



ART OF MATHEMATICS

DISCOVERING THE

# CALCULUS

MATHEMATICAL INQUIRY IN THE LIBERAL ARTS

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with Volker Ecke



# Discovering the Art of Mathematics

## Calculus (Draft)

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# Contents

Acknowledgements	v
Preface: Notes to the Explorer	1
Navigating This Book	3
Chapter 1. Alternating Harmonic Series	5
Chapter 2. What is Area?	9
1. Geoboards and Points	10
2. Magical Shapes	14
3. Archimedes' Circles	15
4. Cutting up the Circle to find its Area...	18
5. Area of Fractals	19
Chapter 3. A Taste of Measure Theory	21
1. Introduction	21
2. Cantor Sets	22
Chapter 4. String Art	29
1. What is String Art?	29
2. Derivatives	31
3. Functions and Algebraic Curves	34
4. Creating String Art	36
5. Further Investigations	38
6. Connections	40
7. Fundamental Theorem of Calculus	43
Chapter 5. Integration	45
1. Quadrature of the Parabola	45
2. Riemann and Cauchy and again the Parabola	47
3. Integration and Art	50
4. Tolstoy's Integration Metaphor	52
5. Cars instead of Planets	53
6. Integration in higher Dimensions	53
7. Further Investigations and Connections	58
Chapter 6. The Banach-Tarski Paradox	61
1. Introduction	61
2. Equidecompositions	63
3. The HyperDictionary	67

Bibliography	69
Index	71

## Preface: Notes to the Explorer

Yes, that's you - you're the explorer.

"Explorer?"

Yes, explorer. And these notes are for you.

We could have addressed you as "reader," but this is not a traditional book. Indeed, this book cannot be read in the traditional sense. For this book is really a guide. It is a map. It is a route of trail markers along a path through part of the world of mathematics. This book provides you, our explorer, our heroine or hero, with a unique opportunity to explore this path - to take a surprising, exciting, and beautiful journey along a meandering path through a mathematical continent named the infinite. And this is a vast continent, not just one fixed, singular locale.

"Surprising?" Yes, surprising. You will be surprised to be doing real mathematics. You will not be following rules or algorithms, nor will you be parroting what you have been dutifully shown in class or by the text. Unlike most mathematics textbooks, this book is not a transcribed lecture followed by dozens of exercises that closely mimic illustrative examples. Rather, after a brief introduction to the chapter, the majority of each chapter is made up of Investigations. These investigations are interwoven with brief surveys, narratives, or introductions for context. But the Investigations form the heart of this book, your journey. In the form of a Socratic dialogue, the Investigations ask you to explore. They ask you to discover the infinite. This is not a sightseeing tour, you will be the active one here. You will see mathematics the only way it can be seen, with the eyes of the mind - your mind. You are the mathematician on this voyage.

"Exciting?" Yes, exciting. Mathematics is captivating, curious, and intellectually compelling if you are not forced to approach it in a mindless, stress-invoking, mechanical manner. In this journey you will find the mathematical world to be quite different from the static barren landscape most textbooks paint it to be. Mathematics is in the midst of a golden age - more mathematics is discovered each day than in any time in its long history. Each year there are 50,000 mathematical papers and books that are reviewed for *Mathematical Reviews!* *Fermat's Last Theorem*, which is considered in detail in Discovering that Art of Mathematics - Number Theory, was solved in 1993 after 350 years of intense struggle. The 1\$ Million Poincaré conjecture, unanswered for over 100 years, was solved by **Grigori Perelman** (Russian mathematician; 1966 - ). In the time period between when these words were written and when you read them it is quite likely that important new discoveries adjacent to the path laid out here have been made.

"Beautiful?" Yes, beautiful. Mathematics is beautiful. It is a shame, but most people finish high school after 10 - 12 years of mathematics *instruction* and have no idea that mathematics is beautiful. How can this happen? Well, they were busy learning mathematical skills, mathematical reasoning, and mathematical applications. Arithmetical and statistical skills are useful skills everybody should possess. Who could argue with learning to reason? And we are all aware, to some degree or another, how mathematics shapes our technological society. But there is something more to mathematics than its usefulness and utility. There is its beauty. And the beauty of mathematics is one of its driving forces. As the famous **Henri Poincaré** (French mathematician; 1854 - 1912) said:

The mathematician does not study pure mathematics because it is useful; [s]he studies it because [s]he delights in it and [s]he delights in it because it is beautiful.

Mathematics plays a dual role as both a liberal art and as a science. As a powerful science, mathematics shapes our technological society and serves as an indispensable tool and language in many fields. But it is not our purpose to explore these roles of mathematics here. This has been done in many other fine, accessible books (e.g. [COM] and [TaAr]). Instead, our purpose here is to journey down a path that values mathematics from its long tradition as a cornerstone of the liberal arts.

Mathematics was the organizing principle of the *Pythagorean society* (ca. 500 B.C.). It was a central concern of the great Greek philosophers like **Plato** (Greek philosopher; 427 - 347 B.C.). During the Dark Ages, classical knowledge was rescued and preserved in monasteries. Knowledge was categorized into the classical liberal arts and mathematics made up several of the seven categories.<sup>1</sup> During the Renaissance and the Scientific Revolution the importance of mathematics as a science increased dramatically. Nonetheless, it also remained a central component of the liberal arts during these periods. Indeed, mathematics has never lost its place within the liberal arts - except in the contemporary classrooms and textbooks where the focus of attention has shifted solely to the training of qualified mathematical scientists. If you are a student of the liberal arts or if you simply want to study mathematics for its own sake, you should feel more at home on this exploration than in other mathematics classes.

“Surprise, excitement, and beauty? Liberal arts? In a mathematics textbook?” Yes. And more. In your exploration here you will see that mathematics is a human endeavor with its own rich history of human struggle and accomplishment. You will see many of the other arts in non-trivial roles: art and music to name two. There is also a fair share of philosophy and history. Students in the humanities and social sciences, you should feel at home here too.

Mathematics is broad, dynamic, and connected to every area of study in one way or another. There are places in mathematics for those in all areas of interest.

The great **Bertrand Russell** (English mathematician and philosopher; 1872 - 1970) eloquently observed:

Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.

It is my hope that your discoveries and explorations along this path through the infinite will help you glimpse some of this beauty. And I hope they will help you appreciate Russell’s claim that:

... The true spirit of delight, the exaltation, the sense of being more than [hu]man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

Finally, it is my hope that these discoveries and explorations enable you to make mathematics a real part of your lifelong educational journey. For, in Russell’s words once again:

... What is best in mathematics deserves not merely to be learned as a task but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement.

Bon voyage. May your journey be as fulfilling and enlightening as those that have served as beacons to people who have explored the continents of mathematics throughout history.

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<sup>1</sup>These were divided into two components: the *quadrivium* (arithmetic, music, geometry, and astronomy) and the *trivium* (grammar, logic, and rhetoric); which were united into all of knowledge by philosophy.

## Navigating This Book

Before you begin, it will be helpful for us to briefly describe the set-up and conventions that are used throughout this book.

As noted in the Preface, the fundamental part of this book is the Investigations. They are the sequence of problems that will help guide you on your active exploration of mathematics. In each chapter the investigations are numbered sequentially. You may work on these investigation cooperatively in groups, they may often be part of homework, selected investigations may be solved by your teacher for the purposes of illustration, or any of these and other combinations depending on how your teacher decides to structure your learning experiences.

If you are stuck on an investigation remember what **Frederick Douglass** (American slave, abolitionist, and writer; 1818 - 1895) told us: “If thee is no struggle, there is no progress.” Keep thinking about it, talk to peers, or ask your teacher for help. If you want you can temporarily put it aside and move on to the next section of the chapter. The sections are often somewhat independent.

Investigation numbers are bolded to help you identify the relationship between them.

Independent investigations are so-called to point out that the task is more significant than the typical investigations. They may require more involved mathematical investigation, additional research outside of class, or a significant writing component. They may also signify an opportunity for class discussion or group reporting once work has reached a certain stage of completion.

The Connections sections are meant to provide illustrations of the important connections between mathematics and other fields - especially the liberal arts. Whether you complete a few of the connections of your choice, all of the connections in each section, or are asked to find your own connections is up to your teacher. But we hope that these connections will help you see how rich mathematics’ connections are to the liberal arts, the fine arts, culture, and the human experience.

Further investigations, when included are meant to continue the investigations of the area in question to a higher level. Often the level of sophistication of these investigations will be higher. Additionally, our guidance will be more cursory.

Within each book in this series the chapters are chosen sequentially so there is a dominant theme and direction to the book. However, it is often the case that chapters can be used independently of one another - both within a given book and among books in the series. So you may find your teacher choosing chapters from a number of different books - and even including “chapters” of their own that they have created to craft a coherent course for you. More information on chapter dependence within single books is available online.

Certain conventions are quite important to note. Because of the central role of proof in mathematics, definitions are essential. But different contexts suggest different degrees of formality. In our text we use the following conventions regarding definitions:

- An *undefined term* is italicized the first time it is used. This signifies that the term is: a standard technical term which will not be defined and may be new to the reader; a term that will be defined a bit later; or an important non-technical term that may be new to the reader, suggesting a dictionary consultation may be helpful.

- An ***informal definition*** is italicized and bold faced the first time it is used. This signifies that an implicit, non-technical, and/or intuitive definition should be clear from context. Often this means that a formal definition at this point would take the discussion too far afield or be overly pedantic.
- A **formal definition** is bolded the first time it is used. This is a formal definition that suitably precise for logical, rigorous proofs to be developed from the definition.

In each chapter the first time a biographical name appears it is bolded and basic biographical information is included parenthetically to provide some historical, cultural, and human connections.

## CHAPTER 1

# Alternating Harmonic Series

As we have already discovered, one of the essential problems of the calculus is to determine the area under a curve. A critically important curve to find the area under is  $y = \frac{1}{x}$  as the area under this curve defines the *natural logarithm*, the inverse of the base  $e$  exponential that describes exponential growth.

Author's Note: Did the McLaurin series predate the sum of the alternating harmonic series sum? One would think so or it would be a triviality - in some sense. So what is the history? How was it that Pitero Mengoli discovered it in 1650?

In this section we will consider the area that defines  $\ln(2)$ . This is the area under the curve between  $x = 1$  and  $x = 2$  as shown in Figure 1. We will try to express this area by approximating it more and more closely via Riemann rectangles.<sup>1</sup>

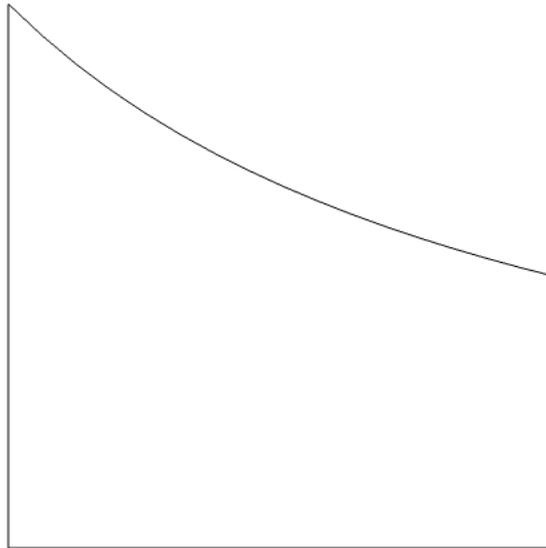


FIGURE 1. Area under the curve  $y = \frac{1}{x}$  for  $1 \leq x \leq 2$ .

Copies of the figures below are included in the appendix for you to work with.

1. What is the area enclosed by the square in Figure 2?
2. In a copy of Figure 3, find and highlight a rectangle whose area is  $\frac{1}{2}$ .
3. Shade a region whose area represents  $1 - \frac{1}{2}$ .

---

<sup>1</sup>This approach is due to **Matt Hudelson** (; -) from “Proof without words: The alternating harmonic series sums to  $\ln(2)$ ”, *Mathematics Magazine*, vol. 83, 2010, p. 294.

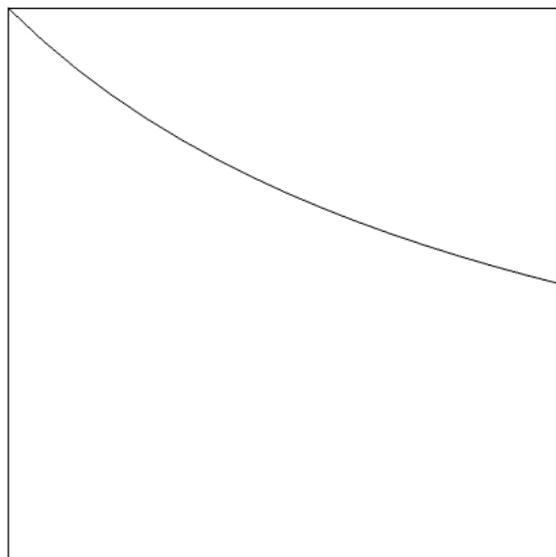


FIGURE 2. First approximation to the area under the curve  $y = \frac{1}{x}$  for  $1 \leq x \leq 2$ .

4. In a new copy of Figure 3, find and highlight a rectangle whose area is  $\frac{1}{3}$ .
5. Shade a region whose area represents  $1 - \frac{1}{2} + \frac{1}{3}$ .
6. After including two more terms,  $\frac{1}{2}$  and  $\frac{1}{3}$ , do you have a nice approximation for the area? Explain.

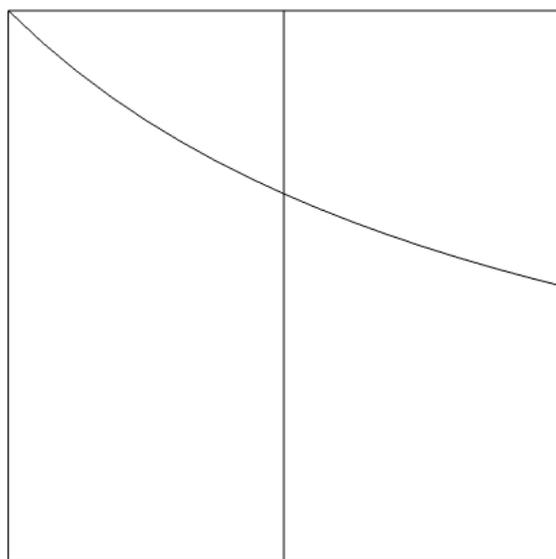


FIGURE 3. Dividing the domain of  $y = \frac{1}{x}$  for  $1 \leq x \leq 2$  in half.

Let's try to continue this process, including terms in pairs.

7. On a copy of Figure 5, highlight rectangles of areas  $\frac{1}{4}$  and  $\frac{1}{5}$  that may be subtracted and added (respectively) to approximate the area under the curve.
8. Shade a region whose area represents  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$ .

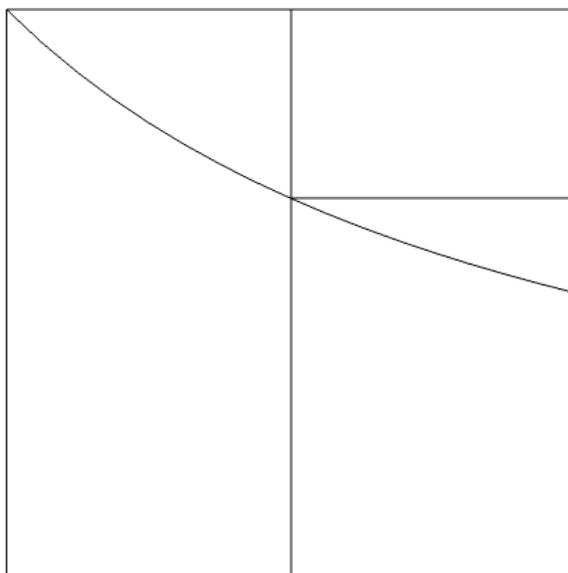


FIGURE 4. Second approximation to the area under the curve  $y = \frac{1}{x}$  for  $1 \leq x \leq 2$ .

9. On another copy of Figure 5, highlight rectangles of areas  $\frac{1}{6}$  and  $\frac{1}{7}$  that may be subtracted and added (respectively) to approximate the area under the curve.
10. Shade a region whose area represents  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7}$ .
11. The vertical lines in Figure 5 are spaced one-quarter of a unit apart. In Figure 6 we added lines so there were lines spaced one-eighth of a unit apart. Could we repeat the process above? Explain in detail.
12. Shade the region that would result from this approximation.
13. Write the sum that explicitly represents this area.
14. Using your observations above, explain how to write  $\ln(2)$  as the sum of an infinite series. (The name of this series is the **alternating harmonic series**.)

Author's Note: Figure 5 looks like logarithmic graph paper. Is it?

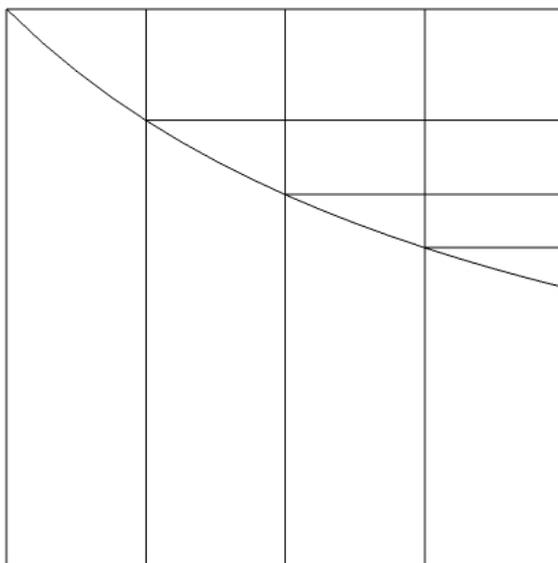


FIGURE 5. Third approximation to the area under the curve  $y = \frac{1}{x}$  for  $1 \leq x \leq 2$ .

