DISCOVERING THE ART OF MATHEMATHEMATICS

PATTERNS

MATHEMATICAL INQUIRY IN THE LIBERAL ARTS

Julian F. Fleron and Philip K. Hotchkiss,
with Volker Ecke and Christine von Renesse

Discovering the Art of Mathematics
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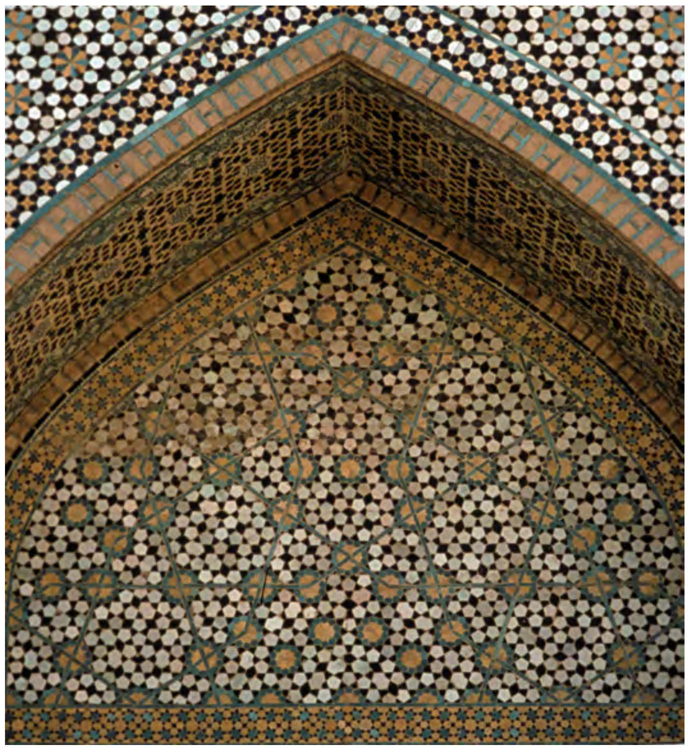

by Julian F. Fleron and Philip K. Hotchkiss

with Volker Ecke and Christine von Renesse

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Somewhere something incredible is waiting to be known.

Carl Sagan (American astronomer and author; 1934 - 1996)

What is mathematics? Ask this question of person chosen at random, and you are likely to receive the answer "Mathematics is the study of number." With a bit of prodding as to what kind of study they mean, you may be able to induce them to come up with the description "the science of numbers." But that is about as far as you will get. And with that you will have obtained a description of mathematics that ceased to be accurate some two and a half thousand years ago!...In fact, the answer to the question "What is mathematics?" has changed several times during the course of history... It was only in the last twenty years or so that a definition of mathematics emerged on which most mathematicians agree: mathematics is the science of patterns.

Keith Devlin (British mathematician and science writer; 1947 -)

Born of man's primitive urge to seek order in his world, mathematics is an ever-evolving language for the study of structure and pattern. Grounded in and renewed by physical reality, mathematics rises through sheer intellectual curiosity to levels of abstraction and generality where unexpected, beautiful, and often extremely useful connections and patterns emerge. Mathematics is the natural home of both abstract thought and the laws of nature. It is at once pure logic and creative art.

Lawrence University Catalogue

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Preface

This book is a very different type of mathematics textbook. Because of this, users new to it, and its companion books that form the Discovering the Art of Mathematics library[1], need context for the book's purpose and what it will ask of those that use it. This preface sets this context, addressing first the Explorers (students), then both Explorers and Guides (teachers) and finishing with important information for the Guides.

0.1 Notes to the Explorer

"Explorer?"

Yes, that's you - an Explorer. And these notes are for you.

We could have addressed you as "reader," but this is not a book intended to be read like a traditional book. This book is really a guide. It is a map. It is a route of trail markers along a path through part of the vast world of mathematics. This book provides you, our explorer, our heroine or hero, with a unique opportunity to explore - to take a surprising, exciting, and beautiful journey along a meandering path through a great mathematical continent.

