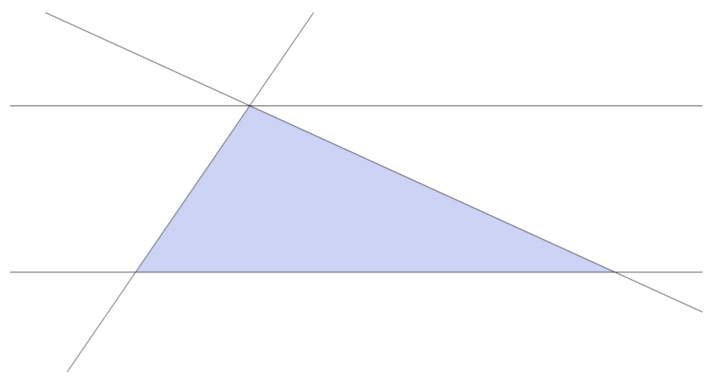


Figure 5.8: The triangle formed by the two different transversals.

5.4.5 Triangle Sum Theorem Conclusions

78. Which of the investigations above help you most in understanding why the Triangle Sum Theorem is true?

79. Which proof of the Triangle Sum Theorem is most compelling to you?

5.5 A Famous Proof

We return now to the question of how many prime number there are.

In Investigation **24** it is hoped that you made the following conjecture:

Conjecture 2. *If the number $2 \cdot 3 \cdot 5 \ldots p_n + 1$ is formed where $2, 3, 5, \ldots, p_n$ are consecutive primes, then any prime divisor of this number must be greater than p_n.*

80. If you form the number $2 \cdot 3 \cdot 5 \ldots p_n + 1$ is formed where $2, 3, 5, \ldots, p_n$ are consecutive primes, can any of the primes $2, 3, 5, \ldots, p_n$ divide this number? Explain.

81. Explain how the previous Investigation proves Conjecture 2.

82. Explain how this establishes, deductively, the following theorem:

Theorem 7. *Let the number $N_n = 2 \cdot 3 \cdot 5 \ldots p_n + 1$ be formed where $2, 3, 5, \ldots, p_n$. Then either:*

- *N_n is prime, or,*
- *N_n is divisible by some prime larger than p_n.*

83. Suppose that there were only finitely many primes. Then there would have to be a largest prime. Denote this largest prime by $p_{Largest}$. If we apply the previous theorem, what happens? What does this tell you about our assumption.

84. Explain how our results prove:

Theorem 8 (Euclid)**.** *There are infinitely many primes.*

85. Does it seems strange to you that you were able to *prove* that there were infinitely many primes without providing a way of explicitly generating prime numbers indefinitely? Explain.

The theorem above was proven by Euclid in his famous collection The Elements. The proof here is the original Euclidean proof. It is among the most famous proofs in all of mathematics.[5]

5.6 A New Kind of Science

In 2002 **Stephen Wolfram** (British Mathematician and Computer Scientist; 1959 -) self-published the magnum opus A New Kind of Science. The importance of this highly anticipated book continues to be vigorously debated. Wolfram is not shy it touting the impact of "his" work:

> Three centuries ago science was transformed by the dramatic new idea that rules based on mathematical equations could be used to describe the natural world. My purpose in this book is to initiate another such transformation, and to introduce a new kind of science that is based on much more general types of rules that can be embodied in simple computer programs.[6]

This "New Science", Wolfram tells us, is:

> . . . one of the more important single discoveries in the whole history of theoretical science. For in addition to opening up vast new domains of exploration, it implies radical rethinking of how processes in nature and elsewhere work.[7]

If one didn't know better the term "crackpot" might seem appropriate. But Wolfram's accomplishments are remarkable. He earned a Ph.D. in physics from the California Institute of Technology at age 20. At age 21 he earned a prestigious MacArthur Fellowship (a.k.a. Genius Award). At 24 he was on the faculty at the Institute for Advanced Study at Princeton University. At 28 he co-founded Wolfram Research which is responsible for the design and development of the profoundly important mathematical software ***Mathematica*** and more recently the computational knowledge engine ***Wolfram Alpha***. This company has made him a millionaire.

Wolfram describes one of his fundamental conclusions in A New Kind of Science as follows:

> One might have thought - as at first I certainly did - that if the rules for a [computer] program were simple then this would mean that its behavior must also be correspondingly simple. For our everyday experience in building things tend to give us the intuition that creating complexity is somehow difficult, and requires rules or plans that are themselves complex. But the pivotal discovery that I made some eighteen years ago is that in the world of programs such intuition is not even close to correct.[8]

[5] For a new, short and very elegant proof see the Further Investigations section of Discovering the Art of Mathematics: Number Theory.

[6] A New Kind of Science, p. 1

[7] A New Kind of Science, p. 2.

[8] A New Kind of Science, p. 2.

The importance of this observation?

> [In human building and engineering] we restrict ourselves to systems whose behavior we can readily understand and predict - for unless we can foresee how a system will behave, we cannot be sure that the system will do what we want. But unlike engineering, nature operates under no such constraint. So there is nothing to stop systems like those at the end of the previous sections from showing up. And in fact one of the important conclusions of this book is that such systems are very common in nature.[9]

For more on this book and its author, see the Further Investigations section.

We introduce here two simple systems that are similar to those that underly Wolfram's "New Science."

5.7 Iteration and the Collatz Conjecture

86. Pick a positive whole number. Then apply the rule below to your number. What is the result of your *evaluation*?

> **Collatz Rule** If the number is even, divide it by two. If the number is odd, multiply by three and then add one.

87. Now apply the Collatz Rule to the result of Investigation **86**. What is the result of your evaluation?

88. Now apply the Collatz Rule to the result of Investigation 87. What is the result of your evaluation?

89. Apply the Collatz Rule at least a dozen more times, each time using the result of the previous application as the next number for which the rule is applied. Describe the results of your evaluations. (One typical way of writing the ***iterates*** is to denote the seed value by N_0, the first iterate by N_1, etc.)

90. Pick another starting value, often called a ***seed***, and repeat Investigations 86 - 89 for this seed value.

91. Repeat 90 for a third seed.

92. Repeat 90 for a fourth seed.

The process of repeatedly applying a rule or function to the previous output is called ***iteration*** of the rule/function. Starting with a specific seed value the sequence of outputs is called the ***orbit*** of the rule/function for this seed value.

In 1937 **Lothar Collatz** (German Mathematician; 1910 - 1990) stated what has become known as the **Collatz conjecture**:

Conjecture 3 (Collatz)**.** *Choose any positive integer seed value. If you iterate the Collatz rule repeatedly, eventually the output will reach 1 and then continue indefinitely as* $1, 4, 2, 1, \ldots$.

93. Compare your results with your peers. Do all of your results agree with Collatz's conjecture? Explain.

[9] A New Kind of Science, p. 40.

The Collatz conjecture remains unsolved to this day and the great character **Paul Erdös** (Hungarian Mathematician; 1913 - 1996) even says of it, "Mathematics is not yet ready for such confusing, troubling, and hard problems."

The Collatz rule can be expressed algebraically as a simple mathematical function:

$$c(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

Let's see what happens when we iterate a simpler function. Let's consider the function

$$s(n) = n + 1.$$

94. Iterate the function s starting from the seed value $n = 1$.

We denote this function by s because it is the *successor function* and starting from a single seed value it generates all of the positive whole numbers. This is important to foundational aspects of the whole numbers and, as seen in the Discovering the Art of the Infinite, to our study of the infinite.

5.8 The Number of Prime Numbers

Here we investigate another iterative process, one which provides a beautiful proof of the infinitude of the prime numbers. The rule for this iteration, which we will call the **Saidak rule**, is as follows. At each stage the result is the product of two numbers:

- The previous result, and,
- The number one larger than the previous result.

Example: Suppose our seed value is N_0=2. Then our iterates are:

$$\begin{aligned} N_1 =& 2 \times 3 = 6 \\ N_2 =& 6 \times 7 = 42 \\ N_3 =& 42 \times 43 = 1806 \\ N_4 =& 1806 \times 1807 = 3263442 \\ &\vdots \end{aligned}$$

95. Find the first five iterates when the seed value is $N_0 = 3$.

96. Choose your own starting seed value N_0. Find the first five iterates corresponding to this seed value.

97. Repeat Investigation **96** for a new seed value of your choice.

98. Repeat Investigation **96** for one more seed value of your choice.

We would like to determine which prime numbers evenly divide each of the iterates. Such divisors are naturally called the **prime factors**. As shown in the example below, we can assemble all of the prime factors into the ***prime factorization*** of each iterate:

$$\begin{aligned} N_0 =& 2 = 2 \\ N_1 =& 6 = 2 \cdot 3 \\ N_2 =& 42 = 2 \cdot 3 \cdot 7 \\ N_3 =& 1806 = 2 \cdot 3 \cdot 7 \cdot 43 \\ &\vdots \end{aligned}$$

99. Notice that in each of the first three iterations, the prime factorization of the iterate includes *exactly one* new prime factor in addition to prime factors of the iterate that came before it. If this continued to happen indefinitely, what would this tell us about the number of prime numbers that exist?

100. Find the prime factorization of the iterate $N_4 = 3263442$. Does it include exactly one more prime factor?

101. Find the prime factorizations of the iterates from Investigation **96**. What do you notice about the prime factors and the number of prime factors that are created as we iterate?

102. Find the prime factorizations of the iterates from Investigation **97**. What do you notice about the prime factors and the number of prime factors that are created as we iterate?

103. Find the prime factorizations of the iterates from Investigation **98**. What do you notice about the prime factors and the number of prime factors that are created as we iterate?

It should be clear that the common factors - prime or non-prime - of a number, n, and its immediate successor $n + 1$ are important in this iterative process.

104. Choose a positive integer n. What common factors do n and $n + 1$ share?

105. Repeat Investigation **104** for several other integers n.

106. State a conjecture about the common factors that any positive integer n and its immediate successor $n + 1$ share.

107. Prove the conjecture in Investigation **106**.

108. Having established this conjecture deductively, this should enable you to definitively establish a result about number of prime factors that result from each iteration of the Saidak rule.

109. Explain how this proves that there are infinitely many prime numbers. Does the seed value, N_0, of the iteration matter?

This is an amazing thing - you have deductively established that there are infinitely many prime numbers.

We've suggested investigations to approach this task. Return for a moment and analyze which steps are absolutely necessary. Which of the investigations were there to help with the learning but are not required as part of a rigorous proof?

110. Write a complete, rigorous, entirely self-contained proof that there are infinitely many primes as economically as you can.

Earlier you rediscovered Euclid's proof of the infinitude of the primes from The Elements. Many, many other proofs exist. The simple and elegant proof rediscovered above is only 5 years old.[10] It was discovered (or should we say invented?) by **Filip Saidak** (; -). He thought of this proof while waiting for a bus at a bus stop![11]

> Proofs aren't there to convince you that something is true - they're there to show you why it is true.
>
> **Andrew Gleason** (American mathematician; 1921 - 2008)

[10] "A new proof of Euclid's theorem," *American Mathematical Monthly*, vol. 113, no. 10, December 2006, pp. 937-8.

[11] Personal communication.

5.9 Connections

5.9.1 Philosophy

One of the fundamental questions of philosophy is the question of ***free will***. Wolfram describes this problem as follows:

> Ever since antiquity it has been a great mystery how the universe can follow definite laws while we as humans still often manage to make decisions about how to act in ways that seem quite free of obvious laws.[12]

111. INDEPENDENT INVESTIGATION: Find out more about the philosophical aspects of Wolfram's thesis in A New Kind of Science. What are the implications of Wolfram's work to the question of freewill?