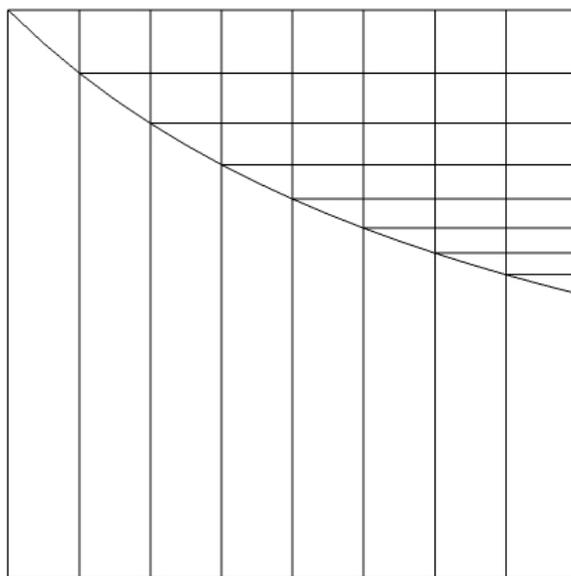


FIGURE 6. Fourth approximation to the area under the curve  $y = \frac{1}{x}$  for  $1 \leq x \leq 2$ .

## CHAPTER 2

# What is Area?

The simplest schoolboy is now familiar with truths for which Archimedes would have sacrificed his life.

**Ernest Renan** (French Philosopher; 1823 - 1892)

Finding areas has been important for a long time. In ancient Egypt (ca. 5000 years ago!) dividing up the land around the river Nile was crucial, since only the land close enough to the river to be flooded could be used to grow crops. The Egyptians knew an astonishing amount of mathematics, they could for example compute *areas under curves*.<sup>1</sup> See Figure 1 for an ancient Egyptian papyrus showing area computation of triangles and area estimation of circles.



FIGURE 1. Rhind Mathematical Papyrus

What areas do we need to be able to compute today? Of course there are many small examples, like finding the area of your living room if you want to buy a new carpet. But think also about finding the area of land that will be flooded when a hurricane hits, or the area of the ocean effected by a big oil spill.

1. Find two more important examples that need area computations.

Mathematicians also enjoy more abstract examples. For instance they wonder what the area is inside Koch's snowflake which is the result of the process started in Figure 2 after infinitely many iterations.

2. Let's remember what you have learned in school: Given a rectangle of dimensions  $x$  and  $y$  how do you compute the area of the rectangle? See Figure 3.

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<sup>1</sup>The advanced state of this math is confirmed by an architectural drawing even older than the Rhind Papyrus that shows that Nilotic engineers had learned to find the area under a curve more than 5,000 years ago. See <http://www.touregypt.net/featurestories/numbers.htm>

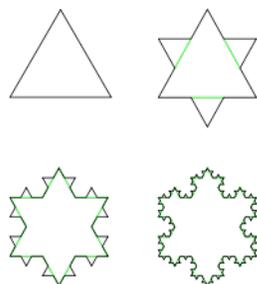


FIGURE 2. The first 4 iterations of Koch's snowflake.



FIGURE 3. The dimensions of a rectangle.

3. Explain why you believe the above formula makes sense. Why do we compute area in this way?

The following investigations will help you understand how we can decide which area formulas make sense. We would like *area* to have the following properties:

- a) If we cut a shape into several pieces, the area of the whole shape should be the same as the sum of the areas of the smaller pieces.
  - b) If two shapes are congruent then their areas should be the same. Two shapes are called *congruent* if one can be transformed into the other by translation (sliding), rotation (turning) or reflection (flipping).
  - c) The area of a square with side length 1 should be equal to 1.
4. Explain why it makes sense to require the above properties a) through c).
  5. Assume that we compute the area of the reactangle in Figure 3 as  $A = x + y$ . Explain why this would not be a good choice for area computation. Use the above properties of area in your argument.
  6. Assume that we compute the area of the reactangle in Figure 3 as  $A = x^2y$ . Explain why this would not be a good choice for area computation. Use the above properties of area in your argument.
  7. Assume that we compute the area of the reactangle in Figure 3 as  $A = y$ . Explain why this would not be a good choice for area computation. Use the above properties of area in your argument.
  8. Assume that we compute the area of the reactangle in Figure 3 as  $A = xy$ . Explain why this would be a good choice for area computation. Use the above properties of area in your argument.

**Definition 1.** The area of a rectangle with dimensions  $x$  and  $y$ , see Figure 3, is defined as  $A = xy$ .

### 1. Geoboards and Points

Before we can handle the complicated area of an oil spill or Koch's snowflake, we need to get some practice with easier shapes. Geoboards are a nice tool to practice rearranging and computing shapes.

See Figure 4. For the following exercises you are encouraged to use a geoboard to help your thinking.



FIGURE 4. A wooden geoboard with rubber bands.

To measure area we decide on a unit square with area 1 and count how many unit squares fit into a given shape. No equations are necessary.

9. Let's choose the smallest square made of 4 pegs on our geoboard as our unit square. How many 1x1 unit squares would fit into the shape in Figure 5?

We might wonder if it matters which square we decide to use as our unit. Let's think about that:

10. How many 2x2 squares would fit into the shape in Figure 5?  
 11. How does your choice of unit square relate to the use of different length and area measurements in, for example, the US and in Europe?

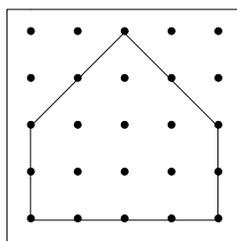


FIGURE 5.

For the next investigations we assume that we use 1x1 unit squares to measure area. If you use equations to compute the area, see if you can find a different way without using any equations. Or see if you can understand **why** the equations you are using actually compute the desired area.

12. For the following shape, see Figure 6, compute the area. Explain your reasoning in detail.  
 13. For the following shape, see Figure 7, compute the area. Explain your reasoning in detail.  
 14. For the following shape, see Figure 8, compute the area. Explain your reasoning in detail.  
 15. For the following shape, see Figure 9, compute the area. Explain your reasoning in detail.  
 16. For the following shape, see Figure 10, compute the area. Explain your reasoning in detail.  
 17. Summarize the strategies you used in the last geoboard investigations. Do you think you can compute the area of **any** shape using your techniques? Explain.

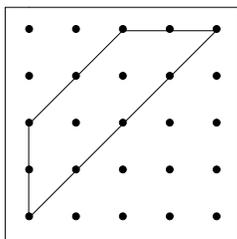


FIGURE 6.

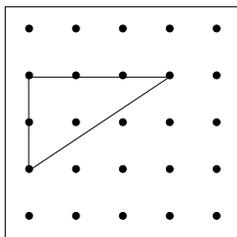


FIGURE 7.

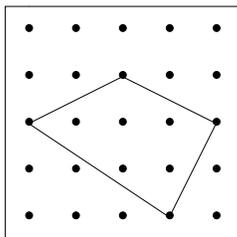


FIGURE 8.

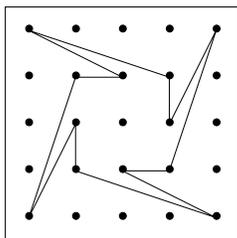


FIGURE 9.

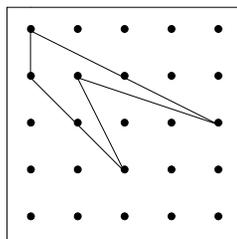


FIGURE 10.

## 2. Magical Shapes

Here is a puzzle for you:

18. Find the area of the large shapes in Figure 11 and Figure 12.

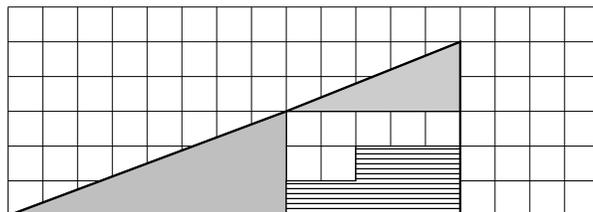


FIGURE 11.

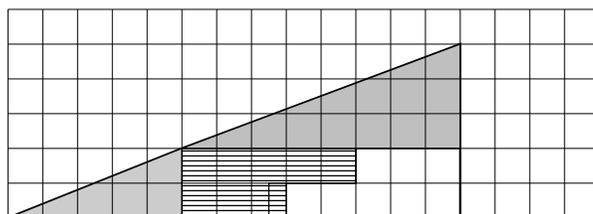


FIGURE 12.

19. Compute the area of the four shapes that the large shape in Figure 11 consists of.  
 20. Compute the area of the four shapes that the large shape in Figure 12 consists of.  
 21. Comparing Investigation 19 and Investigation 20, are you surprised? Why?  
 22. Take some tape and “draw” the shapes and their pieces on a tile floor with large tiles. Look carefully at the situation and explain what is going on in Investigation 21.

**Euclid** (Greek Mathematician; fl 300 BC - ) defined a *point* using the following definition:

A point is that which has no part.

Euclid's book *The Elements* contains all the basic definitions, axioms and theorems of basic geometry, which we now call *Euclidian Geometry*. His book is the second most read book in history! Which one, do you think, is the first most read book?

Mathematicians think of a *point* as being infinitely small. That means we can't really “draw a point” on our paper, we just draw a small disk instead.

23. Why do you think mathematicians want a point to be infinitely small instead of just being a small disk? Think of advantages and disadvantages of the definition.  
 24. Consider the shape in Figure 5. How many points (mathematical points, not pegs!) are inside your shape?  
 25. What is the area of one (mathematical) point? Use the definition of the area of a rectangle in your explanation.  
 26. Using Investigation 24 and Investigation 25, what is the area of the shape in Figure 5? Does this surprise you?  
 27. To compute the area of a shape, do you think we can break the shape into pieces and just add up the area of the pieces? Explain.

It seems that breaking a shape into pieces to find the area is a good idea, since the total area stays the same if we cut a shape apart. Unfortunately we have to be careful if there are “too many pieces that are too small”. There is whole branch of mathematics, called *Measure Theory*, that deals with this kind of problems. We will learn more about this in a later section.

### 3. Archimedes’ Circles

28. Take graph paper and draw a circle of radius 4. Estimate the area, using the boxes on your graph paper as units. Explain your strategies.
29. Compare your results from Investigation 28 with your group. How accurate is your estimation? How can you make more accurate estimates?
30. Make more accurate estimates for the areas of the circle.
31. Can you compute the **exact** area of the circles using your method? Explain why or why not?

**Archimedes** (Greek Mathematician; c. 287 BC - c. 212 BC) had a different idea of estimating the area of a circle with radius  $r = 4$ . He drew different shapes inside the circle of which he could compute the area more easily.

32. If you were Archimedes, which shape would you choose? Explain.
33. Can you compute the area of the shape you chose in Investigation 32? Why or why not? (Assume your circle has radius  $r = 4$ )

To be able to compute the area of the shape in Investigation 32 we need to get some practice in finding areas.

You might remember some equations for area computations from former mathematics classes. Did you just memorize them or did/do you understand **why** they work? Recall that we defined the area of a rectangle with dimensions  $x$  and  $y$  as  $A = xy$ .

34. Now extend the top and the bottom edge of your rectangle and move the top of your rectangle to the right. You have to keep the new shape between the lines. See Figure 13. What is the name of the new shape? Is the area of the new shape the same of different from the area of the rectangle? Explain.

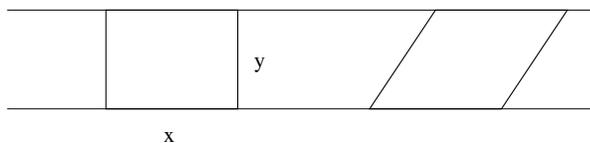


FIGURE 13.

35. Explain how to compute the area of **any** parallelogram.
36. Use your explanation in Investigation 35 to find the area of the parallelogram in Figure 14. Check your work by computing the area of the parallelogram in a different way.
37. Find the area of the triangle in Figure 15 using the area of a parallelogram. Explain. Check your answer using a different method.
38. Explain, how to compute the area of **any** triangle.
39. Using your strategy from Investigation 38, find the area of the second triangle in Figure 16. Explain.
40. Using your strategy from Investigation 38, find the area of the first triangle in Figure 16. What is different or difficult compared to the last investigation? Explain.
41. Recall the Pythagorean theorem and use it to find the area of the first triangle in Figure 16.

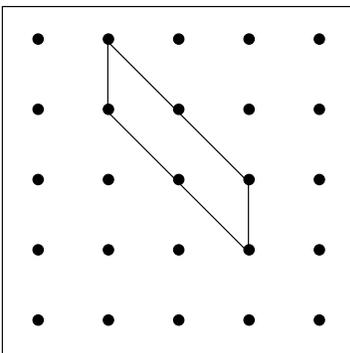


FIGURE 14. Parallelogram on a Geoboard

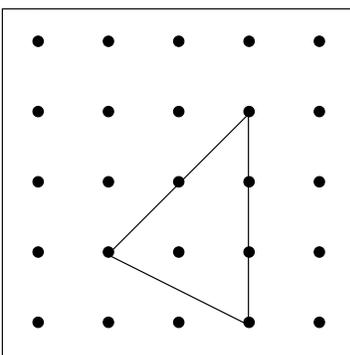


FIGURE 15. Triangle on a Geoboard

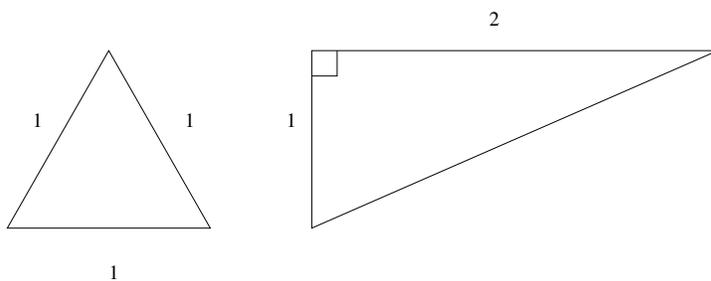


FIGURE 16. Two Triangles

**42. INDEPENDENT INVESTIGATION:** Look at your shape from Investigation **32**. Assume your circles has radius  $r = 4$ . Can you cut your shape into triangles? Can you use those triangles to compute the area of your shape? If there are different ways to cut your shape

into triangles, try finding the one that is most helpful for finding the total area of your shape.

- 43. Classroom Discussion:** Compare the shapes and their area estimates from Investigation 42 with your class mates. Which one do you think is the best estimate? Why? Compare also how you cut your shapes into triangles. Is there a best way to arrange your triangles?

Archimedes inscribed *regular polygons* in the circle. A regular polygon consists of equal length line segments meeting at equal angles. See Figure 17 for some examples.

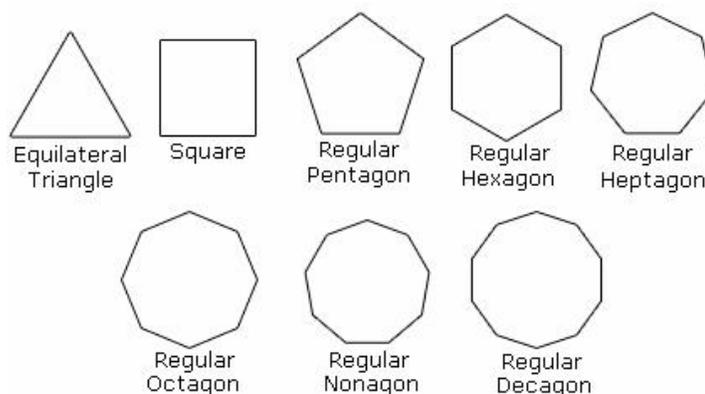


FIGURE 17. Some Regular Polygons

44. Why are regular polygons a good choice for estimating the area of a circle? Explain.  
 45. Compare your shape from Investigation 32 with a regular polygon. How are they similar or different?

The *apothem* of a regular polygon is defined as the line segment from the center of the polygon to the midpoint of one of its sides. See Figure 18.

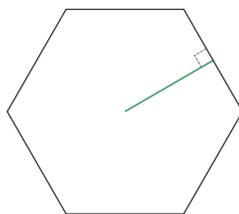


FIGURE 18. Apothem of a Hexagon.

46. Find the length of the apothem in Figure 18. Assume that the length of one side of the hexagon is 1 unit.  
 47. For the hexagon in Figure 18 compute the area using the apothem result from Investigation 46.

**48. INDEPENDENT INVESTIGATION:** Given a circle of radius 4, use Archimedes' method and an inscribed hexagon to compute an estimate of the area of the circle. Now inscribe a dodecagon into the circle by subdividing the sides of your hexagon. Estimate the area of the circle using the area of the dodecagon.

Hint: Draw a picture including the hexagon and the dodecagon. Can you continue this process? Compare your answer with Investigation **28**.

Circles come in very different sizes, so the regular polygons can have different side lengths. To simplify the process we want to find the area of the polygon using variables for the side length and the apothem length. We will call the side length  $s$  and the apothem length  $a$ .

**49.** Label Figure 18 with  $s$  and  $a$  as defined above.

**50.** Find the area of the hexagon as in Figure 18 using  $s$  and  $a$ . Explain your strategy.

We now understand how to find the area of one regular polygon. Now which one do we use for the estimation of the circle? Which one did Archimedes choose?

**51.** How many different polygons are there? Draw a few inside the circle and decide which one is the best to be inscribed the circle for an area estimation. Explain.