"Surprising?" Yes, surprising. You will be surprised to be doing real mathematics. You will not be following rules or algorithms, nor will you be parroting what you have been dutifully shown in class or by the text. Unlike most mathematics textbooks, this book is not a transcribed lecture followed by exercises that mimic examples laid out for you to ape. Rather, the majority of each chapter is made up of Investigations. Each chapter has an introduction as well as brief surveys and narratives as accompaniment, but the Investigations form the heart of this book. They are landmarks for your expedition. In the form of a Socratic dialogue, the Investigations ask you to explore. They ask you to discover mathematics. This is not a sightseeing tour, you will be the active one here. You will see mathematics the only way it can be seen, with the eyes of the mind - your mind. You are the mathematician on this voyage.

"Exciting?" Yes, exciting. Mathematics is captivating, curious, and intellectually compelling if you are not forced to approach it in a mindless, stress-invoking and mechanical manner. In this journey you will find the mathematical world to be quite different from the static barren landscape most textbooks paint it to be. Mathematics is in the midst of a golden age - more mathematics is being discovered now than at any time in its long history. Each year there are 50,000 mathematical papers and books that are reviewed for *Mathematical Reviews*! Fundamental questions in mathematics - some hundreds of years old and others with \$ 1 Million prizes - are

[1] All available freely online at `http://artofmathematics.org/books`.

being solved. In the time period between when these words were written and when you read them important new discoveries adjacent to the path laid out here have been made.

"Beautiful?" Yes, beautiful. Mathematics is beautiful. It is a shame, but most people finish high school after 10 - 12 years of mathematics *instruction* and have no idea that mathematics is beautiful. How can this happen? Well, they were busy learning arithmetical and quantitative skills, statistical reasoning, and applications of mathematics. These are important, to be sure. But there is more to mathematics than its usefulness and utility. There is its beauty. And the beauty of mathematics is perhaps its most powerful, driving force. As the famous **Henri Poincaré** (French mathematician; 1854 - 1912) said:

> The mathematician does not study pure mathematics because it is useful; [s]he studies it because [s]he delights in it and [s]he delights in it because it is beautiful.

Mathematics plays a dual role as a liberal art and as a science. As a powerful science, it shapes our technological society and serves as an indispensable tool and as a language in many fields. But it is not our purpose to explore these roles of mathematics here. This has been done in other fine, accessible books. Instead, our purpose is to journey down a path that values mathematics for its long tradition as a cornerstone of the liberal arts.

Mathematics was the organizing principle of the *Pythagorean society* (ca. 500 B.C.). It was a central concern of the great Greek philosophers like **Plato** (Greek philosopher; 427 - 347 B.C.). During the Dark Ages, classical knowledge was preserved in monasteries. The classical **liberal arts** organized knowledge in two components: the ***quadrivium*** (arithmetic, music, geometry, and astronomy) and the ***trivium*** (grammar, logic, and rhetoric) which were united by philosophy. Notice the central role of mathematics in both components. During the Renaissance and the Scientific Revolution the importance of mathematics as a science increased dramatically. Nonetheless, it also remained a central component of the liberal arts during these periods. Indeed, mathematics has never lost its place within the liberal arts except in contemporary classrooms and textbooks where the focus of attention has shifted solely to its utilitarian aspects. If you are a student of the liberal arts or if you want to study mathematics for its own sake, you should feel more at home on this expedition than in other mathematics classes.

"Surprise, excitement, and beauty? Liberal arts? In a mathematics textbook?" Yes. And more!

In your exploration here you will see that mathematics is a human endeavor with its own rich history of struggle and accomplishment. You will see many of the other arts in non-trivial roles: art, music, dance and literature. There is also philosophy and history. Students in the humanities and social sciences, you should feel at home here too. There are places in mathematics for anyone to explore, no matter their area of interest.

The great **Betrand Russell** (English mathematician and philosopher; 1872 - 1970) eloquently observed:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.

We hope that your discoveries and explorations along this mathematical path will help you glimpse some of this beauty. And we hope they will help you appreciate Russell's claim:

> ...The true spirit of delight, the exultation, the sense of being more than [hu]man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

Finally, we hope that your discoveries and explorations enable you to make mathematics a part of your lifelong educational journey. For, in Russell's words once again:

> ...What is best in mathematics deserves not merely to be learned as a task but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement.

Bon voyage. May your journey be as fulfilling and enlightening as those that have beaconed people to explore the many continents of mathematics throughout humankind's history.

0.2 Navigating This Book

Intrepid Explorer, as you ready to begin your journey, it may be helpful for us to briefly describe basic customs used throughout this book.