5.9.2 Religion

One central argument for the existence of God is called the ***watchmaker argument***. While this argument has ancient roots, the writings of **William Paley** (English Priest; 1743 - 1805) play a prominent role in the debate over this argument as they were explicit and reached a large audience. His statement of this argument begins as follows:

> When we come to inspect the watch, we perceive... that its several parts are framed and put together for a purpose, e.g. that they are so formed and adjusted as to produce motion, and that motion so regulated as to point out the hour of the day; that if the different parts had been differently shaped from what they are, or placed after any other manner or in any other order than that in which they are placed, either no motion at all would have been carried on in the machine, or none which would have answered the use that is now served by it... The inference we think is inevitable, that the watch must have had a maker – that there must have existed, at some time and at some place or other, an artificer or artificers who formed it for the purpose which we find it actually to answer, who comprehended its construction and designed its use.[13]

And his striking conclusion:

> The marks of design are too strong to be got over. Design must have had a designer. That designer must have been a person. That person is GOD.[14]

The watchmaker argument plays an important, perhaps defining, role in contemporary controversy over the teaching of *creationism* and *intelligent design* in American public schools. Of course, generations of scientists - including **Charles Darwin** (English biologist; 1809 - 1882) who said that Paley's works "gave me as much delight as did Euclid. The careful study of these works, without attempting to learn any part by rote, was the only part of the Academical Course which, as I then felt and as I still believe, was of the least use to me in the education of my mind."[15] - have fought against these views. Wolfram claims that his "New Science" has central relevance in this debate. He says:

[12] A New Kind of Science, p. 750.
[13] From Natural Theology: or, Evidences of the Existence and Attributes of the Deity, Collected from the Appearances of Nature.
[14] From Natural Theology: or, Evidences of the Existence and Attributes of the Deity, Collected from the Appearances of Nature.
[15] From his Autobiography of Charles Darwin.

> One of the most striking features of the natural world is that across a vast range of physical, biological and other systems we are continually confronted with what seems to be immense complexity. And indeed throughout most of history it has been taken almost for granted that such complexity - being so vastly greater than in the works of humans - could only be the work of a supernatural being. But my discovery that many very simple programs produce great complexity immediately suggest a rather different explanation. For all it takes is that systems in nature operate like typical programs and then it follows that the behavior will be complex.[16]

112. INDEPENDENT INVESTIGATION: Without finding out more about it, what are your personal views on the watchmaker argument? Now find out a bit more about this argument. In particular, what other roles have the sciences and mathematics played in this debate? As the focus of this book is proof, logic, truth, and certainty, focus at least some of your efforts on these topics. How, if at all, does this new information effect and/or inform your personal views?

5.9.3 Biology

Wolfram claims that his "New Science" has important things to say about biology.

> Vast amounts are now known about the details of biological organisms, but very little in the way of general theory has ever emerged. Classical areas of biology tend to treat evolution by natural selection as a foundation - leading to the notion that general observations about living systems should normally be analyzed on the basis of evolutionary history rather than abstract theories. And part of the reason for this is that traditional mathematical models have never seemed to come even close to capturing the kind of complexity we see in biology. But the discoveries in this book show that simple programs can produce a high level of complexity.[17]

While many (most?) would agree with this statement, Wolfram's claims about his theory are much more grandiose:

> Most obvious examples of complexity in biological systems actually have very little to do with adaptation or natural selection. and instead what I suspect is that they are mainly just another consequence of the very basic phenomenon that I have discovered in this book in the context of simple programs: that in almost any kind of system many choices of underlying rules inevitably lead to behavior of great complexity.[18]

113. INDEPENDENT INVESTIGATION: To get a sense for the type systems that "evolve" from simple programs, go online and play the *Game of Life* - a *cellular automata* of exactly the type that form the basis of Wolfram's "New Science". The Game of Life was invented by **John Horton Conway** (English Mathematician; 1937 -) in 1970. Similarly, go online and find out about *Star Logo* which is used to model large populations and their interactions. Do either of these programs remind you of games that you know? Can you see how programs such as these might impact our ability to view, understand, model, and/or describe large systems - like those that we study in biology?

[16] A New Kind of Science, pp. 2-3.
[17] A New Kind of Science, pp. 8-9.
[18] A New Kind of Science, p. 383.

114. Independent Investigation: Interview someone with expertise in the biological sciences. If they do not know about "simple programs" like those described above, describe these to them. Then ask them what they think about Wolfram's claims as described above.

5.9.4 Mathematical Culture

A stereotype of mathematics is that it is a "dry and arid science" where problems have definitive solutions and there is no room for debate. This is not true. Wolfram's publication of A New Kind of Science provides an interesting opportunity to observe debate in mathematics and the mathematical sciences.

As indicated in the initial description, the publication of A New Kind of Science set off great debate on many fronts.

The review in the *New York Times Book Review* closed by saying, "No one has contributed more seminally to this new way of thinking about the world. Certainly no one has worked so hard to produce such a beautiful book. It's too bad more science isn't delivered in this way."[19]

Yet in the *New York Review of Books* Nobel prize winning physicist **Steven Weinberg** (American physicist; 1933 -) says, "The trouble with Wolfram's conjecture is not only that it has not been proved - a deeper trouble is that it has not even been stated in a form that could be proved."[20] He continues to dismiss much of what Wolfram preaches as already known, saying, "The strongest reaction I have seen by scientists to this new book has been outrage at Wolfram's exaggeration of the importance of his own contributions to the study of complexity."

In the mathematical literature the views have been no less divisive. In the prestigious *American Mathematical Monthly* the well-known **Rudy Rucker** (American mathematician and author; 1946 -) says Wolfram's ideas will "likely be discussed for years to come." He calls the book a "landmark achievement." He describe the "paradigm shift that Wolfram proposes is to focus instead on finding simple computations that model reality."[21]

Yet Wolfram's book is openly mocked by **Steven G. Krantz** (American mathematician and author; 1951 -) in the prestigious *Bulletin of the American Mathematical Society*. Krantz quotes noted physicist **Freeman Dyson** (British physicist and mathematician; 1923 -) as saying, "There's a tradition of scientists approaching senility to come up with grand, improbably theories. Wolfram is unusual in that he's doing this in his 40's." Krantz says the book "devolves to a low common denominator of repetition, hand waiving, and pap." He continues to say, "What we are given... is a practical guide to creating empirical computer phenomena. This is not science."[22]

115. Independent Investigation: Find several other legitimate reviews of A New Kind of Science. What do you find interesting about these reviews? What do these reviews and the discussion above tell you about debate and disagreement in mathematics?

116. Independent Investigation: A New Kind of Science is not rare in the way in which it fanned great debate in mathematics. Throughout its history there have been many famous great debates. Find an example of a great debate in mathematics from the last 100 years. Learn enough about it so you can briefly describe it and use it to inform your view about the nature of mathematical progress.

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[19] From "You know that space-time thing? Never mind" by George Johnson, 9 June, 2002.
[20] From "Is the universe a computer?", 24 October, 2002.
[21] Vol. 110, No. 9, November, 2003, pp. 851-861.
[22] Vol. 40, No. 1, October, 2002, pp. 143-50.

5.9.5 Art

In speaking of Art, Wolfram says:

> It seems so easy for nature to produce forms of great beauty. Yet in the past art has mostly just had to be content to imitate such forms. But now, with the discovery that simple programs can capture the essential mechanisms for all sorts of complex behavior in nature, one can imagine just sampling such programs to explore generalizations of the forms we see in nature. Traditional scientific intuition - and early computer art - might lead one to assume that simple programs would always produce pictures too simple and rigid to be of artistic interest. But looking through this book it becomes clear that even a program that may have extremely simple rules will often be able to generate pictures that have striking aesthetic qualities - sometimes reminiscent of nature, but often unlike anything ever seen before.[23]

117. INDEPENDENT INVESTIGATION: Find several different types of art that have been created by "simple programs" that you find aesthetically pleasing, valuable, and/or "striking". Describe why you have chosen each piece of art. For each also provide a brief description of the type of "simple program" that was used to create the art.

[23] A New Kind of Science, p. 11.

Chapter 6

Limits of Knowledge - Proving Impossibility

> The chess-board is the world; the pieces are the phenomena of the universe; the rules of the game are what we call the laws of Nature.
>
> **Thomas Huxley** (English biologist; 1825 - 1895)

> The unique and peculiar character of mathematical reasoning is best exhibited in proofs of impossibility. When it is asserted that doubling the cube (i.e. in constructing the cube root of two) is impossible, the statement does not merely refer to a temporary limitation of human ability to perform this feat. It goes far beyond this, for it proclaims that never , no matter what will anybody ever be able to construct the cube root of two...if the only instruments at his disposal are a straightedge and a compass. No other science, or for that matter no other discipline of human endeavor, can even contemplate anything of such finality.
>
> **Mark Kac** (Polish mathematician; 1914 - 1984)
>
> **Stanislaw Ulam** (Polish mathematician; 1909 - 1984)

In the opening chapter "Doubt" we saw many examples of the ways in which our knowledge was limited. Optical illusions, sensory misperceptions, and paradigms challenged what we know and how we know it. Awareness of these obstacles can often help us move beyond what were once seen as limitations - in our perspectives, our biases, and our closed-mindedness.

In contrast, in this chapter we find limits of knowledge that cannot be surpassed. We begin to see that there are absolute limits to knowledge. We *prove* that certain things are impossible.

How is it possible to prove something is impossible? Perhaps we have simply not figured out the appropriate way to go about it? We need a new approach, a new perspective.

To set the stage, remember that deductive proof begins from a set of axioms or a collection of theorems already established from a set of axioms for the system. This creates what Seymour Papert[1] calls a ***microworld*** - "a small playgrounds of the mind"[2]. **Larry Latour** (Computer Scientist and Educator; -) has referred to as "tiny worlds inside which a student can explore alternatives, test hypotheses, and discover facts that are true about that world."[3]

What we shall discover is that what we know is limited by our epistemological assumptions about the world we live in, about the paradigms that we choose to follow or the axioms we choose for our system.

[1]In Chapter 5 Seymour Papert and his powerful computer programming language for children, Logo, was described. Papert admitted, "Logo does not in itself produce good learning any more than paint produces good art" but was rather a tool to "support the development of new ways of thinking and learning." From Mindstorms, p. xiv.

[2]Clements, 1989.

[3]http://www.umcs.maine.edu/ larry/microworlds/microworld.html

6.1 A Famous Puzzle

> Hey, you wanna focus on the problem at hand?
>
> **Zeus Carver** (Character played by Samuel L. Jackson; -)

The 1995 hit movie Die Hard with a Vengeance focuses on a terrorist who has placed bombs around New York City. Detective John McClane, played by **Bruce Willis** (American actor; 1955 -), is lead by the terrorist on a deadly game of Simon Says. In one scene McClane and Zeus Carver, played by **Samuel L. Jackson** (American actor and director; 1948 -), are lead to a metal briefcase on the side of a fountain in a crowded city playground. McClane opens the briefcase. A video display monitor reads:

> I am a bomb. You have just armed me.

A phone in the briefcase then rings. The terrorists tells McClane and Carver:

> On the fountain there should be two jugs. . . a five gallon and a three gallon. Fill one of the jugs with exactly four gallons of water and place it on the scale and the [bomb] timer will stop. You must be precise, one ounce or more/less will result in detonation. If you're still alive in five minutes. . .

1. Check the exact time right now. Try to solve this problem in five minutes. If you were unable to solve the problem in five minutes, continue until you can solve it. When you are done, report on how long it took you. More importantly, describe how you worked on this problem, what obstacles you faced, and how you were eventually able to solve the problem.

We'd like to work on some similar problems. So let us make the problem a bit more precise. First, you have an unlimited volume of water at your disposal to use. Second, you have *only* the two jugs to use. The required amount of water *must* be exactly contained in one of these jugs.

2. With the same set-up as above, can you fill one of the jugs with exactly 2 gallons? If so, explain how. If not, can you *prove* why you cannot?

3. With the same set-up as above, can you fill one of the jugs with exactly 1 gallon of water? If so, explain how. If not, can you *prove* why you cannot?

Siméon Denis Poisson (French mathematician, scientist, and teacher; 1781 - 1840) is one of the greatest French scientists of all time and is also one of the most important mathematicians of the nineteenth century. He was fragile as a young child and had been predeceased by several older brothers and sisters. His father was a soldier who was discriminated against prior to the French Revolution of 1789. After the revolution the elder Poisson had high hopes that his oldest surviving son would rise to a higher social status than his parents.