**52.** Given **any** regular polygon with  $n$  sides of length  $s$  and apothem  $a$  find the area of the polygon.

**53.** Express the perimeter  $p$  of a regular polygon in terms of  $a$  and  $s$ .

**54.** Using Investigation **52** and Investigation **53**, find the area of **any** regular polygon with perimeter  $p$  and apothem  $a$ . Don't use the side length  $s$  anymore in your final answer.

It is important to be able to estimate, but we would prefer to compute the **exact** area of a circle of a given radius  $r$ .

**55.** If you choose inscribed regular polygons that approxiamte the circle better and better, how does the apothem of the polygon relate to the radius of the circle? Explain.

**56.** If you choose inscribed regular polygons that approxiamte the circle better and better, how does the perimeter of the polygon relate to the circumference of the circle? Explain.

**57.** Using Investigation **55** and Investigation **56**, how can we find the area of a circle given its radius  $r$  and its circumference?

**58.** The circumference of a circle of radius  $r$  can be computed as  $p = 2\pi r$ . See ??? for investigations on how to develop that equation.

**59.** Using the equation for the circumference of the circle and Investigation **57** find a general equation for the area of a circle of radius  $r$ .

**60.** Using the equation for the circumference of the circle and Investigation **57** compute the area of a circle of radius 4. Compare you result with Investigation **48**.

The above approach might seem complicated but remember that this is how Archimedes thought about circles. He was able to compute very good estimates for the area of a circle. He did, however, not use the constant  $\pi$  as we do today.

The process of inscribing regular polygons with more and more sides into the circle until "there is no space left" is a core idea in *Calculus* especialy in the area called *Integration*. We will use the idea in a later section.

#### 4. Cutting up the Circle to find its Area...

This section will show you a very different way of finding the area of a circle. It was found by **Leonardo Da Vinci** (Italian Mathematician, Scientist and Inventor; 1452 - 1519)

**61.** Look at Figure 19. Why is the area of the circle the same as the area of the shape below?

**62.** Why are the dimensions of the shape below  $r$  and  $\pi r$ ?

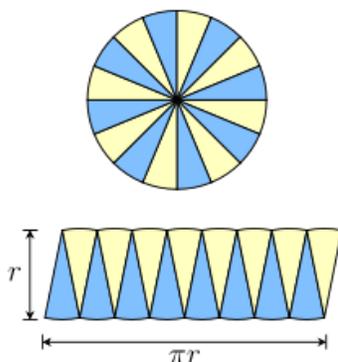


FIGURE 19. A visual proof of the area of a circle of radius  $r$ .

63. What is the estimated area of the shape below? How did you estimate?
64. Using Investigation 61 through Investigation 63, what is your estimate for the area of a circle with radius  $r$ ?
65. How could you change the picture to get an even better estimate for the area of a circle with radius  $r$ ?
66. Can you continue your argument and find the *exact* area of a circle with radius  $r$ ? Explain.
67. Compare your result of Investigation 66 with Investigation 48 and Investigation 60. Do your results agree? Why or why not?

### 5. Area of Fractals

Take an equilateral triangle and assume its area to be 1. Now divide each side into three equal pieces and attach (smaller) equilateral triangles on the middle thirds. See Figure 20.

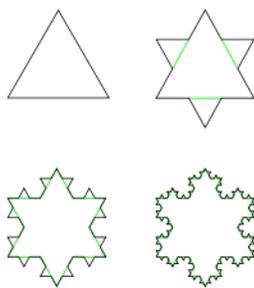


FIGURE 20. The first 4 iterations of Koch's snowflake.

68. What is the area of one smaller triangle? Explain.
69. How many smaller triangles do you need to attach?

Now you keep repeating the same process. Divide each line segment on the outside of the snowflake into three equal pieces and attach (even smaller) equilateral triangles on the middles thirds.

70. What is the area of one even smaller triangle? Explain.
71. How many even smaller triangles do you need to attach?
72. Our goal is to compute the area of the Koch snowflake after *infinitely* many iterations. Do you think the area of Koch's snowflake will be finite or infinite? Explain.
73. Repeating the above pattern, do you notice a pattern in the sizes of the triangles?
74. Repeating the above pattern, do you notice a pattern in the number of triangles you need to attach in each step?

For the following computations you need to know about infinite series, especially the *geometric series*. See Discovering the Art of Mathematics: The Infinite.

75. Write the area of Koch's snowflake as an infinite series.
76. Use your knowledge about the sum of the geometric series to find the area of Koch's snowflake.
77. Does the above result surprise you or not? Explain.

We answered our question about the area of the fractal, but what about the perimeter? Is the perimeter of Koch's snowflake finite or infinite? To make computations easier, let's start with a new construction, in which the **length of each side** of the original triangle is 1.

78. Explain why the area of the large triangle is now no longer equal to 1.
79. Find the perimeter of the first triangle.
80. Find the perimeter of a smaller triangle.
81. Find the perimeter of an even smaller triangle.
82. Find the perimeter of Koch's snowflake after 2 iterations.
83. Find the perimeter of Koch's snowflake after 3 iterations.
84. Find the perimeter of Koch's snowflake after 4 iterations.
85. Write the perimeter of Koch's snowflake as an infinite series.
86. Use your knowledge about the sum of the geometric series to find the perimeter of Koch's snowflake.
87. Does the above result surprise you or not? Explain.

## CHAPTER 3

# A Taste of Measure Theory

It's not denial. I'm just very selective about what I accept as reality.

**Calvin and Hobbes** (American Cartoonist Bill Watterson; 1958 - )

### 1. Introduction

*Length* is one of the most important concepts in calculus (and mathematics), and we generally take the definition of length for granted. However, there are many sets of numbers that are important in calculus and mathematics for which we want to have a way of quantifying how much space they take up even though they may be too spread out for the word length to be appropriate. Because some of these sets of numbers have lots of “holes” in them, mathematicians use the term *measure* for the generalized notion of length we will be discussing in this chapter.

Need more **blah** here.

1. What is the measure (or length) of the closed interval  $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ ? (See Figure 1.) Explain.

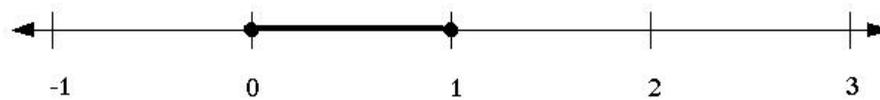


FIGURE 1. The closed Unit Interval,  $[0,1]$

2. What is the measure (or length) of the open interval  $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ ? (See Figure 2.) Explain.

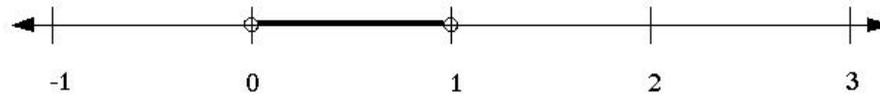


FIGURE 2. The open Unit Interval,  $(0,1)$

3. What is the total measure (or length) of the set  $A = [0, 2] \cup (3, 6) = \{x \in \mathbb{R} : 0 \leq x \leq 2 \text{ or } 3 < x < 6\}$ ? (See Figure 3.) Explain.
4. What should be the measure of the single point set  $P = \{0\}$ ? Explain.
5. What should be the measure of the two point set  $P = \{0, 1\}$ ? Explain.
6. Is your answer to Investigation 5 consistent with your answers to Investigations 1-2? Explain.

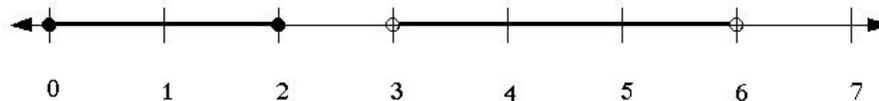


FIGURE 3. The set  $[0, 1] \cup (3, 6)$

7. Use the lengths of the intervals  $[2, 10]$ ,  $(2, 5)$ ,  $(5, 9)$  and  $(9, 10)$  to determine what should be the measure of the four-point set  $\{2, 5, 9, 10\}$ . Explain.
8. Use the ideas from Investigations 1, 2, 5 and 7 to explain why any finite set of numbers has measure zero.

In the above Investigations it was probably fairly easy to determine what was the measure of each of the sets. However, in some cases it isn't clear what the measure should be. In the next set of questions we will look at two sets of numbers that are constructed in similar ways but have, perhaps surprisingly, very different measures.

## 2. Cantor Sets

One of the most important, and interesting, types of sets in all of mathematics are *Cantor Sets*. Although this type of set is named after **Georg Cantor** (German Mathematician; 1845 - 1918), it was first described by **Henry John Stephen Smith** (Irish Mathematician; 1826 - 1883) in an 1875 paper on *integration*. We will start with the most well known Cantor Set, the *Cantor Ternary Set*.

**2.1. The Cantor Ternary Set.** *The Cantor Ternary Set* is a subset of the closed unit interval  $[0, 1]$  and we will construct it in an infinite number of stages. At each stage we will remove more and more from the interval  $[0, 1]$  and the Cantor Ternary Set is what will be left at the end of this infinite process. You might think this infinite process might mean we can never really specify what is in the Cantor Ternary Set, but we can make the description precise enough so that we can be very clear about what is in the Cantor Ternary Set and what is not.

In this section we will be using geometric series, so your instructor should provide you with some information about working with infinite geometric series. One good reference is from our *Discovering the Art of Mathematics* project; Chapter 3 in [Discovering the Art of the Infinite](#) covers geometric series in sufficient detail. In particular, you will need to be able to recognize whether an infinite series is geometric, under what conditions an infinite geometric series will converge and how to determine the sum whenever the series does converge.

Stage 0:

We start with the unit interval  $[0, 1]$ , which we will denote by  $C_0^3$ .

**Note on the notation:** The 3 in the notation is used because the 3 has an important role in constructing the set and this will distinguish this set from other Cantor Sets.

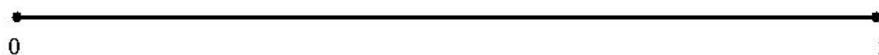


FIGURE 4. Stage 0: The interval  $[0, 1]$

Stage 1:

We then remove the open interval  $(\frac{1}{3}, \frac{2}{3})$ ; that is, we remove all numbers that strictly bigger than  $\frac{1}{3}$  and strictly less than  $\frac{2}{3}$  but leaving the numbers  $\frac{1}{3}$  and  $\frac{2}{3}$ . We denote this stage by  $C_1^3$ :



FIGURE 5. Stage 1:  $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

Stage 2:

The next stage, denoted by  $\mathcal{C}_2^3$ , is obtained by removing the open intervals  $(\frac{1}{9}, \frac{2}{9})$  and  $(\frac{7}{9}, \frac{8}{9})$ , again leaving the end points  $\frac{1}{9}, \frac{2}{9}, \frac{7}{9}$  and  $\frac{8}{9}$ :



FIGURE 6. Stage 2:  $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

Stage 3:

For the next stage,  $\mathcal{C}_3^3$ , we remove the open intervals  $(\frac{1}{27}, \frac{2}{27})$ ,  $(\frac{7}{27}, \frac{8}{27})$ ,  $(\frac{19}{27}, \frac{20}{27})$  and  $(\frac{25}{27}, \frac{26}{27})$ , again leaving the end points  $\frac{1}{27}, \frac{2}{27}, \frac{7}{27}, \frac{8}{27}, \frac{19}{27}, \frac{20}{27}, \frac{25}{27}$  and  $\frac{26}{27}$ :



FIGURE 7. Stage 3:  $[0, \frac{1}{27}] \cup [\frac{2}{27}, \frac{1}{9}] \cup [\frac{2}{9}, \frac{7}{27}] \cup [\frac{8}{27}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{19}{27}] \cup [\frac{20}{27}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{25}{27}] \cup [\frac{26}{27}, 1]$

9. What patterns do you observe in the construction of stages  $\mathcal{C}_1^3 - \mathcal{C}_3^3$ ? Explain.
10. What are the open intervals that will be removed in stage  $\mathcal{C}_4^3$ ?
11. What are the closed intervals that will remain?
12. In your notebook draw a picture of  $\mathcal{C}_4^3$ .
13. What are the open intervals that will be removed in stage  $\mathcal{C}_5^3$ ?
14. What are the closed intervals that will remain?
15. In your notebook draw a picture of stage  $\mathcal{C}_5^3$ .
16. Explain why we can construct the sets  $\mathcal{C}_n^3$  value of  $n$ .

The Cantor Ternary Set is what remains after we let  $n \rightarrow \infty$ . More formally, the **Cantor Ternary Set**,  $\mathcal{C}^3$ , is defined by  $\mathcal{C}^3 = \bigcap_{n=1}^{\infty} \mathcal{C}_n^3$ .

17. Are there any numbers in  $\mathcal{C}^3$ ? That is, are there any numbers that are in  $\mathcal{C}_n^3$  for every  $n$ ? Explain.
18. Based on your answer to Investigation 17 does  $\mathcal{C}^3$  have finitely many or infinitely many points? Explain.
19. What is happening to the lengths of the intervals in  $\mathcal{C}_n^3$  for each  $n$ ? Explain.
20. Based on your answers to Investigations 16 and 19 Does  $\mathcal{C}^3$  contain any interval of positive length?
21. Based on your answer to Investigation 20 what do you believe to be the measure of  $\mathcal{C}^3$ . Explain.

We can use the same strategy as you used in Investigation 8 along with tools for evaluating infinite geometric series (see Chapter 3 in Discovering the Art of the Infinite for more details) to actually determine the measure of  $\mathcal{C}^3$ .

22. What is the length of the interval removed in constructing  $\mathcal{C}_1^3$ ?
23. What is the total length of the intervals removed in constructing  $\mathcal{C}_2^3$  from  $\mathcal{C}_1^3$ ? (Note that your answer should not include the length you identified in Investigation 22.)

24. What is the total length of the intervals removed in constructing  $\mathcal{C}_3^3$  from  $\mathcal{C}_2^3$ ? (Note that your answer should not include the lengths you identified in Investigations **22-23**.)
25. What is the total length of the intervals removed in constructing  $\mathcal{C}_4^3$  from  $\mathcal{C}_3^3$ ? (Note that your answer should not include the lengths you identified in Investigations **22-24**.)
26. What is the total length of the intervals removed in constructing  $\mathcal{C}_5^3$  from  $\mathcal{C}_4^3$ ? (Note that your answer should not include the lengths you identified in Investigations **22-25**.)
27. What patterns do you notice in your answers to Investigations **22-26**? Explain.
28. Use your answer to Investigation **27** to complete the table in Table 1.

Stage	Total Length Removed
$\mathcal{C}_1^3$	$\frac{1}{3}$
$\mathcal{C}_2^3$	
$\mathcal{C}_3^3$	
$\mathcal{C}_4^3$	
$\mathcal{C}_5^3$	
$\mathcal{C}_6^3$	
$\mathcal{C}_7^3$	
$\mathcal{C}_8^3$	
$\vdots$	$\vdots$
$\mathcal{C}_n^3$	

TABLE 1. Total Length of Intervals Removed in Constructing  $\mathcal{C}_n^3$  for  $n = 1, 2, 3, \dots, 8$

29. Use your answer for Investigation **28** to write down an infinite series that represents the total length of open intervals that were removed in constructing  $\mathcal{C}^3$ .
30. Use the methods for evaluating infinite geometric series to determine the sum in Investigation **29**.
31. Using your answers to Investigations **1** and **30** to determine the measure of the Cantor Ternary Set,  $\mathcal{C}^3$ .
32. In light of your answer to Investigation **18**, are you surprised by your answer to Investigation **31**? Explain.

One surprising aspect of the Cantor Ternary Set that makes the set important, is that there are exactly as many numbers in  $\mathcal{C}^3$  as there are in the interval  $[0, 1]$ . That is, we can match each number in  $\mathcal{C}^3$  with exactly one number in  $[0, 1]$  and vice versa. The proof of this fact is beyond the scope of this book, so we will accept this fact without proof.

33. Explain why the above result makes your answer to Investigation **31** even more surprising.

**2.2. The Cantor Quinary Set.** We now consider another Cantor Set the *Cantor Quinary Set*,  $\mathcal{C}^5$ . This set is constructed a manner similar to that of  $\mathcal{C}^3$ , except this time in each stage we partition every interval into fifths and then remove the middle fifth (this is why our notation for the Cantor Ternary Set has a superscript 5).

Stage 0:

We start with the unit interval  $[0, 1]$ , which we will denote by  $\mathcal{C}_0^5$ :



FIGURE 8. Stage 0: The interval  $[0, 1]$

Stage 1:

We then remove the open interval  $(\frac{2}{5}, \frac{3}{5})$ ; that is, we remove all numbers that strictly bigger than  $\frac{2}{5}$  and strictly less than  $\frac{3}{5}$  but leaving the numbers  $\frac{2}{5}$  and  $\frac{3}{5}$ . We denote this stage by  $\mathcal{C}_1^5$ :



FIGURE 9. Stage 1:  $[0, \frac{2}{5}] \cup [\frac{3}{5}, 1]$

Stage 2:

The next stage, denoted by  $\mathcal{C}_2^5$ , is obtained by removing the open intervals  $(\frac{2}{25}, \frac{4}{25})$  and  $(\frac{19}{25}, \frac{21}{25})$ , again leaving the end points  $\frac{2}{25}, \frac{4}{25}, \frac{19}{25}$  and  $\frac{21}{25}$ :



FIGURE 10. Stage 2:  $[0, \frac{4}{25}] \cup [\frac{6}{25}, \frac{2}{5}] \cup [\frac{3}{5}, \frac{19}{25}] \cup [\frac{21}{25}, 1]$

Stage 3:

For the next stage,  $\mathcal{C}_3^5$ , we remove the open intervals  $(\frac{8}{125}, \frac{12}{125})$ ,  $(\frac{38}{125}, \frac{42}{125})$ ,  $(\frac{83}{125}, \frac{87}{125})$  and  $(\frac{113}{125}, \frac{117}{125})$ , again leaving the end points  $\frac{8}{125}, \frac{12}{125}, \frac{38}{125}, \frac{42}{125}, \frac{83}{125}, \frac{87}{125}, \frac{113}{125}$  and  $\frac{117}{125}$ :



FIGURE 11. Stage 3:  $[0, \frac{8}{125}] \cup [\frac{12}{125}, \frac{4}{25}] \cup [\frac{6}{25}, \frac{38}{125}] \cup [\frac{42}{125}, \frac{2}{5}] \cup [\frac{3}{5}, \frac{83}{125}] \cup [\frac{87}{125}, \frac{19}{25}] \cup [\frac{21}{25}, \frac{113}{125}] \cup [\frac{117}{125}, 1]$

34. What patterns do you observe in the construction of stages  $\mathcal{C}_1^5 - \mathcal{C}_3^5$ ? Explain.
35. What are the open intervals that will be removed in stage  $\mathcal{C}_4^5$ ?
36. What are the closed intervals that will remain?