As noted in the Preface, the central focus of this book is the **Investigations**. They are the sequences of problems that will help guide you on your active exploration of mathematics. In each chapter the Investigations are numbered sequentially in bold. Your role will be to work on these Investigation individually or cooperatively in groups, to consider them as part of homework assignments, to consider solutions to selected Investigations that are modeled by your fellow explorers - peers or your teacher - but always with you in an active role.

If you are stuck on an Investigation remember what **Frederick Douglass** (American slave, abolitionist, and writer; 1818 - 1895) told us:

> If there is no struggle, there is no progress.

Or what **Shelia Tobias** (American mathematics educator; 1935 -) tells us:

> There's a difference between not knowing and not knowing *yet*.

Keep thinking about the problem at hand, or let it ruminate a bit in your subconscious, think about it a different way, talk to peers, or ask your teacher for help. If you want you can temporarily put it aside and move on to the next section of the chapter. The sections are often somewhat independent.

Independent Investigations are so-called to point out that the task is more involved than the typical Investigations. They may require more significant mathematical epiphanies, additional research outside of class, or a significant writing component. They may also signify an opportunity for class discussion or group reporting once work has reached a certain stage of completion.

The **Connections** sections are meant to provide illustrations of the important connections between the mathematics you're exploring and other fields - especially in the liberal arts. Whether you complete a few of the Connections of your choice, all of the Connections in each section, or are asked to find your own Connections is up to your teacher. We hope that these Connections sections will help you see how rich mathematics' connections are to the liberal arts, the fine arts, culture, and the human experience.

Further Investigations, when included, are meant to continue the Investigations of the mathematical territory but with trails to significantly higher ground. Often the level of sophistication of these investigations will be higher. Additionally, our guidance will be more cursory - you are bushwhacking on less well-traveled trails.

In mathematics, proof plays an essential role. Proof is the arbiter for establishing truth and should be a central aspect of the sense-making at the heart of your exploration. Proof is reliant on logical deductions from agreed upon definitions and axioms. However, different contexts suggest different degrees of formality. In this book we use the following conventions regarding **definitions**:

- An *Undefined Term* is italicized the first time it is used. This signifies that the term is: a standard technical term which will not be defined and may be new to the reader; a term that will be defined a bit later; or an important non-technical term that may be new to the reader, suggesting a dictionary consultation may be helpful.

- An ***Informal Definition*** is italicized and bold- faced the first time it is used. This signifies that an implicit, non-technical, and/or intuitive definition should be clear from context. Often this means that a formal definition at this point would take the discussion too far afield or be overly pedantic.

- A **Formal Definition** is bolded the first time it is used. This is a formal definition that is suitably precise for logical, rigorous proofs to be developed from the definition.

In each chapter the first time a **Biographical Name** appears it is bolded and basic biographical information is included parenthetically to provide historical, cultural, and human connections.

In mapping out trails for your explorations of this fine mathematical continent we have tried to uphold the adage of **George Bernard Shaw** (Irish playwright and essayist; 1856 - 1950):

> I am not a teacher: only a fellow-traveler of whom you asked the way. I pointed ahead – ahead of myself as well as you.

We wish you wonderful explorations. May you make great discoveries, well beyond those we could imagine.

0.3 Directions for the Guides

Faithful Guide, you have already discovered great surprise, beauty and excitement in mathematics. This is why you are here. You are embarking on a wonderful journey with many explorers looking to you for bearings. You're being asked to lead, but in a way that seems new to many.

We believe telling is not teaching. Please don't tell them. Answer their questions with questions. They may protest, thinking that listening is learning. But we believe it is not.

This textbook is very different from typical mathematics textbooks in terms of structure (only questions, no explanations) and also of expectations it places on the students. They will likely protest, "We're supposed to figure this out? But you haven't explained anything yet!" It is important to communicate this shift in expectations to the students and explain some of the reasons. That's why we have written the earlier sections of this preface, which can help do the explaining for us (and for you).

You need support as well. A shift in pedagogy to a more inquiry-based approach may be subtle for some, but for many it is a great leap. Understanding this we have assembled an online resource to support teachers in the creation and nurturing of successful inquiry-based mathematics classrooms. Available online at `http://artofmathematics.org/classroom` it contains a wealth of information - in many different forms including text, data, videos, sample student work - on many critical topics:

- Why inquiry-based learning?
- How to get started using our books...
- A culture of curiosity
- Learning contracts
- Grouping students
- Choosing materials - Mixing It Up
- Asking good questions
- Creating inquiry-based activities
- Making mistakes
- Cool things
- Proof as sense-making
- Homework stories
- Exams
- Posters
- Assessment: Student Solution Sets
- Evaluating the effectiveness of inquiry-based learning
- ... and much more ...