In Mathematics and the Imagination **Edward Kasner** (American mathematician; 1878 - 1955) and **James Newman** (American mathematician; 1907 - 1966) tell us:

> Poisson's family tried to make him everything from a surgeon to a lawyer, that last on the theory that he was fit for nothing better. One or two of these professions he tackled with singular ineptitude.

They continue by telling us how he found his inspiration:

> It was on a journey that someone posed him a problem similar to the one below. Solving it immediately, he realized his true calling and thereafter devoted himself to mathematics.

The problem, Investigation **4** - Investigation **5**, is a version of the one we have been considering.

4. An 8 liter jug of wine is to be shared among two friends. In addition to the 8 liter jug, they friends have a 5 liter jug and a 3 liter jug. They would like to share the wine equally without spilling any. How is this problem similar to the "Die Hard" problem? How is it different?

5. Solve this problem. As above, when you are done, report on how long it took you. More importantly, describe how you worked on this problem, what obstacles you faced, and how you were eventually able to solve the problem.

Poisson was not the only famous mathematician to be intrigued by this problem. **Niccol Fontana Tartaglia** (Italian Mathematician and Engineer; 1499 - 1557), who will be highlighted soon for his solution of the cubic equation and the great debate that this sparked, also worked on this problem.

Of course, there are many ways we can state problems like this. Let's return to the type of problem posed in Die Hard with a Vengeance - unlimited supply of water and two jugs.

6. Suppose you have an 11 liter jug and a 6 liter jug. Can you measure 5 liters?

7. Can you measure 3 liters? How or why not?

8. Can you measure 8 liters? How or why not?

9. Is there any number of liters (≤ 11) that you cannot measure? Explain in detail.

10. Now suppose you have an 10 liter jug and a 6 liter jug. Can you measure 4 liters?

11. Can you measure 8 liters? How or why not?

12. Can you measure 5 liters? How or why not?

13. Is there any number of liters (≤ 10) that you cannot measure? Explain in detail.

14. Now suppose you have an 15 liter jug and a 6 liter jug. Can you measure 9 liters?

15. Can you measure 12 liters? How or why not?

16. Can you measure 13 liters? How or why not?

17. Is there any number of liters (≤ 15) that you cannot measure? Explain in detail.

18. Do you see a pattern that you can extend to make a conjecture about what units can be measured for a specific choice of jug sizes? Explain precisely.

This problem can be translated into an algebraic problem. Once this is done there is a deep and beautiful connection between this problem and the solution of *Diophantine equations* of the form $n \cdot x + m \cdot y = gcd(n, m)$ where n and m are the sizes of the jugs. (For more see the Further Investigations section.)

19. We have put this puzzle here because we believe that each of these puzzles create a simple microworld. Explain why this may be appropriate.

20. For at least one of the situations you have found an amount that you cannot seem to measure, *prove* rigorously that this amount cannot be measured

21. Have you found a limit to knowledge in this microworld? How firm is this limit? Explain.

6.2 Pythagorean Rationalism

All is number.

Pythagoras (Greek mathematician and philosopher; circa 570 BC - circa 495 BC)

The ***Pythagoreans*** were a cult-like group of followers of Pythagoras who had a significant influence on Greek intellectual culture for several centuries, having a significant influence on greats like Plato and Aristotle. The Pythagoreans were mystics, philosophers, early scientists and musicians. Fundamental to their beliefs was the essential role of number. In addition to their number mysticism, they used number to explain musical scales, to seek to explain the motions of heavenly bodies, and to understand geometry.

By "number" the Pythagoreans meant whole numbers and fractions constructed from whole number ratios. Such numbers are now called **rational numbers**.

22. Do you believe that every number is rational? I.e. can every number be represented as a fraction? Explain carefully.

23. Show that the numbers 7, -6, and 11 can be represented as fractions.

24. Show that the numbers 1 and -1 can be represented as fractions.

25. Correct to three decimal places the number $\sqrt{2} \approx 1.414$. Can you write this latter number as a fraction? Do so or explain why it cannot be done.

26. Correct to ten decimal places the number $\sqrt{2} \approx 1.4142135623$. Can you write this latter number as a fraction? Do so or explain why it cannot be done.

27. Correct to twenty decimal places the number $\sqrt{2} \approx 1.41421356237309504880$. Can you write this latter number as a fraction? Do so or explain why it cannot be done.

28. Correct to thirty decimal places the number $\sqrt{2} \approx 1.414213562373095048801688724209$. Can you write this latter number as a fraction? Do so or explain why it cannot be done.

The process in Investigation **25** - Investigation **28** can be continued indefinitely, getting arbitrarily close to the exact value of $\sqrt{2}$. In fact, the process can be repeated for *every* number which is represented by a finite or even infinite decimal. This property of the rational numbers is called ***density***.

Let us restrict our attention to the rational numbers - they will be our microworld.

Microworlds call for action and exploration, so what can we do?

29. Take several pairs of fractions and multiply each pair together. Is the product of each pair a fraction?

30. Does your work in Investigation **29** prove anything about products of fractions?

31. We can write a general pair of fractions as $\frac{a}{b}$ and $\frac{c}{d}$. What can you say about the numbers a, b, c, d? I.e. what type of numbers are they and are their any limitations on their values?

32. Can you write the product $\frac{a}{b} \cdot \frac{c}{d}$ as a fraction? If so, do so. If not, explain why not.

33. What does Investigation Investigation **32** prove? Explain.

34. Take several pairs of fractions and add each pair together. Is the sum of each pair a fraction?

35. Does your work in Investigation **34** prove anything about sums of fractions?

36. Can you write the sum $\frac{a}{b} + \frac{c}{d}$ as a fraction? If so, do so. If not, explain why not.

37. What does Investigation **36** prove? Explain.

We can continue in this way to show that all of the standard operations of arithmetic - addition, subtraction, multiplication and division - apply for fractions. We can also find additive and multiplicative *inverses*. As such, the rational numbers form what is known as a *field*.

For all of these reasons, a fundamental tenet of Pythagorean mathematics was the following conjecture:

Conjecture 4 (The Rational Number Microworld of the Pythagoreans)**.** *Every number can be represented as a fraction. There are no other numbers.*

It turns out this conjecture is false. There are irrational numbers; numbers that are not commensurable with a unit length. So damning and heretical was this discovery that legend has it that the person who shared this discovery outside of the circle of the Pythagoreans, **Hippasus of Metaponum** (Greek philosopher; circa 5th century B.C. -), was drowned as punishment.

The first irrational number to be found was nothing esoteric or mysterious, but the commonplace $\sqrt{2}$ which is the length of the diagonal of the unit square. Some 2,000 years later mathematicians would discover that in a probabilistic sense the rational numbers are a vanishingly small proportion of the real numbers.

There are many proofs that $\sqrt{2}$ cannot be represented as a fraction. Euclid gave an algebraic proof. There are many different geometric proofs as well. The images below are the basis of some of the geometric proofs.[4]

38. Independent Investigation: Find one of these proofs, understand how it works, and then rephrase it in your own words, as you would explain it to a non-mathematical audience.

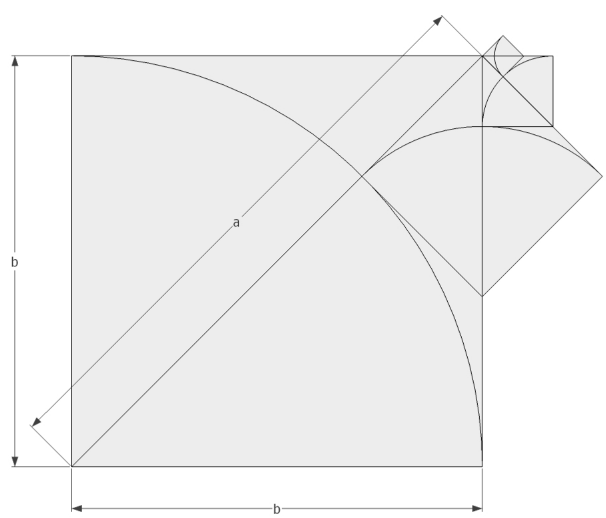

Figure 6.1: Set-up for classical geometric proof of the irrationality of $\sqrt{2}$

[4]These approaches are due to the following mathematicians: Figure 6.1 - classical; Figure 6.2 - "Proof Without Words: $\sqrt{2}$ Is Irrational" by Grant Cairns, *Mathematics Magazine*, Vol. 85, p. 123, 2012; Figure 6.3 - See "Picturing Irrationality" by Steven J. Miller and David Montague, *Mathematics Magazine*, Vol. 85, pp. 110-4, 2012 for history of this result and the author's generalizations; ; Figure 6.4 - From The Book of Numbers, pp. 183-4.

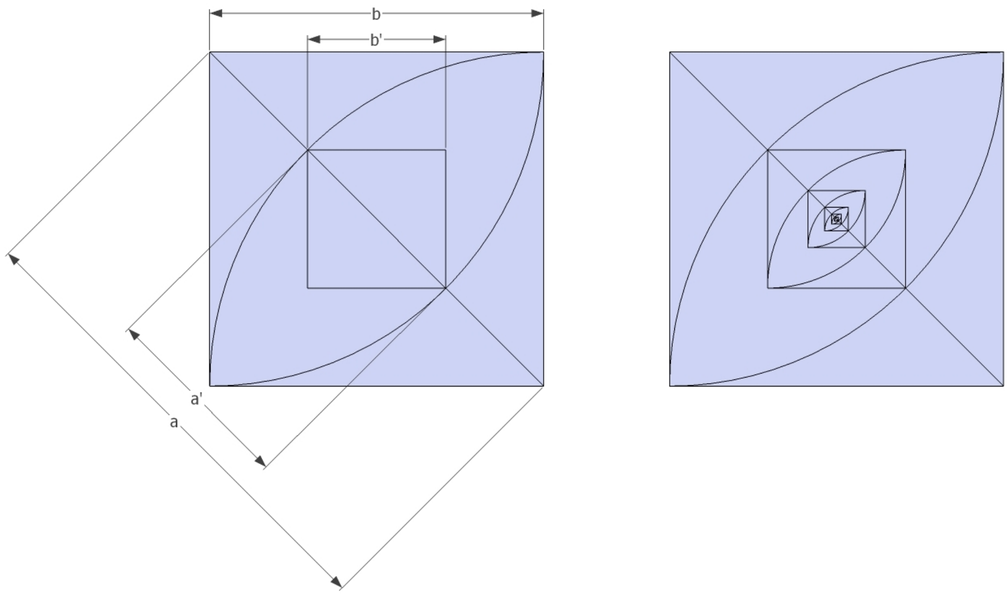

Figure 6.2: Set-up for a "new" geometric proof of the irrationality of $\sqrt{2}$ by Cairns in 2012.

6.3 Three Greek Problems of Antiquity

Coming soon...

6.4 Insolubility of the Quintic by Radicals

More coming soon.

The **quadratic formula** for the solution of the **quadratic equation** $ax^2 + bx + c = 0$ is well known:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Must less well known is the **cubic formula** for the solution of the **cubic equation** $ax^3+bx^2+cx+d = 0$:

$$\begin{aligned} x = &\sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \\ &+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a} \end{aligned}$$

It is natural to ask whether there are similar equations, which are called ***solutions by radicals***, for quartic, quintic,... equations. There is a solution by radicals for the general quartic equation, but it is too enormous to effectively typeset. Remarkably, this is the end of the road. In 1823 **Niels Henrik Abel** (Norwegian mathematician; 1802 - 1829) proved that there is no solution by radicals for the general degree n *polynomial equation* for any $n > 4$.

More coming soon...