37. In your notebook draw a picture of  $\mathcal{C}_4^5$ .
38. What are the open intervals that will be removed in stage  $\mathcal{C}_5^5$ ?
39. What are the closed intervals that will remain?
40. In your notebook draw a picture of stage  $\mathcal{C}_5^5$ .
41. Explain why we can continue this process for each value of  $n$ .

The Cantor Quinary Set is what remains after we let  $n \rightarrow \infty$ . That is, the **Cantor Quinary Set**,  $\mathcal{C}^5$ , is defined by  $\mathcal{C}^5 = \bigcap_{n=1}^{\infty} \mathcal{C}_n^5$ .

42. Are there any numbers in  $\mathcal{C}^5$ ? That is, are there any numbers that are in  $\mathcal{C}_n^5$  for every  $n$ ? Explain.
43. Based on your answer to Investigation 42 does  $\mathcal{C}^5$  have finitely many or infinitely many points? Explain.
44. What is happening to the lengths of the intervals in  $\mathcal{C}_n^5$  for each  $n$ ? Explain.
45. Based on your answers to Investigations 41 and 44 Does  $\mathcal{C}^5$  contain any interval of positive length?
46. Based on your answer to Investigation 45 what do you believe to be the measure of  $\mathcal{C}^5$ . Explain.

We can use the same strategy as you used in Investigation 8 along with tools for evaluating infinite geometric series (see Chapter 3 in Discovering the Art of the Infinite for more details) to actually determine the measure of  $\mathcal{C}^5$ .

47. What is the length of the interval removed in constructing  $\mathcal{C}_1^5$ ?
48. What is the total length of the intervals removed in constructing  $\mathcal{C}_2^5$  from  $\mathcal{C}_1^5$ ? (Note that your answer should not include the length you identified in Investigation 47.)
49. What is the total length of the intervals removed in constructing  $\mathcal{C}_3^5$  from  $\mathcal{C}_2^5$ ? (Note that your answer should not include the lengths you identified in Investigations 47-48.)
50. What is the total length of the intervals removed in constructing  $\mathcal{C}_4^5$  from  $\mathcal{C}_3^5$ ? (Note that your answer should not include the lengths you identified in Investigations 47-49.)
51. What is the total length of the intervals removed in constructing  $\mathcal{C}_5^5$  from  $\mathcal{C}_4^5$ ? (Note that your answer should not include the lengths you identified in Investigations 47-50.)
52. What patterns do you notice in your answers to Investigations 47-51? Explain.
53. Use your answer to Investigation 52 to complete the table in Table 2.
54. Use your answer for Investigation 53 to write down an infinite series that represents the total length of open intervals that were removed in constructing  $\mathcal{C}^5$ .
55. Use the methods for evaluating infinite geometric series to evaluate the sum in Investigation 54.
56. Using your answers to Investigations 1 and 55 to determine the measure of the Cantor Quinary Set,  $\mathcal{C}^5$ .
57. In light of your answers to Investigation 43 and Investigation 31, are you surprised by your answer to Investigation 56? Explain.
58. Using the ideas from this section, describe some other Cantor sets that can be created and determine their measure.

Stage	Total Length Removed
$\mathcal{C}_1^5$	$\frac{1}{5}$
$\mathcal{C}_2^5$	
$\mathcal{C}_3^5$	
$\mathcal{C}_4^5$	
$\mathcal{C}_5^5$	
$\mathcal{C}_6^5$	
$\mathcal{C}_7^5$	
$\mathcal{C}_8^5$	
$\vdots$	$\vdots$
$\mathcal{C}_n^5$	

TABLE 2. Total Length of Intervals Removed in Constructing  $\mathcal{C}_n^5$  for  $n = 1, 2, 3, \dots, 8$

## CHAPTER 4

# String Art

Not being able to touch is sometimes as interesting as being able to touch.

**Andy Goldsworthy** (British Sculptor and Photographer; 1956 - )

### 1. What is String Art?

Look at Andy Goldsworthy's pieces of art: "Woven Branch Circular Arch", Figure 1, and "Poured Icicles", Figure 2.



FIGURE 1. Woven Branch Circular Arch



FIGURE 2. Poured Icicles

1. Which geometric figure does describe best the shape inside the branches in each picture?

It seems amazing that we can take straight pieces of wood and create a round shape with it. How is that possible? And how can we know how to attach the pieces of wood to make this possible? Can we create any curved shape like this? We will start our investigations by drawing a piece of string art that resembles Goldsworthy's.

2. In Figure 3 pick a number on the left vertical number line, say 2, and its reciprocal  $\frac{1}{2}$  on the right vertical number line. Connect the two with a line segment.
3. Choose different numbers on the left number line and their reciprocals on the right number line and connect them.
4. Choose numbers that are fractions on the left number line and proceed in the same way.
5. Choose negative numbers on the left number line and proceed in the same way.
6. Does your picture resemble Goldsworthy's piece of art? How is it the same and how is it different?
7. Repeat the above drawing but move the two vertical lines closer together or further apart from each other. How does this effect your piece of string art?
8. Use the above strategies to create a circle in string art.

Mathematicians call the smooth one-dimensional shape that appears inside the many line segments a *curve*. In the above examples we saw a circle and an ellipse, which are examples of curves.

Examples like the one above are called "String Art" or "Curve Stitching" if they are created using string or yarn, see e.g. Figure 4 through Figure 6.

9. Describe in as much detail as possible how the line segments "touch" the curve in all the above examples of string art.
10. Draw some curve on a piece of paper and try drawing some of the lines you would need for string art. Is this easy or complicated? Explain why.

Mathematicians call lines that touch curves as they do in string art *tangent lines*. We want to create another example of string art, but this time we start with the curve we want to see inside. Let's say we want the curve to be the graph of a parabola  $y = x^2$ , see Figure 7.

x	estimated slope
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	

11. Estimate the slope of the graph of  $y = x^2$  at different  $x$ -values and fill in table 1.
12. Compare the values in the slope column with your peers and see if you can agree on values that show a pattern. Can you for instance predict the slope at  $x = 20$  without having to draw a huge graph?
13. Write the estimated slope as a function  $y = ?$  using the pattern you found.

14. INDEPENDENT INVESTIGATION: Using other graphs, like  $y = x^3$ ,  $y = x^4$ , and  $y = x^5$  try to find a pattern for the slope function. Our goal right now is to predict the slope function without having to graph and estimate anything.



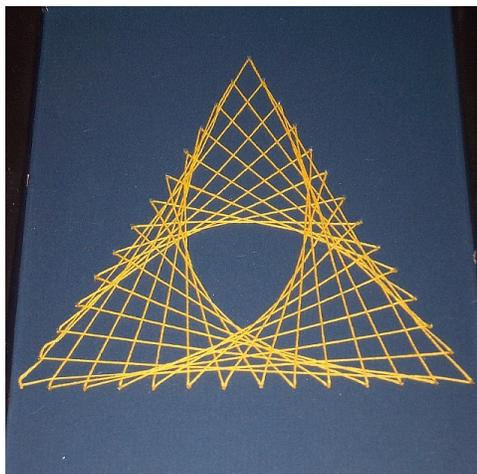


FIGURE 4. Simple Example of String Art



FIGURE 5. Chair with String Art

**Isaac Newton** (English Mathematician and Physicist; 1642 - 1727) and **Gottfried Leibniz** (German Mathematician and Philosopher; 1646 - 1716) but in their time Newton accused Leibniz of plagiarism. They both had different approaches in developing derivatives, Newton coming from a applied physics perspective and Leibniz from a more mathematically formal standpoint.

Now that we found the derivative of our function  $y = x^2$ , we can get the slope at any point we want. How can we use this to create a piece of string art that shows the parabola?

We will draw our string art on GeoGebra, which you can download for free at [www.geogebra.org](http://www.geogebra.org).

15. In GeoGebra, draw two lines  $T1 : y = 8x - 16$  and  $T2 : y = -8x - 16$  by typing the equations in the input field at the bottom of the screen.
16. Find the slope of the parabola  $y = x^2$  at  $x = 1$  and find the equation of the tangent line that goes through the point  $(1, 1)$ .
17. Where does this tangent line intersect the lines  $T1$  and  $T2$ ?
18. Draw you first “string” by connecting the intersection points.

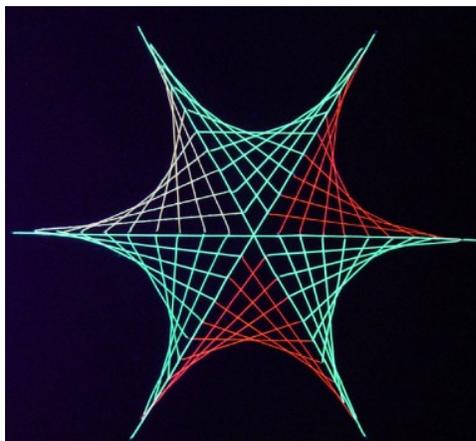


FIGURE 6. String Art on the Inside of a Curve

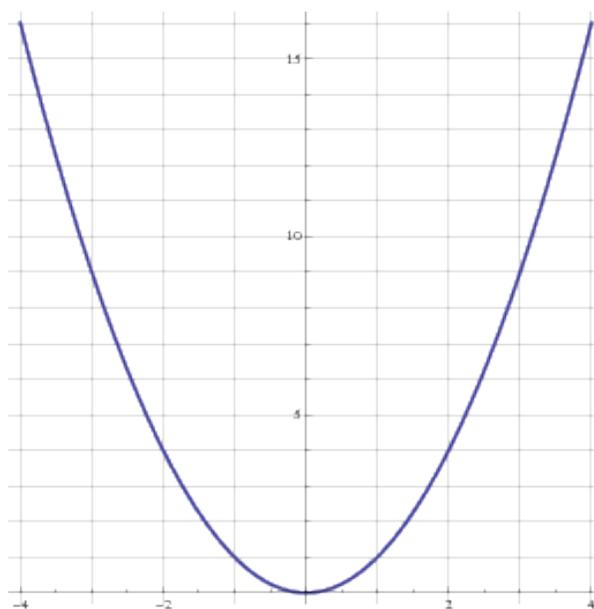


FIGURE 7. Graph of the Parabola  $y = x^2$ .

19. Can you see which other tangent line you can draw with the data you have computed so far? Use symmetry!
20. Continue to choose different  $x$ -values, find the tangent lines, intersection points and draw more strings.
21. After how many strings can you clearly see the parabola?
22. What was hard and what was easy about drawing the strings?

If we want to create more intricate examples of string art with different curves we need to be able to find derivatives of more complicated function. You have discovered how to take the derivative of

powers of  $x$  but there are many other functions we might want to take the derivative of. Fortunately the computer can help us find the derivatives.

23. In GeoGebra type `Derivative[ $x^2, x$ ]` in the Input field at the bottom of the window. We have to write the extra  $x$  in the command, because GeoGebra needs to know the name of the variable. How does GeoGebra show you the derivative?
24. Draw also the function  $y = x^2$  by typing the equation in the Input field at the bottom of the window.
25. Does it make sense to you that the graph of the derivative  $y = x^2$  is *not* tangent to the graph of  $y = x^2$ ? Explain in detail. Now try to take derivatives of more complicated functions like  $y = 7x^2 - 3x^5 - 36x + 6$ .

Let's look at a different way to create a parabola-like shape.

26. In GeoGebra, draw line segments between  $(0, 0)$  and  $(7, 7)$  and between  $(0, 0)$  and  $(-7, 7)$ . Now choose points on your line segments dividing them into *equal* pieces. Each line segment should be divided into the same number of pieces (but you can choose how many). Label the points on the right line segment starting with the label 0 at  $(7, 7)$ . Label all points down to (and including)  $(0, 0)$  with 1, 2, 3, ... Now start on the left line segment with the label 1 at  $(-1, 1)$  and continue labeling to  $(-6, 6)$  with 2, 3, ... Now connect the labels 1 and 1 with a line segment, then the labels 2 and 2, etc. What do you see?
27. We want to convince ourselves that this piece of string art really shows a parabola. Find the parabola that best matches your piece of string art. Recall that a parabola that is symmetric to the  $y$ -axis has the general form  $y = ax^2 + b$ . Explain how you found your best match.
28. Show that all the line segments in your picture are actually tangent lines of your best matching parabola. Explain your strategies. If the line segments are *not* tangent lines, find an even better matching parabola and try again.

### 3. Functions and Algebraic Curves

Looking back at our circles and ellipses from the beginning of the chapter, see Figure 3, we notice that the tangent lines are not as equally spaced as in our last parabola example. Can we find a better solution?

29. Draw a circle with radius 2 in GeoGebra. Looking at the algebra window in GeoGebra, what is the equation of this circle?
30. Using right triangles in your argument, why does it makes sense that this equation will give us a circle? See Figure 8.
31. Now try taking the derivative of your circle equation using GeoGebra. What do you notice?

The problem with the derivative arises, because the circle is not a *function*. Do you remember what a function is? Here is one definition: A **function** is a relation that uniquely associates members of one set (the input) with members of another set (the output).

32. If you describe the parabola  $\{(x, y) \mid y = x^2\}$  with a function, what do you think would be the input set and what would be the output set?
33. If you describe the circle  $\{(x, y) \mid x^2 + y^2 = 1\}$  with a function, what do you think would be the input set and what would be the output set?
34. Using the above definition, explain why the parabola is the graph of a function, but the circle is not.
35. You might remember from high school the vertical line test: *A relation is a function if there are no vertical lines that intersect the graph at more than one point.* Explain why the vertical line test really tests if a relation is a function or not.

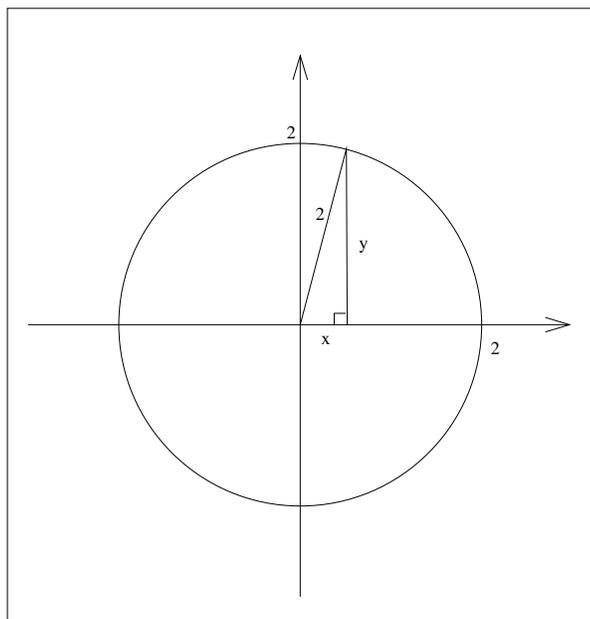


FIGURE 8. Circle of Radius 2

- 36.** Try splitting the circle into pieces that you can describe with functions. Hint: Solve the circle equation for  $y$ .
- 37.** Now use GeoGebra to find the derivative of the pieces of the circle. Explain why the graph of the derivative makes sense to you by looking at the slope of tangent lines of the circle pieces.

Let's see if we fully understand how the derivative works (without using GeoGebra this time). For the graph of a function in Figure 9, draw the graph of the derivative in the empty coordinate system.

Unfortunately, a lot of curves, like the circle or the one in Figure 10, do not arise as graphs of functions. In fact, most “interesting” curves do not. We understand how to take derivatives and draw tangent lines by hand for some functions, but for the more complicated curves we need the help of the computer. In GeoGebra find the *Tangents* tool.

- 38.** In GeoGebra draw the circle  $x^2 + y^2 = 4$ .
- 39.** Now draw a point that is *not* on the circle and use the Tangents tool by clicking the point and then the circle. Explain what you get.
- 40.** Now draw a point that is on the circle and use the Tangents tool by clicking the point and then the circle. Explain what you get.

We were originally asking the question if we can create a piece of string art using GeoGebra that allows equal spacing of the tangent lines, similar to our second parabola string art. The question is which kind of *frame* we use. A *frame* is any shape in the plane that we use to attach our strings to. In the parabola example we used two line segments, the cardioid in Figure 10 uses a circle and in Figure 4 and Figure 5 a triangle and an ellipse-like curve are being used as frames.

**41. INDEPENDENT INVESTIGATION:** Find the best frame for the circle  $x^2 + y^2 = 4$  using GeoGebra. This means you are trying to find a shape with a nice patterns of attaching the strings (tangent lines) to it so that the resulting curve is exactly the circle  $x^2 + y^2 = 4$ .