We wrote the books that make up the Discovering the Art of Mathematics library because they have helped us have the most extraordinary experiences exploring mathematics with students who thought they hated mathematics and had been disenfranchised from the mathematical experience by their past experiences. We are encouraged that others have had similar experiences with these materials. We love to hear success stories and are also interested in hearing about things that might need to be changed or did not work so well. Please feel free to share your stories and suggestions with us: `http://artofmathematics.org/contact`.

0.3.1 Notes for Guides on the Patterns Exploration

This is a beta version of this book. Several chapters are in good shape and have been reviewed and beta tested. Other chapters, and the overall theme and order of the book, are still in progress. Updates will be happening that will be posted only at intermittent intervals. If you would like to know if there are meaningful updates, please contact the authors.

Chapter 1

Introduction - Mathematics as the Art and Science of Patterns

> Mathematics is the science of patterns.
>
> **Keith Devlin** (British mathematician; 1947 -)

> A surprising proportion of mathematicians are accomplished musicians? Is it because music and mathematics share patterns that are beautiful?
>
> **Martin Gardner** (American mathematician; 1914 - 2010)

> Mathematics, in the common lay view, is a static discipline based on formulas...But outside the public view, mathematics continues to grow at a rapid rate...the guide to this growth is not calculation and formulas, but an open ended search for pattern.
>
> **Lynn Arthur Steen** (American mathematician; 1941 - 2015)

1.1 What is a Pattern?

Doubtless you use the word pattern and you have a sense of what a pattern is.

1. Give a brief but concise definition of "pattern".

This entire book is about patterns. Yet as Keith Devlin tells us, "Mathematics is the science of patterns." So this book can only do so much. In the Discovering the Art of Mathematics series there are eleven books. At the heart of each of these books is patterns, patterns which play essential roles in music, dance, art, visualization, number, measurement, games & puzzles, knots, reasoning, and even infinities. Each of these books is a testament to Devlin's claim.

Yet this these books are designed to get you involved in the mathematical adventure. So let's get you started!

2. Find examples of 10 patterns. Spend some time really searching so you are able to find diverse examples with rich and different attributes - a mixture of patterns that are beautiful, surprising, complex, curious, multifaceted, mathematical, and artistic, and that come from many different areas. For each of these 10 patterns provide:

Figure 1.1: Double rainbow in the whirlpools at the base of Niagara Falls.

- A name, either official or one that you think is appropriately descriptive,
- A brief description of the pattern, describing the pattern itself, where it arises, who invented/discovered it, etc.
- A brief description of why you found this pattern compelling enough to include it in your collection.
- A visual image, sample, or rendering of the pattern.

3. Return to your pattern definition in Investigation **1**. Do each of the five patterns you have just found fit within this definition?

Here's a secret - it is not very easy to define what a pattern is. One can try, but as you likely have just seen, if you try to define "patterns" too strictly you will leave out important examples and classes of things that certainly are patterns. On the other hand, if you define "patterns" too broadly there is another problem.

Would you call the stellar constellations like the Big Dipper patterns? If so, here's what Christopher Boone, the 15 year-old hero, a self-described "mathematician with some behavioral difficulties," in the wonderful book Curious Incident of the Dog in the Night-Time by Mark Haddon, would tell you:

> People say that Orion is called Orion beccause Orion was a hunter and the constellation looks like a hunter with a club and a bow and arrow... But this is really silly because it is just stars, and you could join up the dots in any way you wanted, and you could make it look like a lady with an umbrella who is waving, or the coffeemaker which Mrs. Shears has, which is from Italy, with a handle and steam coming out, or like a dinosaur... And there aren't any lines in space, so you could join bits of Orion to bits

of Lepus or Taurus or Gemini and say there were a constellation called the Bunch of Grapes or Jesus or Bicycle (except that they didn't have bicycles in Roman and Greek times, which was when they called Orion Orion). And anyway, Orion is not a hunter of coffeemakers of a dinosaur. It is just Betelgeuse and Bellatrix and Alnilam and Rigel and 17 other stars I don't know the names of. And they are nuclear explosions billions of miles away. And that is the truth.