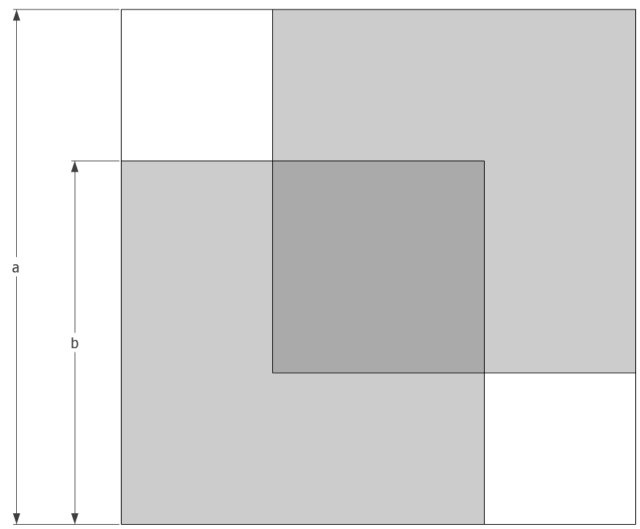

Figure 6.3: Set-up for geometric proof of the irrationality of $\sqrt{2}$ by Tennenbaum in 1950.

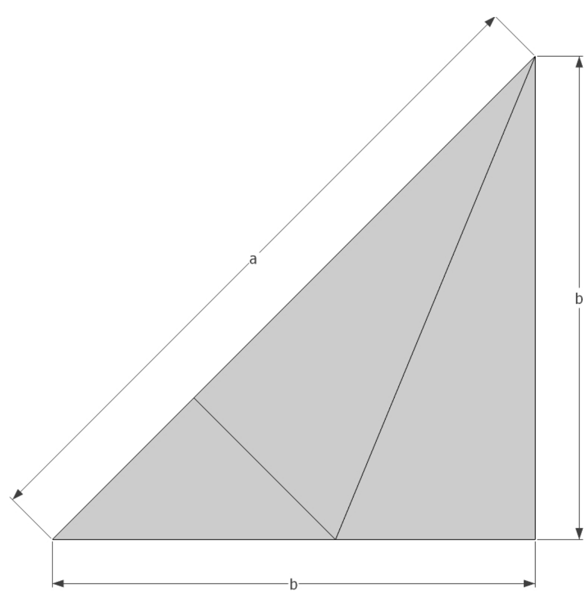

Figure 6.4: Set-up for origami proof of the irrationality of $\sqrt{2}$ by Conway and Guy.

Chapter 7

20th Century Revolutions in Thought

> As the twentieth century draws to a close, it has become increasingly clear that Gdel's famous Incompleteness Theorem for mathematical logic stands alongside Heisenberg's Uncertainty Principle and Einstein's Theory of Relativity as one of the great mathematical achievements of this - or any other - century. Indeed, these three milestones of twentieth century science have much in common. Although all three represent highwater marks in the most rigorous of mathematical sciences, each constitutes at the same time a fountain of philosophical inspiration. In particular, although each is established by formal methods, each demonstrates, in its own peculiar way, a kind of limitation in principle of the relevant formal science. Heisenberg sets limits to our simultaneous knowledge of the position and momentum of the fundamental particles. Einstein, in turn, sets a limit of the speed of light and indeed to the velocity of any information-bearing signal in the universe. And Gdel establishes limits in the ability of any strictly formal, axiomatic mathematical system - in particular, of any computer program - to capture not only all mathematical truth, but even the totality of truths of arithmetic. It is a further, striking fact that all three thinkers draw ontological consequences, about the nature of reality, from what are in effect epistemic premises.[1]
>
> **Palle Yourgrau** (; -)

7.1 Introduction

This is a book about truth, reasoning, certainty and proof. It is essential that it therefore include several of the great modern discoveries that have so fundamentally changed how we think about these matters.

Yourgrau says, "It is a further, striking fact that all three thinkers draw ontological consequences, about the nature of reality, from what are in effect epistemic premises." In other words, what we know is limited by how we know things.

We have seen examples of this already. The Die Hard/Water Jugs Problem does not suggest any deep, philosophical problems. With certain tools only certain jobs can be completed. Yet it is not so different than the situation faced by the Pythagoreans when they found that $\sqrt{2}$ was irrational. Or when you allow only straightedge and compass and learn that this prohibits you from constructively solving any of the Three Greek Problems of Antiquity. Or that one cannot solve the general quintic equation by radicals.

Yet the limitations imposed from within our systems took on much greater importance with several landmark discoveries of the twentieth century. These resulted in some of the greatest paradigm

[1]From the preface to the 1999 edition of Gödel Meets Einstein: Time Travel in the Gdel Universe.

shifts in the history of science and mathematics. We consider several of the most important here:

- Einstein's theory of relativity
- Heisenberg's uncertainty principle
- Determinism, sensitive dependence on initial conditions, fractals, and complexity, and,
- Gödel's incompleteness theorems.

Each of these things will illustrate profound limitations, paradigms shifts in the way we think of truth, reasoning, certainty and proof.

> But in our experience, which is always determined by the goals we have chosen, we always tend to ascribe the obstacles we meet to a mythical reality rather than to the way in which we operate... Whatever we choose as building blocks, be it bricks or Euclid's Elements, determines limiting constraints. We experience these constraints from the 'inside,' as it were, from the brick or the Euclidean perspective. We never get to see the constraints of the world, with which our experience collides. What we experience, cognize, and come to know is necessarily built up of our own building blocks and can be explained in no other way than in terms of our ways and means of building.[2]
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> **Ernst von Glasersfeld** (German philsopher and educational philosopher; 1917 - 2010)

7.2 Cantor's Continuum Hypothesis

> This conviction... is a powerful incentive. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no "we shall not know."
>
> **David Hilbert** (German mathematician; 1862 - 1943)

In 1900 David Hilbert addressed the International Congress of Mathematicians, presenting a list of 23 problems that he believed were the most important problems facing mathematics and which should help guide the subject in the new century. These problems have become known at ***Hilbert's problems***. As part of his address he championed mathematics' ability to transcend other subjects limitations - "in mathematics there is no 'we shall not know.'"

To set the stage for the investigation of the great twentieth century revolutions we begin with a landmark result that is more easily stated than those that will be considered in detail here but one which demolishes Hilbert's maxim and illustrates nicely the nature of the topics to be considered in more detail.

In Discovering the Art of Mathematics - The Infinite infinite sets are considered. As part of these investigations a remarkable result due to **Georg Cantor** (German mathematician; 1845 - 1918) is proven. This result is **Cantor's theorem** which states that the size (more precisely called *cardinality*) of any set is strictly smaller than the size of the *power set* that can be created from the resulting set. If the original set is infinite, this guarantees a strictly larger infinite set. And from it an even larger infinite set can be obtained. In other words, in terms of the sizes of sets, there is an infinite hierarchy of sizes of infinity.

Concretely, one can show that the size of the set $\{1, 2, 3, \ldots\}$ of natural numbers, which is denoted by $\aleph_0$, is a strictly smaller size of infinity than the size of the set $(-\infty, \infty)$ of real numbers, which is denoted by c for continuum. In other words, the two infinities $\aleph_0$ and c satisfy $\aleph_0 < c$. Cantor then asked the natural question: are there any infinities between $\aleph_0$ and c? As early as 1878 Cantor became convinced that there were no sets whose size was between that of $\aleph_0$ and c. This belief became know as the **continuum hypothesis**.

[2]From "An Introduction to Radical Constructivism," p. 37 of The Invented Reality by Paul Watzlawick.

The continuum hypothesis has a very simple structure. The hypothesis is either true - no sets with the given size exist - or the hypothesis is false - there is at least one set with the given size. True or not true. It is like a yes or no question.

Cantor tried to establish this result. By the time of Hilbert's famous presentation in 1900 Cantor had labored unsuccessfully on this problem for 20 years. This had taken its toll on Cantor, as had long-term attacks from other mathematician who accused him of religious heresy by daring to presume that he could appropriately analyze the infinite. Cantor would spend much of the his life after 1899 institutionalized. But his hypothesis was increasingly seen as a legitimate question. In fact, it was the first of Hilbert's problems.

In 1940 **Kurt Gödel** (Austrian mathematician and philosopher; 1906 - 1978) *proved* that the continuum hypothesis could not be proven to be false.[3] One would think then, since the hypothesis could not be proven to be false, that it must be true. But Gödel had already proven his remarkable incompleteness theorems that will be considered later in this chapter and knew better. In 1963 **Paul Cohen** (American mathematician; 1934 - 2007) *proved* that that continuum hypothesis could not be proven to be true! The only conclusion? That this logically simple statement which would seem to require a verdict of true or false was instead undecidable.

The standard tools of set theory had fundamental limits inherent in them, including our ability to answer this natural question about infinite sets.

For the tortured Cantor, it is perhaps appropriate that this hypothesis was the first to be discovered. It is not a human limit that kept him from determining its validity, it is a fundamental epistemological limit that precludes anyone from knowing.

The other major topics considered below are similar in the harsh limitations that their foundations impose on what we can know.

7.3 Einstein's Theory of Special Relativity

1. Classroom Discussion: Are you moving? Not just breathing and having blood flow through your body, but moving in the large.

So wondered the great **Galileo Galilee** (Italian mathematician, physicist, astronomer and philosopher; 1564 - 1642) in his famous Dialogues Concerning the Two Chief World Systems of 1632. Written as a dialogue in the great Socratic style [4] this was, as Galileo explained as part of his dedication of the book to the Grand Duke of Tuscany, a book written so "lovers of truth can draw the fruit of greater knowledge and utility." For "he who looks higher is the more highly distinguished, and turning over the great book of nature (which is the proper object of philosophy) is the way to elevate one's gaze." And where should we point our gaze? To the sky and heavens above: "The constitution of the universe I believe may be set in first place among all natural things that can be known, for coming before all others in grandeur by reason of its universal content, it must also stand above them all in nobility."

Over 300 years Einstein was still lauding the importance of this book:

> Galileo's Dialogue Concerning the Two Chief World Systems is a mine of information for anyone interested in the cultural history of the Western world and its influence upon economic and political development... The leitmotif which I recognize in Galileo's work is the passionate fight against any kind of dogma based on authority. Only experience and careful reflection are accepted by him as criteria of truth. Nowadays it is hard for us to grasp how sinister and revolutionary such an attitude appeared at Galileo's time, when merely to doubt the truth of opinions which had no basis but authority was considered a capital crime and punished accordingly. Actually we are by no means so far removed

[3] Under the axioms of *Zermelo-Fraenkel set theory* which are widely used.

[4] A style which inspires the pedagogical approach of books in this series.

> from such a situation even today as many of us would like to flatter ourselves; but in theory, at least, the principle of unbiased thought has won out, and most people are willing to pay lip service to this principle.[5]
>
> **Albert Einstein** (German physicist and author; 1879 - 1955)

2. Classroom Discussion: What are some of the influences "upon economic and political development" that Galileo's work had?
3. Classroom Discussion: Einstein laments that "we are by no means so far removed from such a situation even today." What are some contemporary areas in which we are compelled not to "doubt the truth of opinions which had [have] no basis but authority"?

For this, Galileo was brought to trial by the Catholic Church's infamous Inquisition. He was found "vehemently suspect of heresy," placed under house arrest for the remainder of his life, and publication of all of his works were banned. The church's ban on Galileo's Dialogue Concerning the Two Chief World Systems was not lifted until 1835!

In his Dialogues, Galileo invites the reader to consider some thought experiments whose impact on our understanding of the nature of motion are profound. They offer a theory of relativity for systems in motion.

One of his more famous thought experiments is as follows:

> Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it.
>
> With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need to throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction.
>
> When you have observed all of these things carefully (though there is no doubt that when the ship is standing still everything must happen this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than towards the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite.
>
> The droplets will fall as before into the vessel beneath without dropping towards the stern, although while the drops are in the air the ship runs many spans. The fish in the water will swim towards the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air.

[5] Foreword to Galileo's Dialogue Concerning the Two Chief World Systems, University of California Press, 1962 edition.

4. What are the implications of this thought experiment on the issues you considered in Discussion **7.3**? What role could such a thought experiment play on the debate about possible nature of the motions of the earth and of the solar system?

We will extend this thought experiment a bit. Consider a mate standing high up in the crows nest atop the mast of a sailing ship. The ship is moving quite quickly although the sea and air are calm. The mate has a very small ball which will not be effected much by wind resistance and has a nearly perfectly elastic bounce. The mate drops the ball, it falls under the forces of gravity until it hits the deck, bounces back up and is caught by the mate when it returns to the height of her hand.