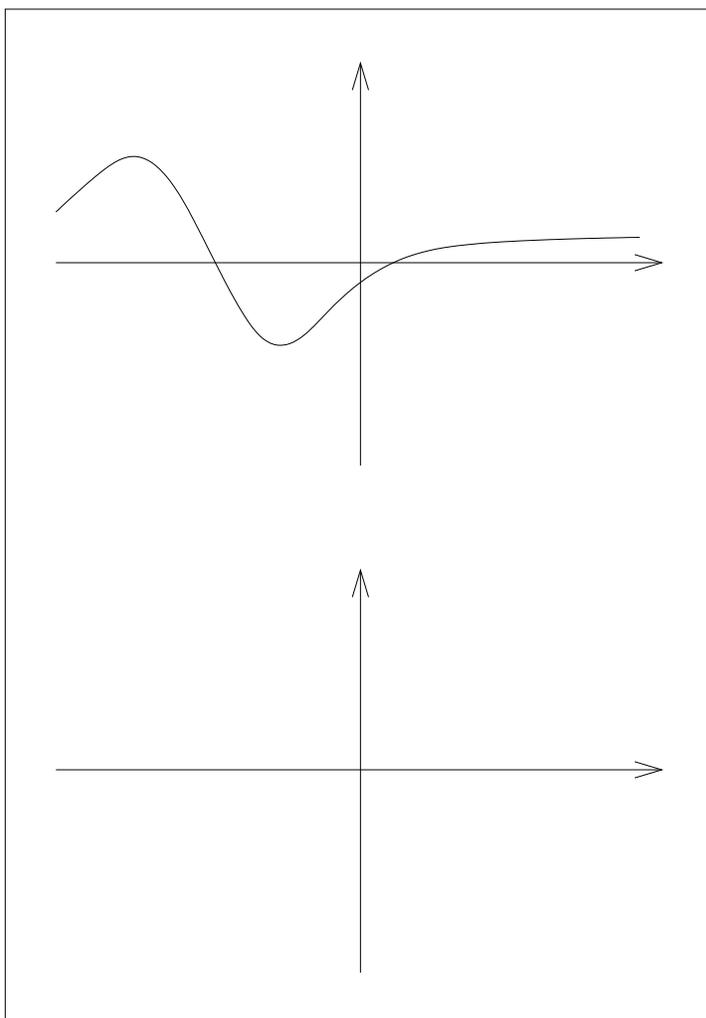


FIGURE 9. Test your Derivative Skills!

42. **Classroom Discussion:** Compare the frames for the circle  $x^2 + y^2 = 4$  and decide which one is the best.
43. Now look at the curve  $x + 2xy - 54x + 216x + y - 54y = 243$  in GeoGebra and create some tangents using the Tangents tool.
44. Type other equations that involve polynomials in  $x$  and  $y$ . Mathematicians call these curves *algebraic curves*. Play with the tangents tool and your curve. Explain what you observe.

#### 4. Creating String Art

Here is a list of some beautiful algebraic curves in the plane:

- Rose Curve :  $(x^2 + y^2)^3 = 4x^2y^2$
- Hyperbola:  $x^2/a^2 - y^2/b^2 = 1$ , choose  $a$  and  $b$
- Nephroid:  $(x^2 + y^2 - 4a^2)^3 = 108a^4y^2$ , choose  $a$

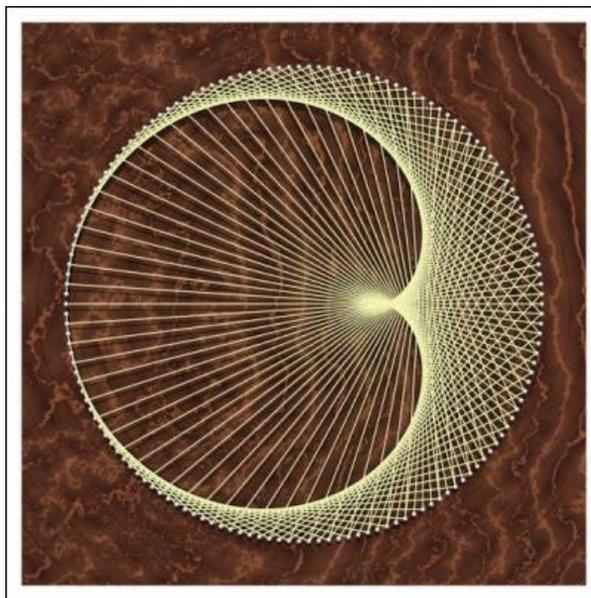


FIGURE 10. Cardioid in String Art

- Lemniscate  $x^4 = x^2 - y^2$
- Folium of Descartes  $x^3 + y^3 - 3axy = 0$ , choose  $a$
- Serpentine Curve  $x^2y + a^2y - abx = 0$ , choose  $a$  and  $b$
- Trisectrix of Maclaurin  $2x(x^2 + y^2) = a(3x^2 - y^2)$ , choose  $a$
- Ambersand Curve  $(y^2 - x^2)(x - 1)(2x - 3) = 4(x^2 + y^2 - 2x)^2$
- Bean Curve  $x^4 + x^2y^2 + y^4 = x(x^2 + y^2)$
- Bicuspid Curve  $(x^2 - a^2)(x - a)^2 + (y^2 - a^2)^2 = 0$ , choose  $a$
- Three-leaved Clover  $x^4 + 2x^2y^2 + y^4 - x^3 + 3xy^2 = 0$
- Deltoid Curve  $(x^2 + y^2)^2 + 18a^2(x^2 + y^2) - 27a^4 = 8a(x^3 - 3xy^2)$ , choose  $a$
- Devil's Curve  $y^2(y^2a^2) = x^2(x^2b^2)$ , choose  $a$  and  $b$
- Hippopede  $(x^2 + y^2)^2 = cx^2 + dy^2$ , choose  $c$  and  $d$
- Limacon  $(x^2 + y^2 - ax)^2 = b^2(x^2 + y^2)$ , choose  $a$  and  $b$
- Astroid  $(x^2 + y^2 - 1)^3 + 27x^2y^2 = 0$
- Butterfly Curve  $x^6 + y^6 = x^2$

**45. INDEPENDENT INVESTIGATION:** Find the graph of a function or an algebraic curve that you really like and use GeoGebra to make your own piece of string art. You don't need to just take two lines to "attach" your strings. You can use a box or circle or anything you want. Be creative! Did you use equal spacing on your line segments or not?

**46. INDEPENDENT INVESTIGATION:** Take your above curve and materials, like wood, nails and string, or paper, thread and a needle to actually make your piece of string art. Be creative!

**4.1. Open Question.** Is it always possible to find a frame of line segments for an algebraic curve so that the tangent lines intercept the line segments with equal spacing? Or at least in a nice pattern?

## 5. Further Investigations

**5.1. Parametrized Curves.** There is yet another way how we can describe curves, by using a *parametrization*. Here is an example:

$$c : t \rightarrow (\cos(t), \sin(t)), 0 \leq t \leq 2\pi.$$

**47.** Consider the parametrized curve above. Plug in different values for  $t$  and plot the resulting points in the  $x, y$  plane. Use a calculator! What do you get?

**48.** Type `Curve[cos(t), sin(t), t, 0, 2 pi]` into the Input line of GeoGebra. Describe what you see.

**49.** What happens when you change the last value in your input to  $1\pi$  or  $0.5\pi$ ?

**50.** Explain why some people like to think of the parameter  $t$  as time.

**51.** Now change the parametrization to

$$c : t \rightarrow (4 \cos(t), 2 \sin(t)), 0 \leq t \leq 2\pi.$$

Which shape do you get?

**52. INDEPENDENT INVESTIGATION:** Find your favorite parametrized curve and create your piece of string art using GeoGebra and real materials.

**5.2. 3-dimensional String Art.** We can also use string to create surfaces in 3 dimensions. The surfaces we can get this way are called *Ruled Surfaces*. See Figure 11 and Figure 12.

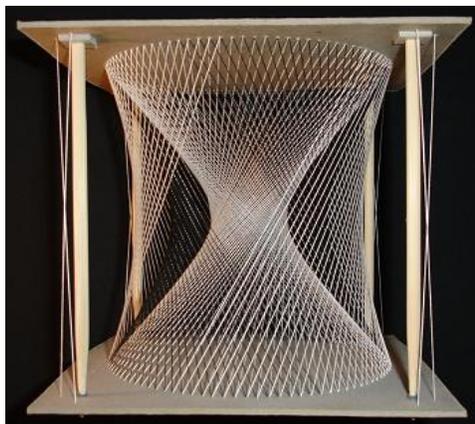


FIGURE 11. Catenoid in Cylinder

**53. INDEPENDENT INVESTIGATION:** Create your own 3-dimensional piece of string art.

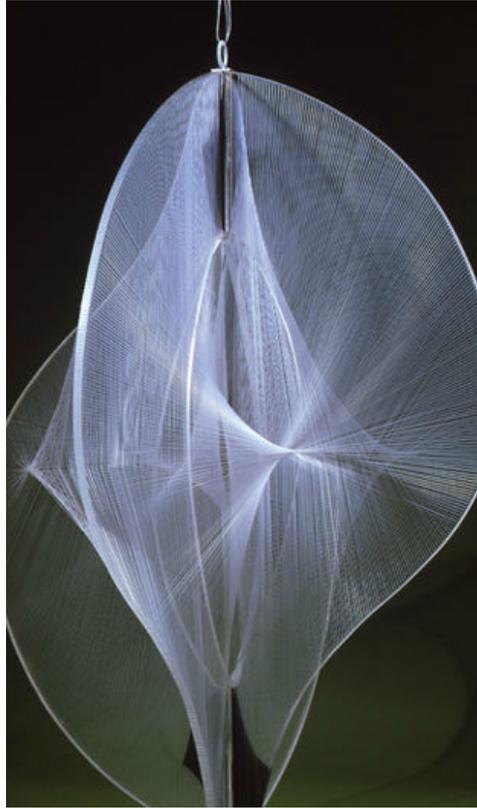


FIGURE 12. Naum Gabo: Linear Construction in Space No. 2

## 6. Connections

**6.1. Newton’s Method and Fractals.** Tangent lines have many practical uses (besides creating beautiful string art!). Newton’s method, for instance, uses tangent lines to find the points where the graph of a function crosses the  $x$ -axis (the so called **zeros** or **roots** of a function).

Consider the function  $f : y = \frac{1}{2}x^3 - \frac{2}{3}x^2$ .

54. Try guessing the zeros of  $f$  and check your guesses. Were you correct or not?
55. Using GeoGebra, choose an  $x$ -value close to one of the zeros of  $f$  and draw the tangent line through the point  $(x, f(x))$ . Mark the value where the tangent line intersects the  $x$ -axis and start with this value as your new  $x$ -value. Repeat this several times recording your  $x$ -values.
56. How do your  $x$ -values relate to the zeros of  $f$ ?
57. Now start Newton’s method from an  $x$ -value that is further away from a zero. What do you notice? Try different  $x$ -values.

We will create a 1-dimensional fractal from Newton’s method. Choose a color for each zero. Color each  $x$ -value depending on its outcome from Newton’s method: if the  $x$ -value converge towards a zero, color  $x$  with the color of that zero.

58. Create the fractal for the function  $f : y = x^3 - 3x^2 - x + 3$  in GeoGebra.

Mathematicians use more than just real numbers, they also work with *complex numbers*. When you apply Newton’s method to complex functions your fractal has two dimensions, like Figure 13.

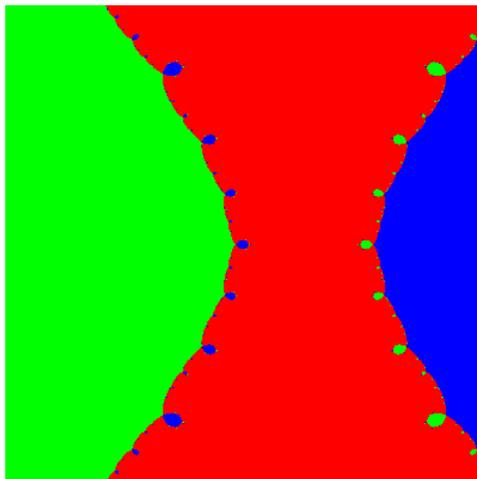


FIGURE 13. Newton Fractal for the Complex Polynomial  $z^3 - 3z^2 - z + 3$

59. Compare your fractal for the function  $f : y = x^3 - 3x^2 - x + 3$  with the fractal for the complex function  $z^3 - 3z^2 - z + 3$  in Figure 13. How are they the same and how are they different?
60. Go to <http://aleph0.clarku.edu/~djoyce/newton/newtongen.html> and create images for different complex polynomials. Do you think they are beautiful?

**6.2. Caustic Curves.** In Figure 14 you can see beams of light shining through a glass of water. When the light beams are reflected or refracted by the glass and the water, we can see the curves that is tangent to the beams. This curve is called a **caustic curve**.

61. How are caustic curves similar to string art?

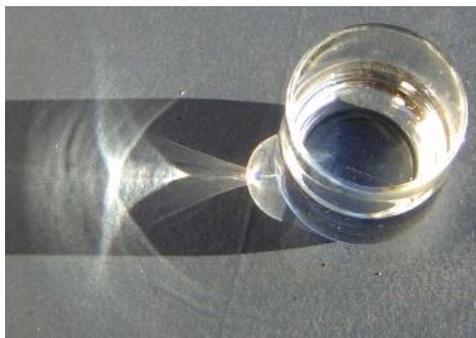


FIGURE 14. Caustic Curve

**6.3. Parabolic Reflectors.** Another example of tangent lines in use are parabolic reflectors, e.g. satellite dishes. Look at Figure 15 for a cross section of a satellite dish.

- 62.** If a microwave comes into a satellite dish, how is it being reflected when it hits the parabola? Explain your reasoning.
- 63.** What happens to all the reflected beams in a satellite dish? Use Figure 15. Why is this useful?
- 64.** Turning the idea around, if  $F$  was a light source, how would the beams of light leave a parabolic reflector? This idea is used in car headlights.

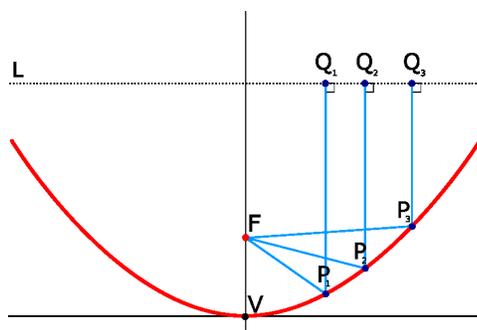


FIGURE 15. Parabolic Reflector: A Satellite Dish or a Car Headlight

**6.4. Elliptical Pool Tables.** Imagine playing pool on an elliptical pool table, or actually playing *billiards*, where there are no pockets in the table. Figure 16 shows some of the different possible paths of a ball in an elliptical pool table.

- 65.** How can you predict how the ball is going to “bounce off” the wall on an elliptical pool table?
- 66.** Draw (by hand) an elliptical pool table and the path of a ball using protractor and ruler. Use Figure 17 to help you draw an ellipse. Explain your strategy of finding and drawing the path.
- 67.** Describe the different mathematical shapes the paths create in Figure 16. Did your path look like one of them?
- 68.** Now draw an elliptical pool table and the path of a ball using GeoGebra. Explain your strategies.
- 69.** Find the angle in which the ball has to start so that the path of the ball is exactly a quadrilateral. Use an ellipse that goes through the points  $(2, 0)$  and  $(0, 1)$  and start the ball at  $(2, 0)$ .

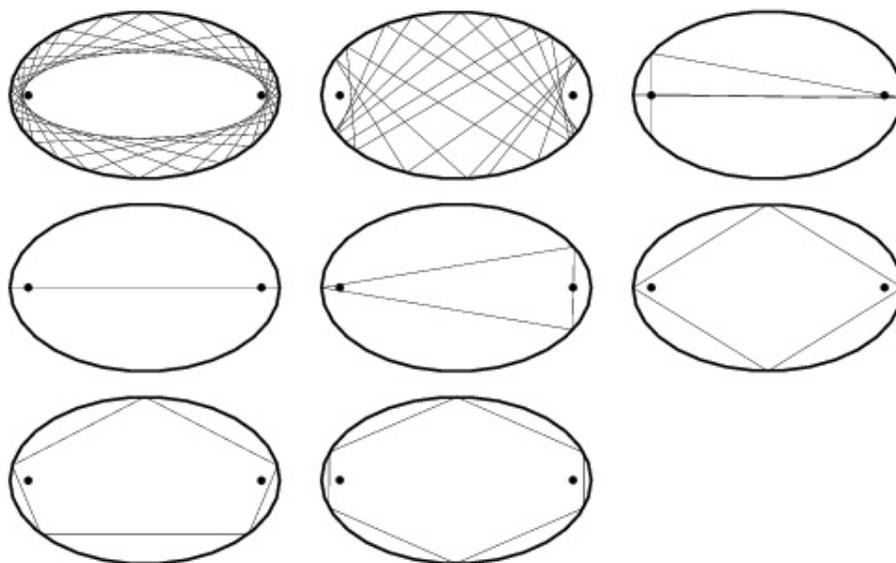


FIGURE 16. Elliptical Pool Table

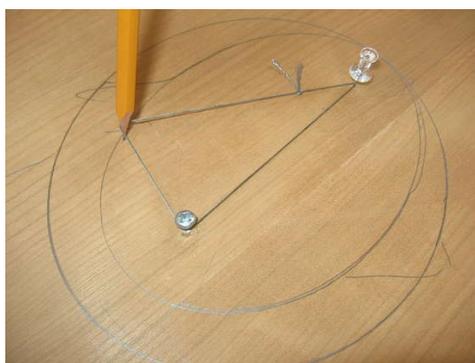


FIGURE 17. How to Draw an Ellipse

70. Find the angle in which the ball has to start so that the path of the ball is exactly a hexagon. Use an ellipse that goes through the points  $(2,0)$  and  $(0,1)$  and start the ball at  $(2,0)$ . You might have to approximate your answer...
71. How are elliptical pool tables related to string art?