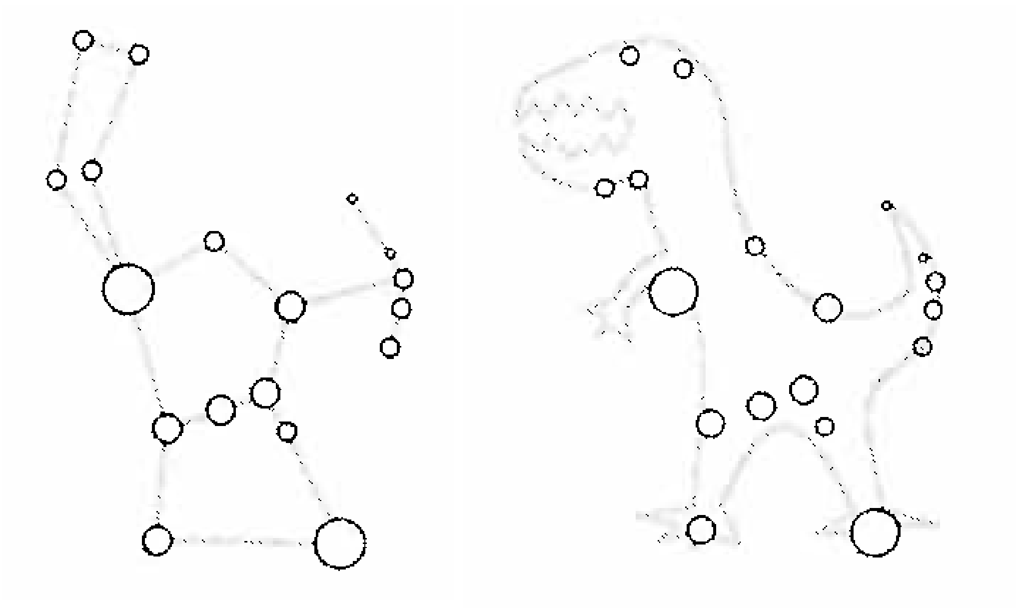


Figure 1.2: Two different views of the constellation Orion.

We'll side with Christopher Boone. Orion is not a pattern - or even a legitimate constellation other than for human amusement. Regarding patterns, if one expands the definition of too broadly then everything is a pattern and the word loses its meaning.

We won't say much more about a definition for "patterns" because mathematicians don't. They have very precise definitions of certain types of patterns, but in general the word "patterns" does not have a precise mathematical definition.[1]

1.2 Orbits on Pool Tables

Many different types of *billiard* games exist. Originally billiard tables did not have pockets, the object of the game involving striking and banking rather than pocketing the balls.

Lewis Carroll (English author and mathematician; 1832 - 1898) was not only the author of famous books such as Alice in Wonderland, but he was a mathematician of significant note. He

[1]For example, while the definitive reference Tilings and Patterns by Branko Grunbaum and G.C. Shephard abounds with definitions of all sorts of important patterns, the word "patterns" itself remains undefined.

included logical conundrums, secret codes, and puzzles in his literature, but his mathematical work extends well beyond this. He published more than a dozen books on significant mathematics under his given name, Charles Dodgson. He was interested in games as well and in 1889 he published rules for playing billiards on a circular table.[2]

Mathematicians since have studied the paths of balls on billiard tables of all different shapes, out of curiosity and because of their connection to important mathematical and physical process:

> Studies of the dynamical systems of billiard type (or simply billiards) form one of the most fascinating and notoriously difficult areas in the modern theory of dynamical systems. Billiards... arise naturally in applications (primarily in classical mechanics, statistical mechanics, optics, acoustics, and quantum physics.) For instance, the Boltzmann gas of elastically colliding balls in a box, which is the most fundamental and venerable model in statistical mechanics, is a billiard system.[3]

In the systems studied by mathematicians there are generally no pockets for the balls to fall into and the ball continues traveling indefinitely. Mathematicians call the paths' of the balls ***orbits***.

It may be surprising, but on Carroll's circular table there are only two types of orbits, as illustrated in Figure 1.3.