5. Suppose you were the captain of the ship and were standing by the rail, directly across from the mast. Precisely describe what the path of the ball would look like to you. How far would the ball appear to travel?

6. Now suppose that you were on the beach, watching the ship speed past and the direction of the boat is perpendicular to your line of site to it. Precisely describe what the path of the ball would look like to you. How far would the ball appear to travel?

7. Did the ball appear to travel the same distance in the two different settings? If so, why? If not, how is this possible?

The situation you have just investigated is a consequence of what is called the *Principle of Galilean Relativity.*

You have probably encountered a related situation before:

8. A fire truck is racing down the street with its sirens blaring. How does the sound of the siren compare to different observers who are:

- Standing on the sidewalk as the truck moves toward them?
- Standing on the sidewalk as the truck moves away from them?
- Driving in the same direction, at the same speed, as the truck?
- Driving in the opposite direction from the truck?

In considering the emergence of non-Euclidean geometry we talked about axioms for a system - the logical starting points for the microworld. By changing the Parallel Postulate new geometries were discovered, geometries that were just as consistent as the usual Euclidean geometry we are used to, and which have subsequently been found to be profoundly useful.

Einstein's theory of relativity is among the more revolutionary *paradigm shifts* in the history of science. It is important to note that the *special theory of relativity*, which we will consider here, relies heavily on non-Euclidean geometry. Indeed, it is no accident that relativity was developed on the heals of significant progress in the study of non-Euclidean geometries. In fact, Einstein's profound results were not the lone work of one genius but involved essential mathematical breakthroughs by important contemporaries of Einstein, including Henri Poincare, Bernard Riemann and Edward Lorenz. While we will not talk about Einstein's *general theory of relativity* of 1915, it has deep mathematical roots as well. One of the shorter descriptions of this theory is as "the geometric theory of gravitation."[6]

It's also interesting to note that the mathematical subtleties of the implications of Einstein's theory of relativity were not always well received nor well understood by Einstein, who said:

> Since the mathematicians have invaded the theory of relativity, I do not understand it myself anymore.

[6]From Wikipedia, http://en.wikipedia.org/wiki/General_relativity, accessed 8/10/2010.

Here we will focus on Einstein's *special theory of relativity* which was developed in 1905. We consider it because it has a profound impact on our question of the limits of knowledge and we can investigate it as a fairly straightforward axiomatic system. Indeed, in his attempt to explain the theories of relativity to the common person in Relativity: The Special and General Theory [7] Einstein begins by discussing axioms:e and we can investigate it as a fairly straightforward axiomatic system. Indeed, in his attempt to explain the theories of relativity to the common person in Relativity: The Special and General Theory [8] Einstein begins by discussing axioms:

> Geometry sets out form certain conceptions such as "plane," "point," and "straight line," with which we are able to associate more or less definite ideas, and from certain simple propositions (axioms) which, in virtue of these ideas, we are inclined to accept as "true." Then, on the basis of a logical process, the justification of which we feel ourselves compelled to admit, all remaining propositions are shown to follow from those axioms, i.e. they are proven. A proposition is then correct ("true") when it has been derived in the recognized manner from the axioms. The question of "truth" of the individual geometrical propositions is thus reduced to one of the "truth" of the axioms. Now it has long been known that the last question is not only unanswerable by the methods of geometry, but that it is in itself entirely without meaning. We cannot ask whether it is true that only one straight line goes through two points. We can only say that Euclidean geometry deals with things called "straight lines," to each of which is ascribed the property of being uniquely determined by two points situated on it. The concept "true" does not tally with the assertions of pure geometry, because by the word "true" we are eventually in the habit of designating always the correspondence with a "real" object; geometry, however, is not concerned with the relation of the ideas involved in it to objects of experience, but only with the logical connection of these ideas among themselves.

He concludes the section by noting, "we shall see that this 'truth' is limited, and we shall consider the extent of its limitation."

9. Classroom Discussion: Albert Einstein is one of the most famous scientists of all times. His scientific contributions were at their peak as he wrote Relativity: The Special and General Theory for general audiences in 1916. What do his efforts to explain these landmark achievements tell you about his view of the common person; the non-scientists? Do you notice some comparison with the goals of the books in this series? Explain.

10. Classroom Discussion: While Einstein continued to do important work in physics throughout his life, he spent considerable amounts of his time later in life exploring the human condition, advocating for peace, promoting education, and engaging in significant philosophical questions - all in very public ways given the renown he had achieved. Find several related quotes by Einstein that you find compelling. Describe them and why you find them compelling.

The next example is directly from Einstein's Relativity: The Special and General Theory.

Suppose a conductor is standing at the exact middle of a long, fast-moving train. A hiker is standing still, next to the tracks, watching the train speed by. Simultaneously flashes of light reach the hiker from bolts of lightning which have hit the very front and very back of the train. The bolts of lightning have left marks on the ground as well and it is later determined that the hiker is standing midway between the lightning strikes.

11. Explain why, from the perspective of the hiker, the bolts of lightning must have hit the train simultaneously.

12. As judged by the hiker, when the bolts of lightning hit the train, where must the conductor have been in relation to the hiker? Explain.

[7] R.W. Lawson translation, published by Methuen and Co. Ltd, 1920.

[8] R.W. Lawson translation, published by Methuen and Co. Ltd, 1920.

13. In the time it takes for the light from the lightning flashes to travel to the conductor, how has the location of the conductor changed?

14. Which flash of lightning will the conductor see first? Explain.

15. Immediately following this thought experiment, Einstein concludes:

> Every reference-body (co-ordinate system) has its own particular time ; unless we are told the reference-body to which the statement of time refers, there is no meaning in a statement of the time of an event.

Explain this in the context of our thought experiment.

16. What implications does this ***relativity of time*** have for our discussion of the limits of knowledge? Explain.

We are now in a position to derive the equations governing the relativity of time. They are little more than a careful quantitative investigation of a situation analogous to Galileo's ship experiment only now our focus is on the relative travel of light. The key assumptions we need are two physical axioms that Einstein called *postulates*. Generally a very empirical science, this chapter of physics resembles mathematics in its derivation and logical structure.

Einstein's postulates are as follows:

- "The same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate"
- "...and also introduce another postulate...namely, that light is always propagated in empty space with a definite [finite] velocity c which is independent of the state of motion of the emitting body."

In analogy with Galileo's ship, consider a rocket ship which is moving quite quickly (say at a speed s) although there is no net acceleration in any direction. A small light source which can emit pulses of light is suspended d units above a mirror that reflects light back to the source.

17. Suppose an astronaut was standing next to the light/mirror apparatus. Precisely describe what path the light's travel would appear to take from the perspective of the astronaut. How far would the light appear to travel in this *reference frame*?

18. Use the constant speed of light, which is usually denoted by the letter c, to find an equation for the length of time in the reference frame of the astronaut that it appears to take for the light to travel from the source to the mirror and then back to the source. Denote this time by t_1.

19. Now suppose that you were on a planet, watching the rocket ship speed past. Explain why, from an appropriate vantage point [9] the path of the light would look as in Figure 7.1 in your reference frame. Explain why the light would appear to you to travel a distance of $2l$.

20. Does the light appear to travel a greater distance to you or to the astronaut? Explain.

21. Because the speed of light is constant in all reference frames, what does Investigation **20** say about the time it took the light to travel from the source to the mirror and back to the source in the two reference frames?

[9] The actual vantage point does not matter as the argument here requires only knowledges of distances so perspective and foreshortening are irrelevant.

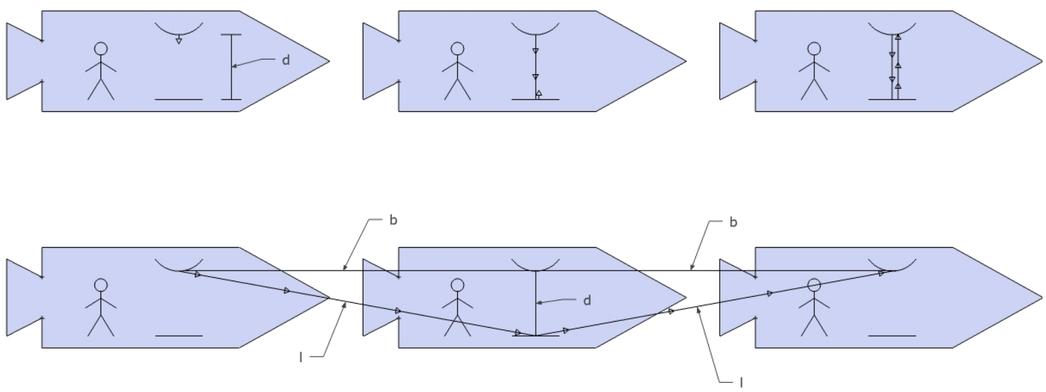


Figure 7.1: A beam of light traveling in a rocket ship as pictured at three points in time as seen by an observer moving with the rocket ship (e.g. the astronaut), top, and the same beam of light seen by an observer which is at rest in relation to the rocket ship, bottom.

We can now use simple Euclidean geometry to find a exact quantitative relationship between the two different time intervals t_1 and t_2.

22. Use the constant speed of light to find an expression for the length of time, in terms of l, in your reference frame that it appears it takes for the light to travel from the source to the mirror and then back to the source. Denote this time by t_2.

23. Find an expression for the length labeled b in Figure 7.1 in terms of t_2 and s, explaining how you obtained this equation.

24. Use the Pythagorean theorem to find a single equation relating l, d, and b.

25. Using results from your investigations above, substitute for each of the *parameters* l, d, and b in Investigation **24** to obtain an equation involving only the quantities s, c, t_1 and t_2.

26. Solve the equation in the previous investigation for t_2 in terms of t_1. Simplify as much as possible.

27. Use your results to fill in the following table which compares the time intervals t_1 and t_2 at different speeds s:

s	t_1	t_2
.01 % c	10 sec	
.1 % c	10 sec	
1 % c	10 sec	
10 % c	10 sec	
50 % c	10 sec	
90 % c	10 sec	
99 % c	10 sec	
99.9 % c	10 sec	
99.99 %c	10 sec	

28. For small speeds, near those that we have been able to historically measure, what do you notice?

29. For large speeds, near the speed of light, what do you notice? What happens if we approach or even exceed the speed of light in this model?

30. In a short essay, describe the implications of some of these results of Einstein to the limits of knowledge.

7.4 The Heisenberg Uncertainty Principle

In classical mechanics the **momentum** of an object is the product of its mass and its velocity. This corresponds to colloquial usage of the term - a heavy car moving quickly has lots of momentum.

Theorem 9. *(Heisenberg Uncertainty Principle) The more accurately we know the position of a particle the less accurately we know its momentum and conversely. More precisely, if Δp is the uncertainty of our measurement of the momentum of a particle and Δx is the uncertainly of our measurement of the position of the same particle then we must always have*

$$\Delta p \cdot \Delta x \geq \frac{h}{2\pi}$$

*where h is the **Planck's constant** which is approximately 6.626×10^{-34} joule-sec.*

There are several important things to note about this result:

- This result is *not* the result of need for better measuring tools - this is a fundamental limit.
- This is not an empirical result which has been postulated on the basis of many experiments. Instead, it is a precise mathematical result which follows in a straightforward manner from the *de-Broglie-Einstein relations* and basic mathematical properties of waves.[10]

7.5 Determinism, Sensitive Dependence on Initial Conditions, Fractals and Complexity

7.5.1 Determinism

> We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at any given moment knew all of the forces that animate nature and the mutual positions of the beings that compose it, if this intellect were vast enough to submit the data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom; for such an intellect nothing could be uncertain and the future just like the past would be present before its eyes.
>
> **Marquis Pierre Simon de Laplace** (French Mathematician; 1749 - 1827)

> Everything proceeds mathematically...if someone could have a sufficient insight into the inner parts of things, and in addition had remembrance and intelligence enough to consider all the circumstances and take them into account, he would be a prophet and see the future in the present as in a mirror.
>
> **G.W. Leibniz** (German philosopher and mathematician; 1646 - 1716)

7.5.2 Fractals

> The physical universe is basically an iterated system, so actually it is surprising we have made the progress we have, using only simple evaluation. The equations have been around forever. The physical universe has been USING them almost forever. The equations have as part of their very nature things like fixed points, period cycles, chaotic

[10] See e.g. Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles, 2nd edition, 1985, by Robert Eisberg and Robert Resnick, pp. 69-77.