## 7. Fundamental Theorem of Calculus

In string art we can see that the curve that fits the tangent lines is unique! This is a version of the first fundamental theorem of calculus proved first by **Isaac Barrow** (English Theologian and Mathematician; 1630 - 1677), see Figure 18.



FIGURE 18. Isaac Barrow



## CHAPTER 5

# Integration

My husband is a physicist. He was “embarrassed” to marry someone who never took calculus. On our first Christmas he gave me this big, fat calculus book. On our second Christmas I gave him a writer’s notebook - full of all of the answers to the questions in the calculus textbook. Doing calculus for love is a better reason than we generally give kids in school.

**Susan Ohanian** (Public School Teacher and Freelance Writer; 1946 - )

### 1. Quadrature of the Parabola

**Archimedes of Syracuse** (287 BC; 212 BC - Greek mathematician) was one of the greatest mathematicians of all time, see Figure 1.

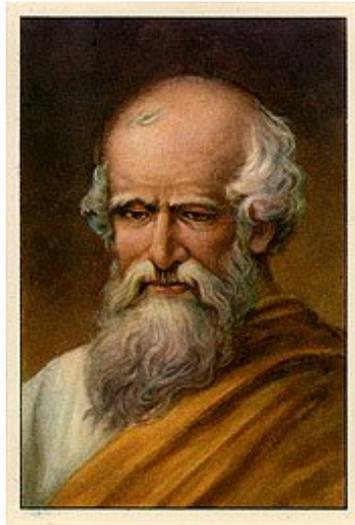


FIGURE 1. Archimedes

Not only did he plant the seeds for many ideas now known as calculus, he also invented all kinds of machines using screws, pullies and levers. Many of his inventions were used in the war of his hometown Syracuse against the Romans. The area in mathematics he was most interested in was geometry. We will investigate one of his beautiful solutions to a geometric problem as an entryway into thinking about area.

Archimedes “simple” problem was the following: compute the area of a parabolic segment, see Figure 2. The next investigations will lead you through his approach of finding the area. To make it a little easier we will look at a particular parabola,  $y = 16 - x^2$ , and compute the area between that graph and the  $x$ -axis.

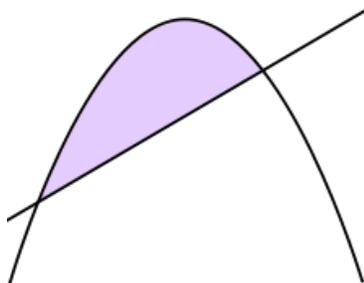


FIGURE 2. A General Parabolic Segment

1. Take the equation  $y = 16 - x^2$  and graph it on graph paper.
2. Estimate the area between the parabola and the  $x$ -axis using your grid paper.

Archimedes' key idea was to use a *method of exhaustion*. He filled the area under the curve with triangles in such a way that he could predict the area of all the triangles and hence the area of the parabolic segment.

For the following investigations we suggest to use GeoGebra (<http://www.geogebra.org>) to compute the areas of the triangles. You probably remember the area formula for triangles from high school? It is base times height divided by 2, or as an equation  $A = \frac{bh}{2}$ . While this is correct it is not always possible to use this equation, as you will see below. If you do want to compute the areas by hand, you can use Pick's theorem (see chapter ???) or use the equation at <http://www.mathopenref.com/coordtrianglearea.html>.

3. Draw the parabola  $y = 16 - x^2$  in GeoGebra by typing the equation in the command line on the bottom of the window.
4. Draw the triangle  $T_1$  with vertices  $(-4, 0)$ ,  $(0, 4)$  and  $(0, 16)$  in GeoGebra using the polygon tool. Compute the area by hand using  $A = \frac{bh}{2}$ . Now use the area tool in GeoGebra to compute the area. Did you get the same answer?
5. Now draw the triangle  $T_2$  with vertices  $(-4, 0)$ ,  $(-2, 12)$ , and  $(0, 16)$  and compute its area by hand and using GeoGebra. What do you notice?
6. How does the  $x$ -coordinate of the new point  $(-2, 12)$  relate to the  $x$ -coordinates of the old points  $(-4, 0)$  and  $(0, 16)$ ?
7. Find another triangle of the same area as  $T_2$  under the parabola (use symmetry).
8. Can you find the next smaller triangle  $T_3$  Archimedes would have used? How many triangles of the same area are there?
9. Write the area of all triangles so far as a sum:  $A = 64 + ???$ . How will the pattern continue? Write down the next 4 terms in the sum.
10. How many more triangles areas do we need to compute to find the total area of the parabolic segment?

Archimedes was just discovering how to formally handle infinitely many objects (2000 years before anyone else reinvented it!). When he published his result he was using a different technique though to confirm the supposed value of the area. He used a *Proof by Contradiction*, showing that the area of the parabolic segment could not be less and could not be more than the supposed value. Read more about *proofs by contradiction* in Discovering the Art of Mathematics: Reasoning, Truth, Logic and Certainty.

To finish Archimedes' solution we need to understand how to find the value of a geometric series. If you have read the chapter *Grasping Infinity* in the book Discovering the Art of Mathematics: The

Infinite, you can continue with Investigation **11**. If not, here is a quick summary:

Mathematicians call an infinite sum a *series*. Series in which you multiply each addend by the same number  $r$  to get to the next addend are called **geometric series**, e.g.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Here each term is multiplied by  $r = \frac{1}{2}$  to get to the next. If you have played with series before you will know that often we have no idea which value the series converges to (if any). So the following result is very special and useful: If  $|r| < 1$  the value of the geometric series  $1 + r + r^2 + r^3 + \cdots$  is equal to  $\frac{1}{1-r}$ . Or, as mathematicians write formally,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

If you wonder why this is true (and as a true mathematician you should! Never believe statements without a good argument!) work through the investigations in the chapter *Grasping Infinity* in the book Discovering the Art of Mathematics: The Infinite. Now you are ready to continue following Archimedes' thinking:

- 11.** Find the value of the geometric series in Investigation **9**.
- 12.** Compute the area between the parabola  $16 - x^2$  and the  $x$ -axis using Archimedes' triangles and the geometric series.
- 13.** What are advantages of using Archimedes' triangles in the above computations? What are disadvantages?

## 2. Riemann and Cauchy and again the Parabola

Men pass away, but their deeds abide.

**Augustin-Louis Cauchy** (French Mathematician; 1789 - 1857)

If only I had the theorems! Then I should find the proofs easily enough.

**Bernhard Riemann** (German Mathematician; 1826 - 1866)

Without GeoGebra it would have been pretty difficult to compute the areas of all the triangles in the parabolic segment in Section 1. We want to see if it would be easier to use different shapes to approximate the area.

- 14.** Looking at the parabolic segment between  $y = 16 - x^2$  and the  $x$ -axis, which shapes would you have chosen to compute the area? Explain your thinking.
- 15. Classroom Discussion:** Compare the different strategies for finding area under the parabola using different shapes by looking at advantages and disadvantages.

We will use GeoGebra to investigate some ways to approximate the area. You might have thought of these yourself in the investigation above.

- 16.** Draw the function  $f(x) = 16 - x^2$  in GeoGebra. Now use the command **UpperSum[f,-4,4,8]**. How does this approximate the area between the parabola and the  $x$ -axis?
- 17.** Change the 8 in **UpperSum[f,-4,4,8]** to other values and observe what happens. How can you get a more accurate approximation of the area? Explain.
- 18.** Now use the command **LowerSum[f,-4,4,8]**. How does this approximate the area between the parabola and the  $x$ -axis?
- 19.** Change the 8 in **LowerSum[f,-4,4,8]** to other values and observe what happens. How can you get a more accurate approximation of the area? Explain.

20. How can you use the values from the upper sum and the lower sum to get an even better approximation for the area? Let GeoGebra compute your approximation to see if it is actually better.

This method of computing the area under a curve was invented by **Bernhard Riemann** (German Mathematician; 1826 - 1866), that is why the sum of the rectangle areas are called ***Riemann Sums***. Riemann was a brilliant (but very shy) mathematician who laid the groundwork for *Differential Geometry*, an vibrant area of mathematics that analyzes smooth shapes in higher dimensions. See Figure 3 for a picture of Riemann’s minimal surface.

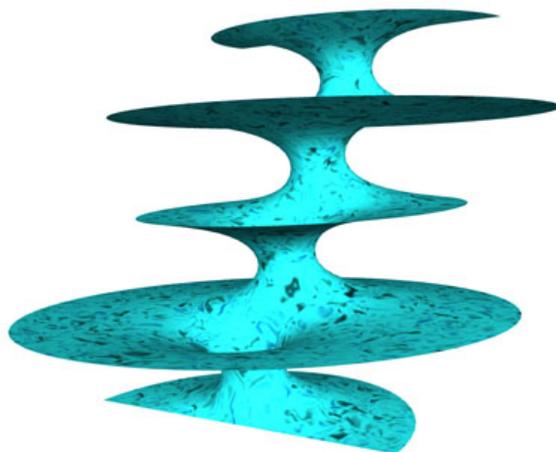


FIGURE 3. Riemann’s Minimal Surface

To get ab glimpse of what mathematicians do in differential geometry you can watch the beginning of the video [http://www.youtube.com/watch?v=8qGM8HA1\\_pI](http://www.youtube.com/watch?v=8qGM8HA1_pI) which shows the proof of the Willmore conjecture, a problem just solved by Fernando Coda Marques and Andre Neves in 2012.

It seems as though this new Riemann Sum method will not easily compute the precise area for us since we have to add so many rectangle areas. But actually there is something else happening here, which is really amazing - and you are about to discover it yourself!

If mathematicians are looking for patterns and structure, they often look at simpler objects first. In our case we will look at simpler functions.

21. Change the function in GeoGebra to  $f(x) = 2x$ . Change the left point of the interval to  $a = 0$  and the right point of the interval to  $b = 1$ . What is the best approximation of the area under the graph if you have 100 rectangles?
22. Now change the right end point to  $b = 2$ . Again, what is the best approximation for the area?
23. Now change the right end point to  $b = 3$ . Again, what is the best approximation for the area?
24. Now change the right end point to  $b = 4$ . Again, what is the best approximation for the area?
25. Record your values in the following table.

end points $b$	area under the graph between $a = 0$ and $b$
1	
2	
3	
4	

26. Do you notice a pattern in the table? How would the next entries continue? Explain. (If you can not find the pattern, read the chapter about linear and quadratic growth in the book Discovering the Art of Mathematics: Patterns.)
27. Use your pattern to find the area under the graph of  $f(x) = 2x$  from  $a = 0$  to *any*  $b$ . Your answer should contain  $b$ .

We hope to find a general pattern for polynomial functions. A **polynomial** is a sum of terms, each consisting of a power of the variable  $x$  multiplied by some constant. For example:  $f(x) = 4x^5 + 2x^3 - 26$  is a polynomial of degree 5.

**28. INDEPENDENT INVESTIGATION:** Repeat the above experiment for other functions. You might want to split up the work and let different groups work on different functions.

- a.  $f(x) = 3x^2$
- b.  $f(x) = 4x^3$
- c.  $f(x) = 5x^4$
- d.  $f(x) = 1$
- e.  $f(x) = x$
- f.  $f(x) = x^2$
- g.  $f(x) = x^3$
- h.  $f(x) = x^4$

Can you see a pattern for the area? If I have a function  $f(x) = x^n$  how do I find its area between  $a = 0$  and any  $b$ ?

Using your conjectures from above, can you determine the following areas (without using GeoGebra or any other help)?

29. Find the area under  $f(x) = x^5$  between 0 and  $b$  using your above conjectures. Explain your reasoning.
30. Find the area under  $f(x) = 3x^5$  between 0 and  $b$  using your above conjectures. Explain your reasoning.
31. Find the area under  $f(x) = 3x^5 + 1$  between 0 and  $b$  using your above conjectures. Explain your reasoning.
32. Find the area under  $f(x) = x^5 + x^8$  between 0 and  $b$  using your above conjectures. Explain your reasoning.

For many functions  $f$  you can now compute a different function depending on  $b$ . This second function has a name, it is called an **antiderivative of  $f$** . For example  $f(x) = x^2$  has an antiderivative  $g(b) = \frac{b^2}{2}$ .

33. Use your knowledge about derivatives from Chapter 4 to explain why the second function is called an antiderivative of  $f$ .
34. Is it surprising to you that the computation of area can have a strong connection to derivatives? Explain.

The amazing connection between areas and derivatives was first discovered in the 16th century. It was stated and proven as the **Fundamental Theorem of Calculus** in the 18th century. **Augustin-Louis Cauchy** (French Mathematician; 1789 - 1857) was the first to prove the result rigorously in 1823.

Let's summarize what we know so far:

If we want to compute the area under the graph of a function  $f$  from  $a = 0$  to  $b$  we need to find an antiderivative of  $f$  and evaluate it at  $b$ .

35. Compute the area under the parabola  $f(x) = 16 - x^2$  using anti-derivatives. Compare your answer to your original value for the area from Archimedes' method. What do you notice?

There is something we don't understand yet when the left end point of our interval is not  $a = 0$ . We will use GeoGebra to explore the areas for different values of  $a$ .

36. Find the area under the graph of  $f(x) = 1$  from  $a = -1$  to  $b = 1$ .  
 37. Find the area under the graph of  $f(x) = 1$  from  $a = -2$  to  $b = 2$ .  
 38. Find the area under the graph of  $f(x) = 1$  from  $a = -3$  to  $b = 3$ .  
 39. Find the area under the graph of  $f(x) = 1$  from  $a = -4$  to  $b = 4$ .  
 40. Remembering your investigations from before, what is an antiderivative  $g$  of  $f(x) = 1$ ?  
 41. Fill the following table with the required values and see if you can detect a pattern how we can use the antiderivative to find the values in the area column.

a	b	g(a)	g(b)	area
-1	1			
-2	2			
-3	3			
-4	4			

42. Find the area under the graph of  $f(x) = 3x^2$  from  $a = -1$  to  $b = 1$ .  
 43. Find the area under the graph of  $f(x) = 3x^2$  from  $a = -2$  to  $b = 2$ .  
 44. Find the area under the graph of  $f(x) = 3x^2$  from  $a = -3$  to  $b = 3$ .  
 45. Find the area under the graph of  $f(x) = 3x^2$  from  $a = -4$  to  $b = 4$ .  
 46. Remembering your investigations from before, what is an antiderivative  $g$  of  $f(x) = 3x^2$ ?  
 47. Fill the following table with the required values and see if you can detect a pattern how we can use the antiderivative to find the values in the area column.

a	b	g(a)	g(b)	area
-1	1			
-2	2			
-3	3			
-4	4			

48. Make a conjecture: How do we find the area under the graph of a function  $f$  between  $a$  and  $b$ ?  
 49. **Classroom Discussion:** Compare your conjectures for the area computation under the graph of a function  $f$  between  $a$  and  $b$  and agree as a class on one of them.  
 50. Using the above conjecture find the antiderivative of  $16 - x^2$  and find the area under the parabola  $16 - x^2$  between  $a = -4$  and  $b = 4$ . Compare your result with you previous answer in Investigation 12.  
 51. Can you see why using the fundamental theorem of Calculus to find area is so powerful? What is the advantage over Archimedes' method? Explain in detail.

### 3. Integration and Art

The idea of Riemann sums seems to be present in many different areas and objects. Look for instance at the church Hallgrímskirkja in Reykjavik, see Figure 4.

52. Where do you see a connection between Riemann sums and the church? Explain.

**Robert Smithson** (American Artist; 1938 - 1973) was part of the minimalism movement in which the artist uses minimal forms and concepts to expose more of the essence of a piece of art.

Walker Art Center (<http://www.walkerart.org>) describes the piece as follows:

*Leaning Strata is the visual manifestation of an extensive set of investigations Smithson was conducting during the mid-1960s, which included geology, astronomy, perspective, mapping, and the nature*



FIGURE 4. Church Hallgrímskirkja in Reykjavik, Iceland

*of time and matter. The title suggests a geological configuration. The stepping of the elements in the form, if continued according to the system established (i.e., moving at a regular rate away from the implied center), would conclude in a spiral.*

**53.** Explain how the “Leaning Strata” by Robert Smithson, see Figure 5, is similar and different from a Riemann sum.

The area under the graph of a function between  $a$  and  $b$  is called a *definite integral* and was denoted by Riemann as

$$\int_a^b f(x)dx.$$

The concept of integration (together with the concept of derivatives, see Chapter 4) was developed independently by **Isaac Newton** (English physicist and mathematician; 1642 - 1727) and **Gottfried Leibniz** (German mathematician; 1646 - 1716) in the late 17th century. Their (and your) above discovery in formal Riemann notation would be:

$$\int_a^b f(x)dx = F(b) - F(a),$$

with  $F$  being an antiderivative of  $f$ .

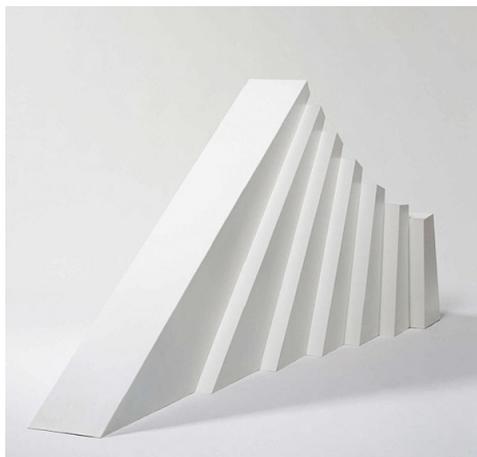


FIGURE 5. Robert Smithson: Leaning Strata, 1968

#### 4. Tolstoy's Integration Metaphor

**Leo Tolstoy** (Russian Writer; 1828 - 1910) wrote his famous novel *War and Peace* from 1863 to 1869. It is one of the longest novels ever written, taking place during the war between France and Russia in 1812. It is more than historical fiction though, containing many philosophical ideas. Did you know that mathematics and philosophy are closely related? Many mathematicians were philosophers and vice versa! The following quote shows how Tolstoy uses modern mathematical ideas to explain his idea of the study of history.