> cycles, basins of attraction, etc., so you can be sure all these things are manifested in the physical universe INCLUDING FRACTALNESS. To say therefore that fractals have nothing to do with anything and have not explained or proven useful in our understanding of the universe is more a statement about the people who are working with fractals rather than a statement about the pertinence of fractals to the world at large. Fractals are so pertinent to the universe no one can see it yet. Long time ago, they thought math did not pertain either. The "why" was God. The "why" might still be God, but if it is, then clearly God is a mathematician of significant merit, and no doubt a fractal enthusiast.
>
> **Homer Wilson Smith** (; -)

Coming soon - Newton's method, an inquiry-based example based on a real-valued cubic.

7.5.3 Sensitive Dependence on Initial Conditions

Essay Write a brief essay of one- to two-pages which describes two ways in which your life might have been fundamentally changed by sensitive dependence on initial conditions:

- One that actually happened and your were aware of its importance as it happened, and,

- Another that either did not happen or you were totally unaware of its significance in real time.

> [Von Neumann] said, 'The computer will enable us to divide the atmosphere at any moment into stable and unstable regions. Stable regions we can predict. Unstable regions we can control. . . ' Von Neumann, speaking in 1950, said it would take only ten years to build computers powerful enough to diagnose accurately the stable and unstable regions of the atmosphere. Then, once we had accurate diagnosis, it would take only a short time for us to have control. He expected that practical control of the weather would be a routine operation within the decade of the 1960's. Von Neumann, of course, was wrong.[11]
>
> **Freeman Dyson** (; -)

> Edward Lorenz discovered in 1963 that the solution of the equations of meteorology are often chaotic. . . Lorenz was a meteorologist and generally regarded as the discoverer of chaos. He discovered the phenomena of chaos in the meteorological context and gave them their modern names. But in fact I had heard the mathematician Mary Cartwright, who died in 1998 at the age of 97, describe the same phenomena in a lecture in Cambridge in 1943, twenty years before Lorenz discovered them. She called the phenomena by different names, but they were the same phenomena. . . I heard all about chaos from Mary Cartwright seven years before I heard von Neumann talk about weather control, but I was not far-sighted enough to make the connection.[12]
>
> **Freeman Dyson** (; -)

> This wild star, it is now three centuries since, with clasped hands, and with streaming eyes, at the foot of my beloved - I spoke it - with a few passionate sentences - into birth. Its brilliant flowers are the dearest of all unfulfilled dreams and its raging volcanoes are the passions of the most turbulent and unhallowed of hearts.[13]

[11]From "Birds and Frogs," *Notices of the AMS*, Vol. 56, No. 2, February, 2009.
[12]From "Birds and Frogs," *Notices of the AMS*, Vol. 56, No. 2, February, 2009.
[13]From "The Power of Words" by Edgar Allen Poe.

> And there was a feel. His flesh twitched. His hands twitched. He stood drinking the oddness with the pores of his body. Somewhere, someone must have been screaming one of those whistles that only a dog can hear. His body screamed silence in return. Beyond this room, beyond this wall, beyond this man who was not quite the same man seated at this desk that was not quite the same desk lay an entire world of streets and people. What sort of world it was now, there was no telling.[14]

7.5.4 Complexity

> Ever since antiquity it has been a great mystery how the universe can follow definite laws while we as humans still often manage to make decisions about how to act in ways that seem quite free of obvious laws... It finally now seems possible to give an explanation for this. And the key, I believe... is that even though a system may follow definite underlying laws its overall behavior can still have aspects that fundamentally cannot be described by reasonable laws... The only way to work out how the system will behave is essentially to perform this computation - with the result that there can fundamentally be no laws that allow one to work out the behavior more directly. And it is this, I believe, that is the ultimate origin of the apparent freedom of human will.[15]
>
> **Stephen Wolfram** (; -)

Coming soon - Conway's Game of Life.

7.6 Gödel's Incompleteness Theorems - Knights and Knaves

Note: These questions are from Forever Undecided by Raymond Smullyan, Alfred A. Knopf, New York, 1987.

Knights and Knaves is a logic puzzle due to **Raymond Smullyan** (American mathematician, logician, philosopher, and magician; 1919 -). These puzzles take place on a fictional island, which we will call Smullyan Island, that consists of two types of citizens: **Knights**, who always tell the truth and **Knaves** who always lie. The goal of the puzzle is to determine the type of citizen each person is based on their statements. This structure gives a microworld where we can easily explore the truth, falsity or undecidability of statements. In particular we will use this setting to understand two of the most important results in mathematics, the two Incompleteness Theorems of **Kurt Gödel** (Austrian Logician, Mathematician and Philosopher; 1906 - 1978) .

Here is an example:

> While on Smullyan Island, you meet two people, Bob and Steve.
> Steve says, "Bob is a knave."
> Bob says, "Exactly one of us is knight."
> What are Steve and Bob?

The technique used to solve these types of puzzle is illustrated in the following series of questions.

31. We begin by assuming Steve is a Knight. (We could have started with the assumption that Steve is a Knave, but for our current purposes beginning with the assumption Steve is a Knight is better.)

[14] From "A Sound of Thunder" by Ray Bradbury.
[15] From A New Kind of Science, p. 750.

(a) What does the fact that Steve is a Knight tell us about the truth of his statement? Explain.

(b) What does you answer to Investigation **31a** tell you about Bob's type? Explain

(c) What does your answer to Investigation **31b** tell you about the truth of Bob's statement? Explain.

(d) What do your answers to Investigation **31b** - Investigation **31c** tell you about the types for both Steve and Bob? Explain

(e) Since we are assuming Steve is a Knight, what does your answer to Investigation **31d** tell you about Bob's type? Explain.

(f) Can your answers to Investigation **31b** and Investigation **31e** both be true? Explain.

(g) Since your answers to both Investigation **31b** and Investigation **31e** followed from the assumption that Steve is a Knight, what can we conclude about the correctness of our assumption that Steve is a Knight? Explain.

32. Having reached our conclusion in Investigation **31g** based on the assumption that Steve is a Knight, we now assume Steve is a Knave.

(a) What does the fact that Steve is a Knave tell us about the truth of his statement? Explain.

(b) What does you answer to Investigation **32a** tell you about Bob's type? Explain

(c) What does your answer to Investigation **32b** tell you about the truth of Bob's statement? Explain.

(d) What do your answers to Investigation **32b** - Investigation **32c** tell you about the types for both Steve and Bob? Explain

(e) Since we are assuming Steve is a Knave, what does your answer to Investigation **32d** tell you about Bob's type? Explain.

(f) Can your answers to Investigation **32b** and Investigation **32e** both be true? Explain.

(g) Since your answers to both Investigation **32b** and Investigation **32e** followed from the assumption that Steve is a Knave, what can we conclude about the correctness of our assumption that Steve is a Knave? Explain.

33. Based on your answers to Investigation **31a** - Investigation **32g** what are the types for Steve and Bob? Explain.

34. Use a reasoning process similar to what you used in Investigation **31** - Investigation **33** to solve the following puzzle:

> While on Smullyan Island you meet three people Annie, Betty and Carrie.
>
> Annie says, "Carrie is a Knave"
>
> Betty says, "None of us are Knaves"
>
> Carrie says, "Betty is a Knight."
>
> What are Annie, Betty and Carrie?

7.6.1 Conditional Statements

One of the most common form of mathematical statements is the ***conditional statement***. These are statements of the form "If p then q" where p and q are statements that are either true or false. We call p the **hypothesis** and q the **conclusion**.

Below are several examples of conditional statements.

A. If you are registered for 12 or more credits then you are considered a full time student at Westfield State.

B. If you live in Massachusetts then you live in the United States.

C. If 6 divides the whole number x then 3 divides the whole number x.

D. If 3 divides the whole number x then 6 divides the whole number x.

E. If the whole number n is even and greater than 2, then we can find two primes p_1 and p_2 so that $p_1 + p_2 = n$.

F. If I am a Knight then there is gold on Smullyan Island.

With conditional statements, it is important to understand the relationship between the truth or falseness of hypothesis, the conclusion and the conditional statement.

35. For each of the statements **A** - **F**, identify the hypothesis and the conclusion.

36. For each of the statements **A** - **F**, what do you think it means for the conditional statement to be true? Explain.

37. For each of the statements **A** - **F**, what do you think it means for the conditional statement to be false? Explain.

38. For each of the statements **A** - **D**, is the conclusion true every time the hypothesis true? Explain.

39. Suppose we believe that the conditional statements **E** and **F** are true? What can we conclude when we believe the hypothesis is true? Explain.

Use your answers to Investigation **35** - Investigation **39** to answer the following questions about the general conditional statement "If p then q;" where p and q are statements that are either true or false.

40. What do we need to know about p and q in order for the conditional statement, "If p then q" to be true?

41. What do we need to know about p and q in order for the conditional statement, "If p then q" to be false?

42. If we believe the conditional statement, "If p then q" to be true, and we believe p to be true, what can we conclude about q?

43. If we believe the conditional statement, "If p then q" to be true, and we believe q to be false, what can we conclude about p?

Now let us return to Smullyan Island and explore some of the subtleties of conditional statements on the Island.

44. Can a Knave say, "If I am a Knight, then there is gold on Smullyan Island?" Explain.

45. Can a Knave say, "If I am a Knight, then 1+1=3?" Explain.

46. Can a Knave make any statement of the form, "If I am a Knight, then q," for any statement q? Explain.

47. Can a Knight say, "If I am a Knight, then there is gold on Smullyan Island" if there is no gold on the island? Explain.

48. Can a Knight say, "If I am a Knight, then 1+1=3?" Explain.

49. Can a Knight say, "If I am a Knight, then q" for any false statement q? Explain.

50. Suppose a native of Smullyan Island makes a statement of the form "If I am a Knight, then q" for some statement q; based on your answers to Investigation **44** - Investigation **49**, what can you conclude about the native and the truth or falseness of q? Explain.

51. Suppose you meet a native who you believe is a Knight, and the native says, "There is gold on Smullyan Island." Do you go searching for gold? Explain.

52. Suppose you meet a native who you believe is a Knave, and the native says, "There is gold on Smullyan Island." Do you go searching for gold? Explain.

53. Can any inhabitant of Smullyan Island ever say "I am not a Knight."? Explain.

54. Suppose you meet a native who you believe is a Knight, and the native says, "You will believe that I have a pot of gold." What do you believe? Explain.

55. Suppose you meet a native who you believe is a Knight, and the native says, "You will never believe that I have a pot of gold." What do you believe? Explain.

56. Suppose you meet a native who you believe is a Knight, explain why you believe whatever the native says.

Our investigation of Gödel's Incompleteness Theorems will revolve around a native of Smullyan Island making the statement "You will never believe I am a knight." In order to completely examine this scenario, we need to examine how various reasoners will deal with this situation. Our first case will deal with the situation in which the reasoner, whom we will call Raymond.[16], believes he is incapable of making a mistake. That is, if Raymond believes a particular statement, then, as far as he is concerned, the statement is really true. For example, if Raymond ever believes a native of Smullyan Island is a Knight, then because he thinks he is incapable of making mistakes, he concludes that the native really is a Knight and therefore everything that person says is true.

57. Suppose the reasoner, Raymond, believes that he is incapable of making a mistake. He meets a native who says, "You will never believe that I am a Knight." Suppose Raymond begins with the case that the native is a Knave. What can he conclude from the native's statement? Explain.

58. Based on your answer to Investigation **57** and the fact that Raymond thinks he is incapable of making mistakes, what does he conclude the native to be? Explain.

59. Explain why your answer to Investigation **58** gives rise to a contradiction.

[16]We need to use a third party as the reasoner because to reach the appropriate conclusions you, the student, need to be "outside" the system.

60. What does your answer to Investigation **59** mean about Raymond's assumption that the native is a Knave? Explain.

61. Based on your reasoning in Investigation **57** - Investigation **60** what does Raymond now believe the native to be? Explain

62. Based on the native's statement, "You will never believe I am a Knight", what is the true status of the native? Explain.

63. Based on your answers to Investigation **57** - Investigation **62** explain why a reasoner who believes they are always accurate and who meets a native who says, "You will never believe I am a Knight" will eventually become inaccurate.

Most people are not so conceited that they believe they will never make mistakes. However, in this chapter we are interested in conclusions that are logical possibilities, even when they may seem unlikely. The conclusions you made in Investigation **57** - Investigation **63** illustrate this. A more bizarre possibility is illustrated in the next series of questions. For these questions we will assume that our reasoner, Raymond, is what we will call ***regular***. That means he believes that it is impossible for him to believe something and, at the same time, believe that he does not believe it. For example, since Raymond is regular, he believes that it is impossible for him to believe a native is a Knave and, at the same time, believe that he does not believe the native is a Knave.

As before, Raymond goes to Smullyan Island and meets a native who says, "You will never believe I am a Knight."

64. Suppose Raymond ever believes the native is a Knight. Then based on your answer to Investigation **56** what will Raymond also believe? Explain.

65. Since Raymond believes he is regular, what does your answer to Investigation **64** mean about his original belief that the native is a Knight? Explain.

66. Based on your answer to Investigation **65** what will Raymond now believe to be the type of the native? Explain.

67. Based on your answer to Investigation **66** is the native's original statement true or false? Explain.

68. Based on your answer to Investigation **67**, what does Raymond now believe to be the type of the native? Explain.

69. Based on your answers to Investigation **65** and Investigation **67** is Raymond regular? Explain.

The argument you worked through in Investigation **64** - Investigation **69** may seem completely unrealistic, but since the rules of the island hold, this conclusion was forced on us. Most reasoners would, understandably, at this point decide that there is something wrong with the rules of the island. However, because we are assuming that the rules of the island hold, no matter what the results are, we are forced to conclude that, at lease theoretically, it is possible for a person to both believe and not believe something. This possibility is essentially what lies at the heart of Gödel's Incompleteness Theorems.

Now that we have an understanding of the rules for Smullyan Island we are almost ready begin our investigation of Gödel's Incompleteness Theorems. However, before we do that we need to consider ***conditional statements***.

We call a reasoner **inconsistent** if there is some statement p, such that the reasoner believes both p and its negation, denoted $\sim p$. We therefore call a reasoner **consistent** if there is no statement p such that the reasoner believes both p and its negation $\sim p$.

We are now ready to explore Gödel's seminal results using the Knights and Knaves from Smullyan Island.[17] Both results are based on the conclusions we can draw from a native of the Island whose says, "You will never believe that I am a Knight." [18]

Gödel's First Incompleteness Theorem via Knights and Knaves

A consistent reasoner visits Smullyan Island and meets a native who says, "You will never believe that I am a Knight."

70. Suppose the reasoner believes the native is a Knight. Explain why it is reasonable to conclude that the reasoner believes that she believes the native is a Knight.

71. Since the reasoner believes that the native is a Knight, what must she also believe, based on what the native said? Explain.

72. Explain why your answers to Investigation **70** - Investigation **71** show that the reasoner will thus never actually believe the native is a Knight since she is consistent.

73. Alternatively, suppose that the reasoner at some point believes that the native is a Knave. What must she also believe, based on what the native said? Explain.

74. Explain why it is reasonable to conclude from your answer to Investigation **73** that the reasoner must also believe that the native is a Knight.

75. Explain why your answers to Investigation **73** - Investigation **74** show that the reasoner will thus never actually believe the native is a Knave since she is consistent.

76. Use your answers to Investigation **70** - Investigation **75** to explain why a reasoner visiting Smullyan Island will never believe that this native is Knight nor that this native is a Knave.

77. Explain why, despite your answer to Investigation **76**, we know that the native must be either a Knight or a Knave.

Your answers to Investigation **70** - Investigation **77** illustrate Gödel's First Incompleteness Theorem in the Knights and Knaves setting:

Theorem 10 (Gödel's First Incompleteness Theorem in the Knights and Knaves setting (Smullyan, pp. 173-174))**.** *A normal, consistent, stable reasoner of type 1 comes to the island and believes the rules of the island. She meets a native who says: "You will never believe that I am a Knight. Then the reasoner's belief system is incomplete. That is, the reasoner will never believe the native is a Knight and will never believe that the native is not a Knight.*

Here is an informal statement of Gödel's First Incompleteness Theorem.

Theorem 11 (Gödel's First Incompleteness Theorem)**.** *In any sufficiently strong formal system, there are true arithmetical statements that are undecidable within that system.*

How does Theorem 11 relate to Theorem 10? Let's begin by parsing the important phrases in Theorem 11. The phrase "Sufficiently strong system" refers to a collection of statements where the truth or falseness of most of these statements can be determined from a set of *axioms* and previously established true statements using the rules of logic. In the Knights and Knaves setting our ***axioms***, or basic assumptions, are

[17] By this point students need to be very comfortable with conditional statements.

[18] It is logically impossible for a native of Smullyan Island to say, "You will never know that I am a Knight" or "You will never correctly believe that I am a Knight." (Smullyan pp. 67-71)

1. Every native of Smullyan Island is a Knight or a Knave.

2. Knights always tell the truth.

3. Knaves always lie.

4. The rules of the Island always hold.

The rules of logic and our assumptions that

a. If the reasoner believes something, then they believe that they believe it.

- This is what Smullyan calls ***normal***.

b. If the reasoner believes they believe something, then they believe it.7

- This is what Smullyan calls ***stable***.

create a system that is "sufficiently strong". Mathematics, and in particular, number theory, is such a "sufficiently strong" system as well. In fact, if we replace the word believe by provable in assumptions (7.6.1) and (7.6.1) above and the sentence, "You will never believe that I am a Knight" by "The truth of this sentence can not be proved;" we essentially get the proof of Gödel's First Incompleteness Theorem.

Gödel's Second Incompleteness Theorem via Knights and Knaves

As in Gödel's First Incompleteness Theorem, a reasoner who believes they are consistent and will remain consistent, visits Smullyan Island and meets a native who says, "You will never believe that I am a Knight."

78. Explain why your answers to Investigation **70** - Investigation **72** show that the what the native said is true.

79. Since what the native said is true, what does the reasoner conclude about the native? Explain.

80. What does your answer to Investigation **79** mean that the reasoner believes about the type of the native? Explain.

81. What does your answer to Investigation **80** mean about the truth of the natives statement? Explain.

82. What does your answer mean that the reasoner will have to believe about the native? Explain.

83. Explain why your answers to Investigation **80** and Investigation **82** show that because the reasoner believes that she is consistent, she in fact becomes inconsistent.

Your answers to Investigation **78** -Investigation **83** illustrate Gödel's Second Incompleteness Theorem in the Knights and Knaves setting:

Theorem 12 (Gödel's Second Incompleteness Theorem in the Knights and Knaves setting (Smullyan, p. 101))**.** *A reasoner of type 4 visits Smullyan Island and meets a native of the island who says: "You will never believe that I am a Knight." Then, if the reasoner is consistent, she can never know that she is consistent; or stated otherwise, if the reasoner ever believes that she cannot be inconsistent, she will become inconsistent!*

Here is an informal statement of Gödel's Second Incompleteness Theorem.

Theorem 13 (Gödel's Second Incompleteness Theorem)**.** *Any sufficiently strong formal system, cannot prove its own consistency within the system. Stated another way, if a sufficiently strong formal system can prove its own consistency then it is inconsistent!*

The connections between Theorem 12 and Theorem 13 are the same as those between Theorems 10 and 11 on page 114.

Impact of Gödel's Incompleteness Theorems

Gödel's Incompleteness Theorems have had an enormous impact beyond mathematics. These theorems have entered into the public consciousness in a way few mathematical and scientific results have.

People who quote Gödel's First Incompleteness Theorem recognize that it says something about the limits of mathematical, and hence, human knowledge.

To use Smullyan's ideas we need to have the following assumptions:

1. The *reasoner* or *student* needs to believe all **tautologies**; that is, any proposition that can be "established purely on the the basis of truth table rules for the logical connectives." (Smullyan, p. 43)

2. If, for any propositions p and q, the reasoner believes p and $p \to q$, then (s)he will believe q. (Smullyan, p. 90)

- For any proposition p, Smullyan uses the notation Bp to denote, "the reasoner believes p".
- Using the above notation, condition (2) can be written as $[Bp \; \& \; B(p \Rightarrow q)] \;\Rightarrow\; Bq$.

3. The reasoner believes that if (s)he ever believes both p and $p \to q$, then (s)he will believe q. (Smullyan, p. 90)

- $B[Bp \; \& \; B(p \Rightarrow q)] \;\Rightarrow Bq$

4. If the reasoner believes p then (s)he believes (s)he believes p. I think this says that the reasoner is aware of what they believe. (Smullyan, p. 90).

- $Bp \;\Rightarrow BBp$.
- Smullyan calls this condition **normal**.

5. The reasoner believes that believing $p \Rightarrow$ (s)he believes (s)he believes p. (Smullyan, p. 90)

- $B(Bp \Rightarrow BBp)$

6. The reasoner believes the rules of the island.

These seem reasonable assumptions that we can make about our students. However most students do not understand *modus ponens*, condition (2) above, so we will have to go over this.

- Smullyan calls a reasoner **inconsistent** if for some proposition p, the reasoner believes both p and $\sim p$. (Smullyan p. 93)
- Smullyan calls a reasoner **type 1** if [s]he satisfies conditions (1) and (2) above. (Smullyan pp. 68-69)
- Smullyan calls a reasoner **peculiar with respect to a proposition** p if [s]he believes p and believes that [s]he doesn't believe p. (Smullyan p. 81)

- Smullyan calls a reasoner **type 1*** if [s]he satisfies conditions (1) - (2) above and believes that if [s]he ever believes $p \Rightarrow q$ then whenever [s]he believes p then [s]he will believe q. (Smullyan p. 83)
- Smullyan calls a reasoner **type 2** if [s]he satisfies conditions (1) - (3) above. (Smullyan pp. 89-90)
- Smullyan calls a reasoner **type 3** if [s]he satisfies conditions (1) - (4) above. (Smullyan p. 90)
- Smullyan calls a reasoner **type 4** if [s]he satisfies conditions (1) - (5) above. (Smullyan p. 90)
- Smullyan calls a reasoner **peculiar** if [s]he believes p and believes [s]he doesn't believe p.
- Smullyan defines a reasoner to be **reflexive** if for every proposition q, there exists a proposition p such that the reasoner will believe $p \equiv (Bp \;\Rightarrow\; q)$.
- Smullyan calls a reasoner **conceited** if for every proposition p, [s]he believes $Bp \;\Rightarrow p$.
- Smullyan calls a reasoner **modest** if, for every proposition p, whenever [s]he believes $Bp \;\Rightarrow p$ [s]he believes p.
- Smullyan calls a reasoner **stable** if for every proposition p, if [s]he believes that [s]he believes p, then [s]he really does believe p.

7.6.2 Smullyan's version of Gödel's Second Incompleteness Theorem

Gödel's Second Incompleteness Theorem states that any system based on arithmetic can not prove its own consistency (with in the system). Here is the corresponding theorem in the Knights and Knaves setting:

Theorem 14 (Gödel's Second Incompleteness Theorem)**.** *Suppose a native of the island says to a reasoner of type 4: "You will never believe that I am a Knight." Then if the reasoner is consistent, [s]he can never know that [s]he is consistent; or stated otherwise, if the reasoner ever believes that [s]he cannot be inconsistent, [s]he will become inconsistent!*

Proof. Suppose the reasoner does believe that [s]he is (and will remain consistent). The reasoner reasons: " Suppose I ever believe that the native is a Knight. Then I'll believe what [s]he said- I'll believe that I don't believe that [s]he is a Knight. But also, if I believe [s]he's a Knight, then I'll believe that I *do* believe [s]he's a Knight (since I'm normal [condition 4 above]). Therefore, if I ever believe that [s]he's a knight, then I'll believe both that I do believe [s]he's a Knight and that I don't believe [s]he's a Knight, which means I'll be inconsistent. Now I'll never be inconsistent, hence I will never believe [s]he's a Knight. [S]He said that I would never believe [s]he's a Knight, and what he said was true, hence [s]he is a Knight"

At this point, the reasoner believes the native is a Knight, and since [s]he is normal (condition 4 above), he will then know that [s]he believes this. Hence the reasoner will continue: "Now I believe [s]he is a Knight. [S]He said that I never would, hence [s]he made a false statement, so he not a Knight."