The movement of humanity, arising as it does from innumerable arbitrary human wills, is continuous.

To understand the laws of this continuous movement is the aim of history...

Only by taking infinitesimally small units of observation (the differential of history, that is, the individual tendencies of men) and attaining to the art of integrating them (that is, finding the sum of these infinitesimals) can we hope to arrive at the laws of history. (page 918)

**Leo Tolstoy** (Russian Writer; 1828 - 1910)

Stephen Ahearn states in his paper about Tolstoy's metaphor: "Thus, to understand the laws governing history, we must "integrate" the wills of all people. Once we are able to carry out this integration, the historical laws will be apparent." Tolstoy probably didn't know of Riemann's work, but there are clear connections that you will think about in the next investigations<sup>1</sup>:

54. What are Tolstoy's variables?
55. Why does Tolstoy point out that the movement of humanity is continuous?
56. What in Tolstoy's metaphor corresponds to a Riemann sum?
57. What part of the integral corresponds to "taking infinitesimally small units for observation"?

---

<sup>1</sup>Investigations from Stephen T. Ahearn's paper: Tolstoy's Integration Metaphor from *War and Peace*. July 2004.

58. Does the metaphor work or does it fail as a metaphor?  
59. How do you feel about this use of mathematics to illustrate historical ideas?

60. INDEPENDENT INVESTIGATION: Find at least one other person that was (or is) interested in both, mathematics and philosophy. Describe the person's life and try to explain in your own words some of his or her philosophical and mathematical ideas.

### 5. Cars instead of Planets

After examining a philosophical connection to the idea of integration, we want to consider a “real life” problem. Many questions that inspired the development of calculus came from physics, for instance **Isaac Newton** (English Physicist and Mathematician; 1642 - 1727) studying Kepler's laws of the movement of planets. Since those laws are beyond the scope of this book, we will study the movement of your car instead.

61. Assume you drove your car for 4 hours at a speed of 30 miles per hour. How far did you drive?  
62. Graph the function of your speed and see if you can find the value of the distance that you drove somewhere in the picture.  
63. Assume you drove your car for 2 hours at 40 miles per hour and for 2 hours at 20 miles per hour. How far did you drive?  
64. Graph the function of your speed and see if you can find the value of the distance that you drove somewhere in the picture.  
65. In reality, your car doesn't just start at 40 miles per hour, right? Draw the graph of a speed function that is more reasonable. How would you find the distance you drove using the speed curve? Explain in detail.  
66. Explain the connection between integration and driving a car.

### 6. Integration in higher Dimensions

Now that we can use integration and the fundamental theorem of calculus to compute area, we can wonder how this generalizes to higher dimensions. <http://www.math.brown.edu/~banchoff/multivarcalc2/multivarcalc2-4.html> has a nice java applet that lets you see how a Riemann sum approximates the volume under a graph in three dimensions. See also Figures 6-7.

Creating these Riemann Sums in three dimensions is based on the idea of Riemann sums in two dimensions and is also very similar to the idea of slice forms, see Figure 8. You can read the chapter about slice forms in the book Discovering the Art of Mathematics: Art and Sculpture to learn how to create your own.

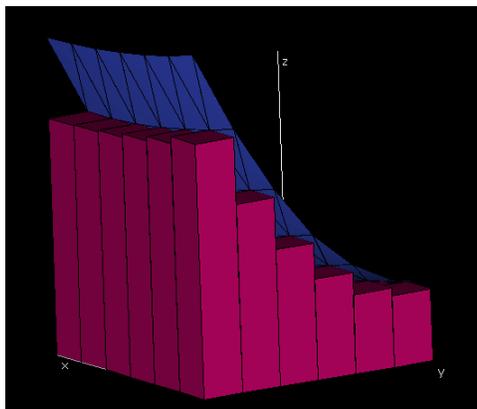


FIGURE 6. Riemann Sum under the Graph of  $f(x) = x^2 + 0.1y^2 + 0.2$

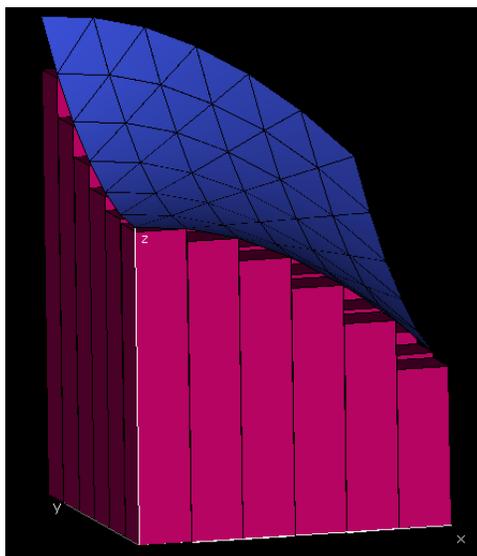


FIGURE 7. Riemann Sum under the Graph of  $f(x) = 1 - 0.5x^2 + 0.5y^2$

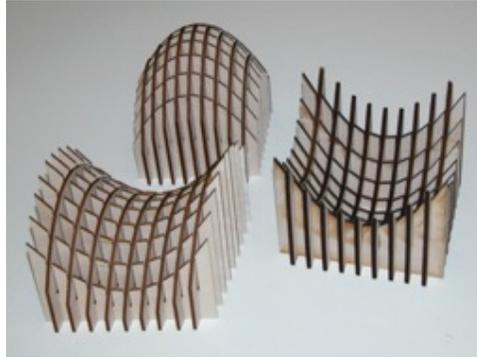


FIGURE 8. Slice Forms

67. Explain the connection between slice forms and Riemann sums for graphs in three dimensions. How are the ideas similar and how are they different?
68. Explain the connection between integration in three dimensions and Smithson's sculpture in Figure 9.



FIGURE 9. Robert Smithson: Map on Mirror Passaic, 1967

69. Consider the lego structure build at Westfield State University in Figure 10. The structure approximates the graph of the function  $f(x, y) = 5 \cos(x^2 + y^2) + 6$ , see Figure 11. How can you use lego pieces to explain Riemann Sums?

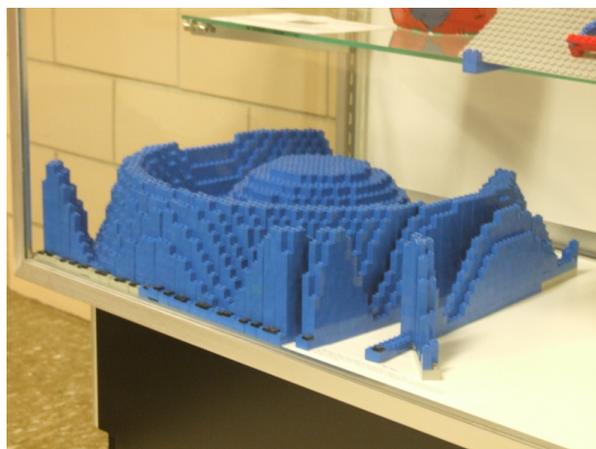


FIGURE 10. Legos and Integration

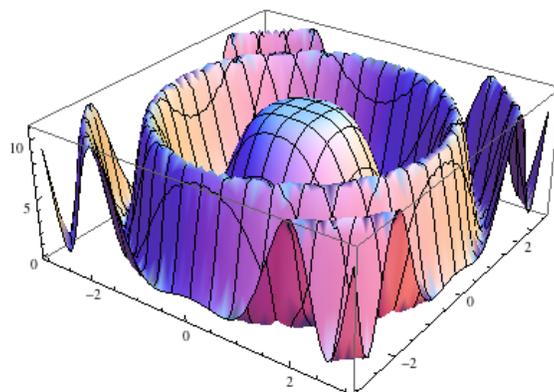


FIGURE 11. The graph of  $f(x, y) = 5\cos(x^2 + y^2) + 6$

## 7. Further Investigations and Connections

Learning is experience. Everything else is just information.

**Albert Einstein** (German born Physicist; 1879 - 1955)

You might have noticed that the text talks about **an** antiderivate instead of **the** antiderivative. Why is that?

**70.** Can you find more antiderivatives for  $f(x) = x^2$  than just  $g(b) = \frac{b^2}{2}$ ? How many are there?

**71.** Will every function have more than one antiderivative? Explain.

There is another problem with our theory about integration and area computation that we have avoided so far:

**72.** Find the area between the graph of  $f(x) = x^2$  and the  $x$ -axis using integration. What do you notice about the sign of your result? How can we “fix” this problem?

**73.** Try your idea by computing the area between the function  $h(x) = (x - 3)(x + 4)(x + 1)$  and the  $x$ -axis between  $x = 0$  and  $x = 4$ . Draw a graph on graph paper to see if your result is reasonable.

The next investigations will help you see what else integration is connected to.

**74.** Find the mathematical equation for the graph of the frontline of the church Hallgrímskirkja, see Figure 12. The book *The Nature of Mathematics* by Karl J. Smith claims that it follows a normal curve. Do you think this is true? There are models in google sketchup of the church that might be helpful in answering this question. Can you trust just measuring the heights of the rectangles in the picture? Why or why not?



FIGURE 12. Frontview of Church Hallgrímskirkja in Reykjavik, Iceland

We will assume for the moment that the curve really a normal curve, also called a *normal distribution*:

$$f(x) = \frac{1}{2\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

If we wanted to estimate the material needed for all the “steps” on each side of the church we could use our ideas of integration. Unfortunately it is not easy at all to compute the antiderivative for the normal

distribution. To find an approximation you can use the idea of series. In fact, you have to understand complex numbers to really understand the mathematics involved. The answer is given by the error function  $erf(x)$ , see Figure 13.

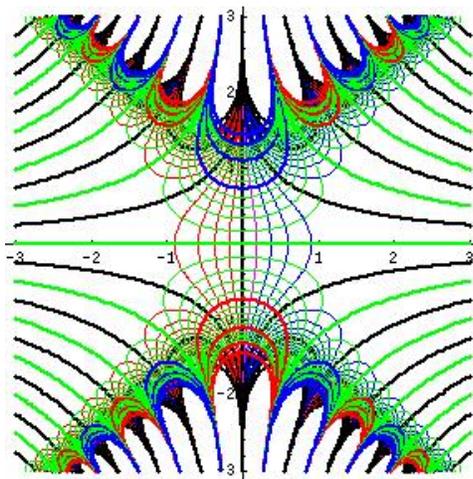


FIGURE 13. Complex Error Function  $Erf(z)$ .



## CHAPTER 6

# The Banach-Tarski Paradox

Perhaps the greatest paradox of all is that there are paradoxes in mathematics.

**Edward Kasner and James Newman** (American Mathematicians; - )

Major paradoxes provide food for logical thought for decades and sometimes centuries.

**Nicholas Bourbaki** (Fictional French Mathematician; - )

Since human beings have never encountered actually infinite collections of things in our material existence, all of our attempts to deal with them must involve projecting our finite experience... Therefore, we must rely on logical reasoning...and then be prepared to accept the consequences of our reasoning, regardless of whether or not they conform to our intuitive feelings.

**W. P. Berlinghoff and K. E. Grant** (American Mathematicians; - )

### 1. Introduction

As you have seen, our understanding of the infinite has led to some surprising and counter-intuitive results: infinite series that converge, infinite sets that have the same cardinality as the unit interval,  $[0,1]$  but have measure zero; and other infinite sets that contain no intervals but have positive measure. In this chapter we consider another counter intuitive result, the *Banach-Tarski Paradox*. We begin with a popular puzzle.

1. In Figure 1 is a square made up from *tangrams*, a seven piece popular dissection puzzle from China that is also a common manipulative in many elementary classrooms. What is the area of the square formed from the seven pieces and what is the area of each piece? Be sure to explain how you computed the area of each piece.

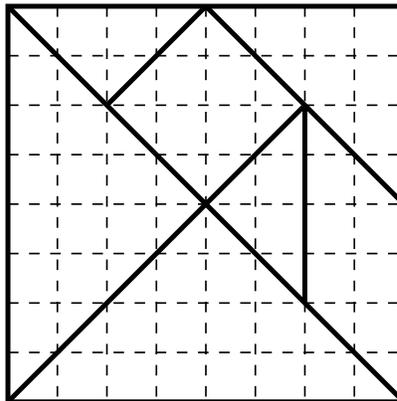


FIGURE 1. Tangram Puzzle Pieces

2. Is the area of the whole square equal to the sum of the area of the pieces? Explain.

In the back of the book is a larger copy of Figure 1. Carefully cut out the seven pieces so you may use them in the following questions.

3. In Figure 2 is a picture made from tangrams of a runner. Make this picture with your tangrams. What is the area of the runner figure? Does runner have the same as the area of the square in Figure 1? Explain.

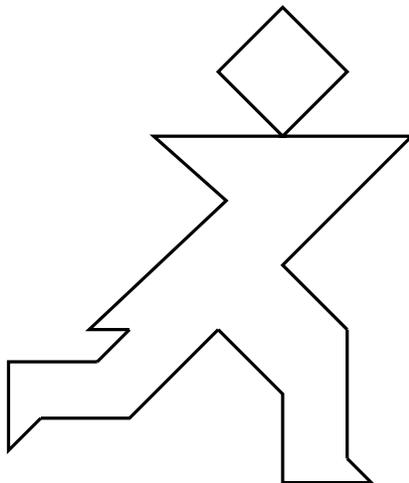


FIGURE 2. Tangram Runner Puzzle

4. In Figure 3 is a two square tangram paradox . Explain why this is a paradox.

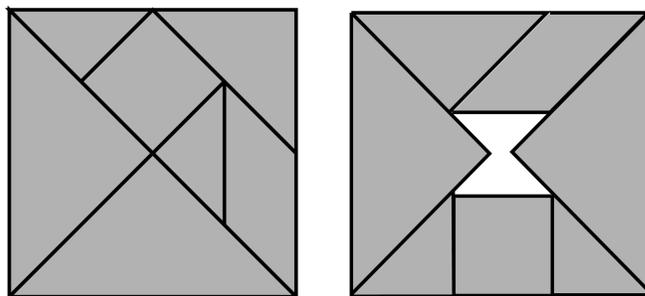


FIGURE 3. Tangram Square Paradox

5. Explain how you can resolve the paradox in Figure 3.
6. If a region with finite area or a solid with finite volume is cut up (*dissected*) into a finite number of pieces and those pieces are rearranged (without changing them in any way) into new shapes can the area or volume change? Why or why not?

Your answer to Investigation 6 illustrates our intuition about the relationship between the area (or volume) of the whole and the area of the pieces that make up the whole. There is a sense that when a region or a solid is cut up into pieces and the pieces are rearranged, nothing can be gained or lost. The next section has several examples that will challenge our intuition about dissections.

## 2. Equidecompositions

The tangram examples in Section 1 illustrate one of the two important concepts we will be using in this chapter. We say that two sets,  $X$  and  $Y$  are *equidecomposable* if we can cut both  $X$  and  $Y$  into the same number of non-overlapping finite pieces such each piece of  $X$  is congruent to exactly one piece of  $Y$ . The term *congruent* means that the two pieces are identical in shape and size and that we can transform one piece into the other by only using some combinations of the following *rigid motions*:

1. A *translation*; i.e shifting the entire piece a certain distance in a specific direction as shown in Figure 4.

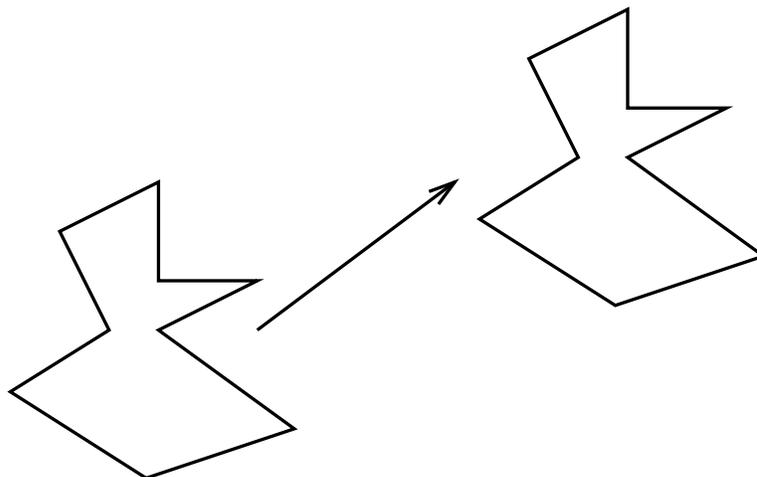


FIGURE 4. A Translation

2. A *rotation*; i.e rotating the entire piece through a specific angle as shown in Figure 5.
3. A *reflection*; i.e flipping the entire piece about a line or a point as shown in Figure 6.
7. We denote the *natural numbers*  $= \{1, 2, 3, 4, \dots\}$  by the symbol  $\mathbb{N}$ . Let  $M = \{1, 2, 3, 4, 6, 7, 8, 9, \dots\}$  (the natural numbers minus 5),  $S = \{4, 5, 6, 7, 8, 9, \dots\}$ ,  $T = \{2, 4, 6, 8, \dots\}$ ,  $U = \{-2, -1, 0, 1, 2, 3, \dots\}$  and  $V = \{\dots, -4, -3, -2, -1\}$ . To which of the sets  $M$ ,  $S$ ,  $T$ ,  $U$  and  $V$  (if any) is  $\mathbb{N}$  congruent? Explain your reasoning.
8. Do any of your answers to Investigation 7 surprise you? Explain.
9. The other important concept we will need in this chapter is called *shifting to infinity*. Use your answers to Investigation 7 to explain what this means.