At this point the reasoner believes that the native is a Knight and also believes that the native is not a Knight, and so [s]he is now inconsistent. (Smullyan pp. 101-102.) □

Connection to Gödel's second Completeness Theorem

Here is Smullyan's description of how the Knights and Knaves setting of Gödel's Second Incompleteness Theorem (Smullyan, pp. 107 - 111) connects back to mathematics.

The types of systems, S, investigated by Gödel had the following properties:

1. There is a well-defined set of propositions expressible in S.

2. S has various axioms and logical rules making certain propositions *provable* in the system. We thus have a well-defined subset of the set of propositions of the system- namely, the set of *provable* propositions of the system.

3. For any proposition $p \in S$, the proposition that p is *provable* in S, denoted Bp, is itself a proposition of the system. (Note Bp may or may not be provable in S.)

Systems of type 1, 1*, 2, 3 and 4, are defined in an analogous way to the way we did for reasoners (using the symbolic representations).

Smullyan calls a system, S, a **Gödelian system** if there is a proposition $p \in S$ such that $p \equiv \sim Bp$ is provable in S.

The Knight and Knaves result is a specific example of a Gödelian system of type 4 and, in fact, Gödel's Second Incompleteness Theorem proves that all Gödelian systems of type 4 can not prove their own consistency.

Examples of Gödelian systems of type 4 include:

- Russell and Whitehead's system in *Principia Mathematica.*
- First Order Peano Arithmetic

These systems therefore, can not prove their own consistency.

7.6.3 Smullyan's version of Gödel's First Incompleteness Theorem

Gödel's First Incompleteness Theorem states that any sufficiently strong formal system there are true arithmetical statements that are undecidable in the system. Here is the corresponding theorem in the Knights and Knaves setting:

Theorem 15 (Gödel's First Incompleteness Theorem)**.** *A normal, consistent, stable reasoner of type 1 comes to the island and believes the rules of the island. [S]he meets a native who says: "You will never believe that I am a Knight. Then the reasoner's belief system is incomplete. That is, the reasoner will never believe the native is a Knight and will never believe that the native is not a Knight.*

Proof. First note that since the native said, "You will never believe I am a Knight", the reasoner will believe "the native is a Knight iff [s]he doesn't believe the native is a Knight." Stated symbolically: if K represents the statement "The native is a Knight", then the reasoner believes $K \equiv \sim BK$.

Suppose the reasoner believes K. Then, being normal, [s]he will believe BK. [S]he will also believe $\sim BK$ (since [s]he believes K and believes $K \equiv \sim BK$ and [s]he is of type 1), hence [s]he will be inconsistent. Therefore, if [s]he is consistent, [s]he will never believe K.

Since the reasoner is of type 1 and believes $K \equiv \sim BK$, the reasoner also believes $\sim K \equiv BK$. Now suppose the reasoner ever believes $\sim K$, then [s]he believes BK. Being stable, [s]he will then believe K and hence become inconsistent (since [s]he believes $\sim K$). Therefore being both stable and consistent, [s]he will never believe $\sim K$.

This is hard to translate into non-symbolic language here is what I think we get:

Suppose the reasoner believes the native is a Knight. Being normal, the reasoner will believe that [s]he believes the native is a Knight. On the other hand, being of type 1, the reasoner will believe what the native says, namely that they will never believe the native is a Knight. Thus the reasoner believes [s]he believes the reasoner is a Knight and believes [s]he doesn't believe the native is a Knight. Since the reasoner is inconsistent, [s]he will never believe the native is a Knight.

Since the reasoner is of type 1 and believes "the native is a Knight iff [s]he doesn't believe the native is a Knight"; the reasoner also believes "the native is not a Knight iff [s]he believes the native is a Knight." Suppose the reasoner at some point does not believe the native is a Knight; then [s]he believes that [s]he believes the native is a Knight; being stable, the reasoner then believes that the native is a Knight and hence become inconsistent (since [s]he believes the native is not a Knight). Therefore being both stable and consistent, [s]he will never believe the native is not a Knight.

□

Connection to Gödel's First Completeness Theorem

Here is how I see the Knights and Knaves setting of Gödel's Second Incompleteness Theorem (Smullyan, pp. 107 - 111) connects back to mathematics.

Gödel essentially proved any consistent, normal, stable Gödelian system, S of type 1 must be incomplete. More specifically, if S is a consistent, normal, stable Gödelian system of type 1, and there is a proposition $p \in S$ such that $p \equiv \sim Bp$ is provable in S, then neither p nor $\sim p$ is provable in S.

Since Russell and Whitehead's system from *Principia Mathematica* and First Order Peano Arithmetic are normal, stable Gödelian systems of type 4, they are normal, stable Gödelian systems of type 1 and hence if they are consistent, they will contain undecidable propositions.

7.6.4 Printer Version of Gödel's Second Incompleteness Theorem:

From http://rationalwiki.org/wiki/Essay:Godel's_incompleteness_theorem_simply_explained

You have a magic printer, when you type a true statement into the computer, it will print the statement out. If you type in a false statement, it will not print it out. For example, if you type in the statement, "Today is Independence Day in the United States" on July 4, the printer prints out "Today is Independence Day in the United States". If you type in the statement, "Today is Independence Day in the United States" on August 4, the printer prints out nothing.

84. Suppose you type in the statement "$2 + 2 = 4$". What will the printer do? Explain.

85. Suppose you type in the statement "$2 + 2 = 5$". What will the printer do? Explain.

86. Suppose you type in the statement "The printer can print the statement '$2 + 2 = 4$'." What will the printer do? Explain.

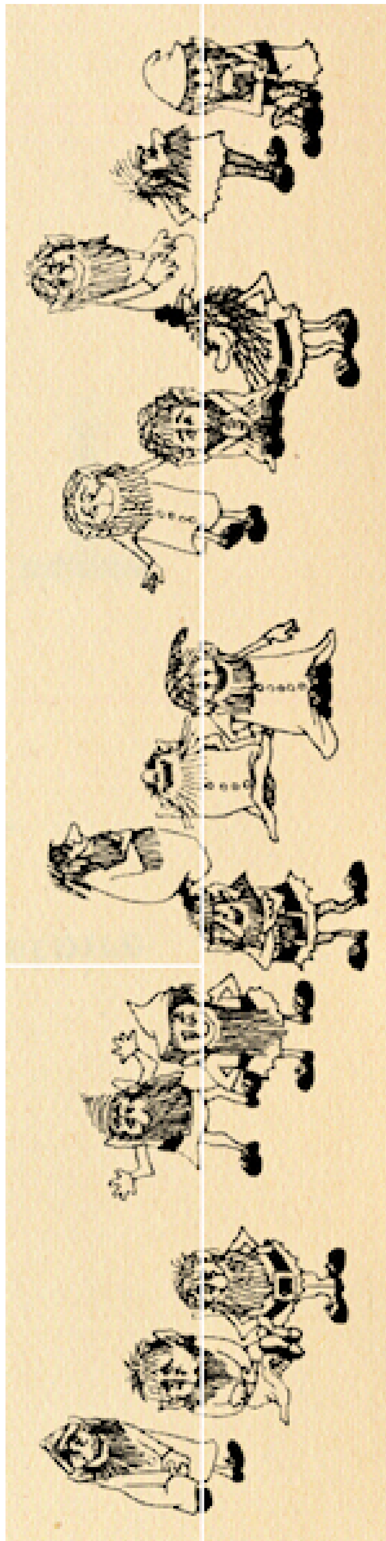
87. Suppose you type in the statement "The printer can print the statement '$2 + 2 = 5$'." What will the printer do? Explain.

88. Suppose you type in the statement "The printer can not print the statement '$2 + 2 = 4$'." What will the printer do? Explain.

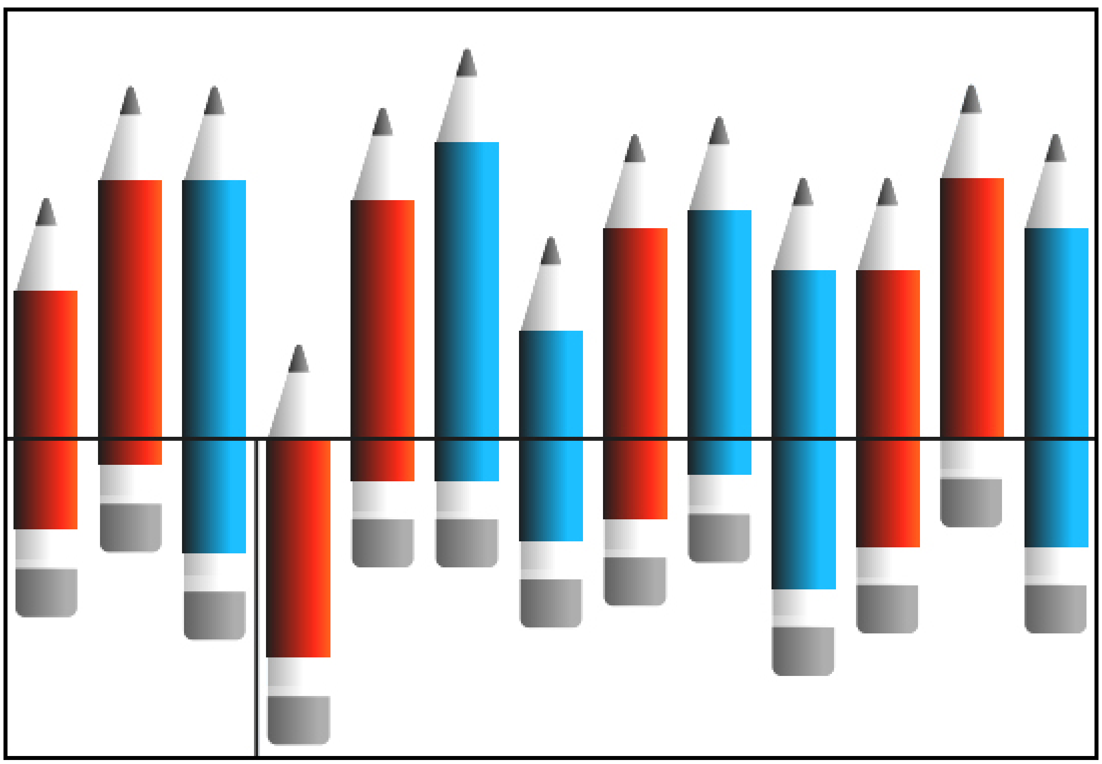
89. Suppose you type in the statement "The printer can not print the statement '$2 + 2 = 5$'." What will the printer do? Explain.

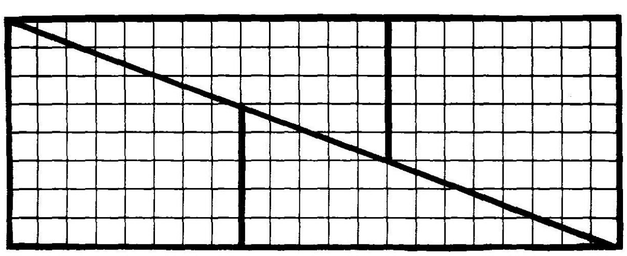
Chapter 8

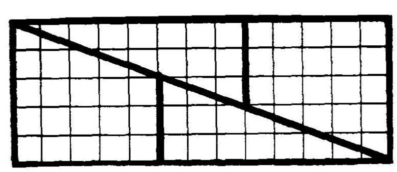
Appendix




SAILING UNDER
FALSE COLORS
PRIZE PUZZLE
BY
SAM LOYD







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