**Include something about Hilbert's Hotel here?**

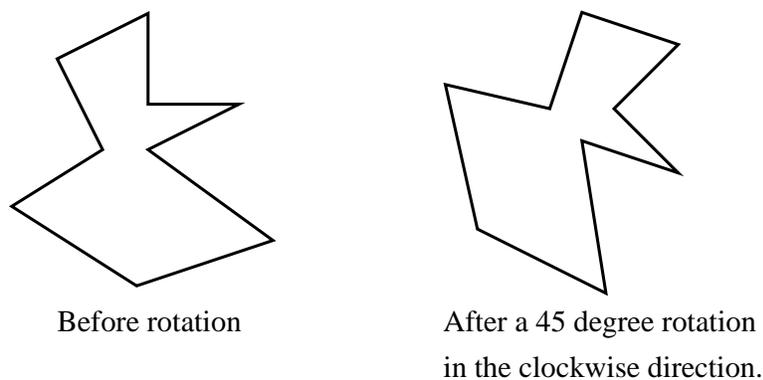


FIGURE 5. A rotation of  $45^\circ$  in the clockwise direction.

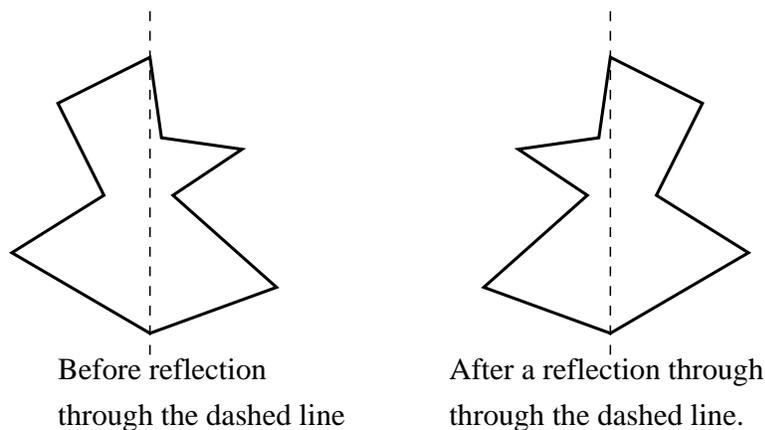


FIGURE 6. A Reflection.

- 10.** Show that the natural numbers,  $\mathbb{N}$ , and  $M = \{1, 2, 3, 4, 6, 7, 8, 9, \dots\}$  (the natural numbers minus the number 5) are equidecomposable.

**Hint:** Break both  $\mathbb{N}$  and  $M$  into two pieces such that one pair of pieces from each are identical and the other pair of pieces are congruent by a shift to infinity (i.e. a translation).

- 11.** Why might people find your answer to Investigation **10** surprising? Explain.

Our next example, showing that a circle is equidecomposable to a circle minus a point, is similar to Investigation **10** but since it is done on a circle, this adds a layer of complexity.

- 12.** In Figure 7 is a circle of radius 1. Cut a piece of string whose length is equal to the radius, then beginning at  $P_0$ , mark off a point  $P_1$  that is 1 unit (the length of the string) along the circle away from  $P$  in the clockwise direction.

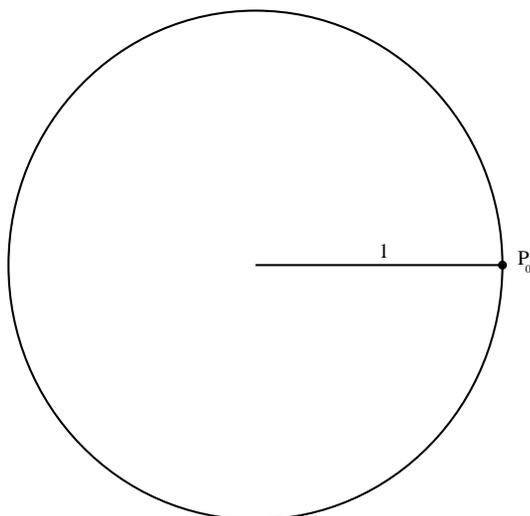


FIGURE 7. A Circle of Radius 1.

- 13.** Find and mark off a point  $P_2$  on the circle that is one unit from  $P_1$  along the circle in a clockwise direction. Continue in this manner to plot the points  $P_3, P_4, P_5, P_6, P_7, P_8, P_9$  and  $P_{10}$  on the circle that are 1 unit from the previous point along the circle in a clockwise direction.

Let  $P$  be the set of points on the circle that come from the (infinite) continuation of the procedure in Investigations **12-13**. That is  $P = \{P_0, P_1, P_2, P_3, \dots\}$ . We want to do a shift to infinity on this set of points like we did in Investigation **10**. However, there is a potential problem.

- 14.** How might the set  $P$  differ from  $\mathbb{N}$  in a way that might make shifting to infinity not possible?

The potential problem you identified in Investigation **14** does not occur because  $\pi$  is an *irrational number*; that is, we can not find whole numbers  $p$  and  $q$  (with  $q \neq 0$ ) so that  $\pi = \frac{p}{q}$ . In the next few questions you will explore why the fact that  $\pi$  is irrational means that our set  $P$  must be infinite. (The proof that  $\pi$  is irrational is beyond the scope of this book, but it is worth noting that the first proof of the irrationality of  $\pi$  is due to **Johann Heinrich Lambert** (Swiss Mathematician; 1728 - 1777) who proved it in 1761.)

- 15.** Suppose  $P_n = P_k$  for some pair of whole numbers  $n$  and  $k$  with  $n > k$  as illustrated in Figure 8. We are going to measure the distance between  $P_k$  and  $P_n$  in two different ways. The first way uses the fact that the distance along the circle between successive points  $P_i$  and  $P_{i+1}$  is 1. Using this fact, what is the distance along the circle between the points  $P_k$  and  $P_n$ ?

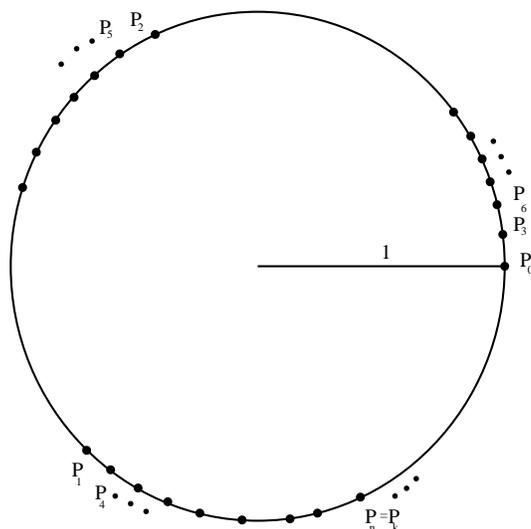


FIGURE 8.  $P_n = P_k$  for some  $n$  and  $k$ .

16. Another way to compute the distance along the circle between  $P_k$  and  $P_n$  is to use the circumference formula for the circle. Since  $P_n = P_k$ , we know that we will have gone around the circle some whole number of times, say  $L$  times; use this and the circumference formula for a circle to determine the distance along the circle between  $P_n$  and  $P_k$ .
17. Use your answers to Investigations 15-16 to show that if  $P_n = P_k$  then we can find whole numbers  $p$  and  $q$  so that  $\pi = \frac{p}{q}$ ; i.e.,  $\pi$  would have to be a rational number.
18. Use your answers to Investigations 15-17 and the fact that  $\pi$  is irrational to explain why all the points  $P_i$  in  $P$  are distinct; and hence, why the set  $P$  is infinite.

We are now ready to show that a circle and a circle minus a point are equidecomposable. We will let  $C$  denote the circle and let  $C'$  denote the circle minus  $P_0$  as shown in Figure 9.

19. Use the set  $P$  and the technique of shifting to infinity to show that  $C$  and  $C'$  are equidecomposable.  
**Hint:** As you did in Investigation 10, break both  $C$  and  $C'$  into two pieces such that one pair of pieces from each are identical and the other pair of pieces are congruent by a shift to infinity.

While the results to Investigation 10 and Investigation 19 may seem a bit surprising to you, there is a similar result that is even more surprising, the **Banach-Tarski Paradox**. Informally, the Banach-Tarski Paradox says that it is possible to take a pea cut it up into a finite number of pieces and using only the rigid motions described on page 63 resemble them to a ball the size of the sun. A more formal version of the theorem is the following:

**Theorem 1** (The Banach-Tarski Theorem). *It is possible to divide a solid ball into a finite number of pieces and then using only rigid motions, reassemble the pieces in such a way as to create two solid balls whose size and volume are the same as the original ball.*

This result first appeared in a 1924 paper entitled *Sur la décomposition des ensembles de points en parties respectivement congruentes* (Translation: On the decomposition of sets of points in respectively

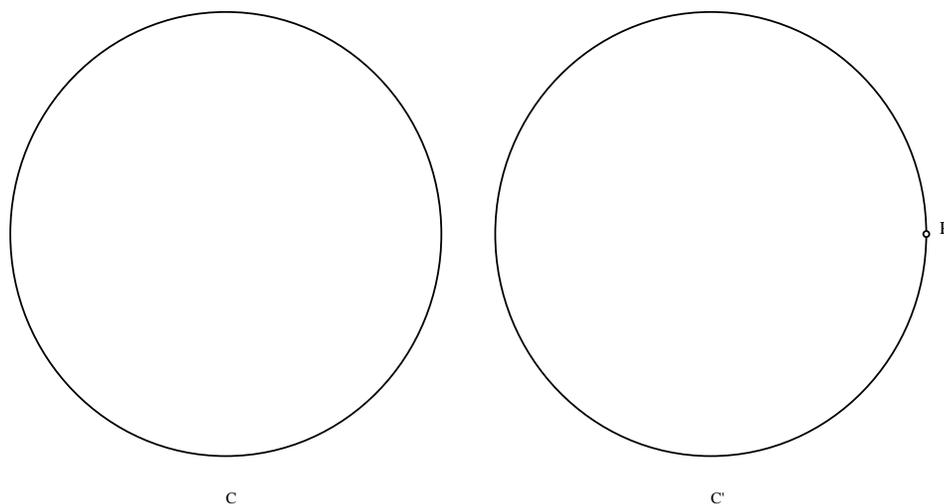


FIGURE 9. Circles  $C$  and  $C'$ .

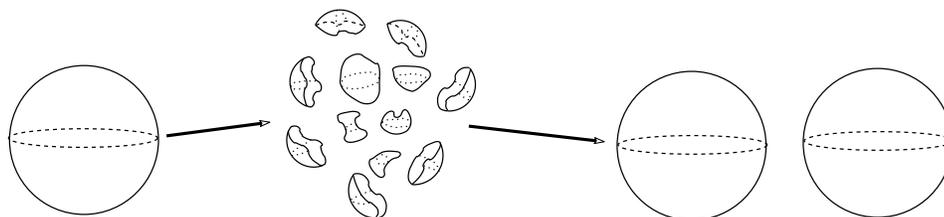


FIGURE 10. The Banach-Tarski Theorem

congruent parts) by **Stefan Banach** (Polish Mathematician; 1892 - 1945) and **Alfred Tarski** (Polish Mathematician; 1902 - 1983). While the technical aspects of this result are beyond the scope of this book, the following metaphor<sup>1</sup> will give you a sense of the ideas behind this remarkable result.

- 20.** Do you find the result of the Banach-Traski Paradox surprising? Explain?  
**21.** Do you

### 3. The HyperDictionary

The company hyper.com has decided to create the worlds most extensive online dictionary, the **HyperDictionary**. This dictionary will contain all possible words in the English language without accompanying definitions. That is, it will contain all the words we could possibly encounter in the English language; words like EQUIDECOMPOSABLE and SEQUESTRATION; as well as made up words such as AVRACADAVRA (from the Harry Potter books) and SUPERCALIFRAGILISTICEXPIALIDOCIOUS (from the movie *Mary Poppins*); and non-sensical words like DGBJKRTSPQXZ. hyper.com decides to put the dictionary on one big page.

- 22.** What will be the first 5 words in the Dictionary?  
**23.** How many words will be in the Dictionary before the word AB? Explain.

<sup>1</sup>Adapted from Wapner, Leonard, The Pea and the Sun: A Mathematical Paradox, A. K. Peters, Ltd., Wellesley, MA, 2005, pp. 135-138.

**24.** How many words will be in the Dictionary between the word AB and the word AC? Explain.

This dictionary has some very interesting properties that are worth exploring. While the Dictionary technically contains only individual words it will also contain complete sentences and definitions, if you know how to look for them.

**25.** Why will Virgil's famous saying, "Love conquers all" appear in the HyperDictionary? Explain.

**26.** Why will the definition, "A square is a four sided figure with equal sides and equal angles" appear in the HyperDictionary? Explain.

**27.** Why will the incorrect definition, "A square is a flying monkey" also appear in the HyperDictionary? Explain.

**28.** Explain why Hermann Melville's book, Moby Dick will appear in its entirety in the HyperDictionary.

**29.** Will anything you would ever want to know appear in the HyperDictionary? Explain.

As hyper.com gets set to have the HyperDictionary go live, concerns are raised about how long it will take for the page to upload on a browser. In an effort to decrease the loading time, hyper.com decides to break the Dictionary into 26 separate pages, one for each letter. The first page will consist of all possible words that begin with A; the second will list all possible words that begin with B; the third will list all possible words that begin with C and so on.

**30.** What will be the first 5 words on the A page?

**31.** What will be the first 5 words on the B page?

**32.** What will be the first 5 words on the Z page?

As hyper.com once again gets set to have the HyperDictionary go live, more concerns are raised about the length of time it will take for each page to upload on a browser. In another effort to decrease the upload time for each page, the authors decide to eliminate the first letter of every word on each page.

**33.** What will now be the first 5 words be on the A page? Explain.

**34.** What will now be the first 5 words be on the B page? Explain.

**35.** What will now be the first 5 words be on the Z page? Explain.

**36.** In what ways will the 26 pages be the same and in what ways will they be different? Explain.

**37.** How do these 26 pages now compare to the original HyperDictionary? Explain.

**38.** Why are your answers to Investigations **36-37** paradoxical?

**39.** The manner in which each of the 26 pages were modified corresponds to which one of the rigid motions on page 63? Explain.

**40.** Explain how your answers to Investigations **30-39** give us a metaphor for Theorem 1, the Banach-Tarski Paradox.

**41.** What does the Banach-Tarski Paradox suggest I should be able to do if I had a pound of gold? Explain.

**42.** Why do you think no one has been able to do what you stated in your answer to Investigation **41**? Explain.

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# Index

- algebraic curves, 36
- alternating harmonic series, 7
- and Hobbes, Calvin, 21
- and James Newman, Edward Kasner, 61
- and K. E. Grant, W. P. Berlinghoff, 61
- antiderivative of  $f$ , 49
- apothem, 17
- Archimedes, , 15
- area, 10
- areas under curves, 9
  
- Banach, Stefan, 67
- Banach-Tarski Paradox, 61, 66
- Barrow, Isaac, 43
- billiards, 41
- Bourbaki, Nicholas, 61
  
- Calculus, 18
- Cantor Quinary Set, 26, 27
- Cantor Sets, 22
- Cantor Ternary Set, 22, 24
- Cantor, Georg, 22
- Cauchy, Augustin-Louis, 47, 49
- caustic curve, 40
- complex numbers, 40
- congruent, 10, 63
- curve, 30
  
- Da Vinci, Leonardo, 18
- definite integral, 51
- derivative, 31
- Differential Geometry, 48
- dissected, 62
- Douglass, Frederick, 3
  
- Einstein, Albert, 58
- equidecomposable, 63
- $\operatorname{erf}(x)$ , 59
- Euclid, , 14
- Euclidian Geometry, 14
  
- Fermat's Last Theorem, 1
- formal definition, 4
- frame, 35
- function, 34
  
- Fundamental Theorem of Calculus, 49
  
- geometric series, 47
- Goldsworthy, Andy, 29
  
- Hudelson, Matt, 5
  
- informal definition, 4
- Integration, 18
- integration, 22
- irrational number, 65
  
- Lambert, Johann Heinrich, 65
- Leibniz, Gottfried, 32, 51
- Length, 21
  
- measure, 21
- Measure Theory, 15
- method of exhaustion, 46
  
- natural logarithm, 5
- natural numbers, 63
- Newton, Isaac, 32, 51, 53
- normal distribution, 58
  
- of Syracuse, Archimedes, 45
- Ohanian, Susan, 45
  
- parametrization, 38
- Perleman, Grigori, 1
- Plato, , 2
- Poincaré, Henri, 1
- point, 14
- polynomial, 49
- Proof by Contradiction, 46
- Pythagorean society, 2
  
- quadrivium, 2
  
- reflection, 63
- regular polygons, 17
- Renan, Ernest, 9
- Riemann Sums, 48
- Riemann, Bernhard, 47, 48
- rigid motions, 63
- roots, 40
- rotation, 63

Ruled Surfaces, 38  
Russell, Bertrand, 2

series, 47  
shifting to infinity, 63  
Smith, Henry John Stephen, 22  
Smithson, Robert, 50

tangent lines, 30  
tangrams, 61  
Tarski, Alfred, 67  
The Cantor Ternary Set, 22  
Tolstoy, Leo, 52  
translation, 63  
trivium, 2

undefined term, 3

zeros, 40