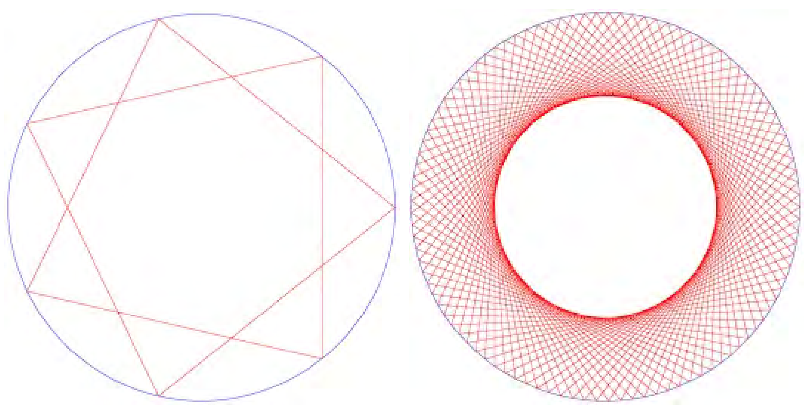


Figure 1.3: Representative orbits on a circular pool table - either *periodic* (left) or *irrational* (right).

Similarly surprising, on elliptical pool tables there are also only two types of orbits, as shown in Figure 1.4.[4]

The geometry of these orbits need *tangent lines* from calculus to determine the actual trajectories and these images of orbits share important connections to the string art pieces that are created in chapter of that name in Discovering the Art of Mathematics - Calculus.

It's not just physical applications that propel mathematicians to study dynamical systems like billiard tables. They may unlock some of the deepest secrets in mathematics, among them

[2] The Universe in a Handkerchief by Martin Gardner, p. 148-9.

[3] Book review by L. Bunimovich of Chaotic Billiards, *Bulletin of the American Mathematical Society*, Vol. 46, No. 4, October 2009, p. 683.

[4] For more see "Billiards in the Round" by Ivars Peterson, *MAA Online*, at `www.maa.org/mathland/mathland_3_3.html` and there references therein.

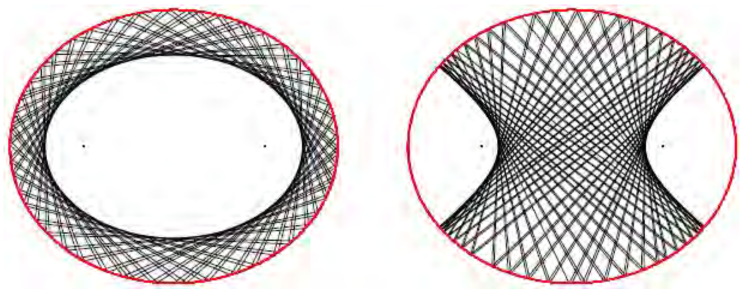


Figure 1.4: Representative orbits on an elliptical pool table - enclosing an interior ellipse in which the ball does not travel (left) or avoiding an exterior hyperbola in which the ball does not travel (right).

the *Riemann hypothesis*, a hypothesis more than 150 years old. This hypothesis is one of the *Millennium Prize Problems* with $1 million U.S. dollars awaiting would-be solvers. For more on this connection see "A Prime Case of Chaos" by Barry Cipra.[5] For more on the Riemman hypothesis see "Class Numbers: A Bridge Between Two $ 1 Million Dollar Problems" in Discovering the Art of Mathematics - Number Theory.

Our investigations will take place on pool tables with the following properties:

- The table is rectangular.
- There are four pockets, one at each corner.
- The ball is shot from the lower, left corner of the table.
- The ball rebounds at a 45 degree angle from each wall it strikes and continues to roll until it reaches a pocket.

Figure 1.5 shows two examples of the orbits of the ball on pool tables of this type. Notice that we denote the dimensions by $l \times w$ with the length horizontal and the width vertical.

4. On graph paper make 15 pool tables and draw the orbit of the ball on each table. Each square on the graph paper should represent one unit. Label the dimensions of your tables. Strive to make many tables of many different sizes and proportions.

5. On the scale of the individual squares that make up the graph paper, what does the path of the ball look like?

6. On a ***local scale*** of the individual squares on your graph paper, what do you notice about the path of the ball?

7. Will this pattern, or some related pattern, continue when a single ball is shot on a real pool table? Explain.

[5] Originally appearing What's Happening in the Mathematical Sciences, Vol. 4, American Mathematical Society, 1999 but also available online at `http://www.ams.org/samplings/math-history/prime-chaos.pdf` .

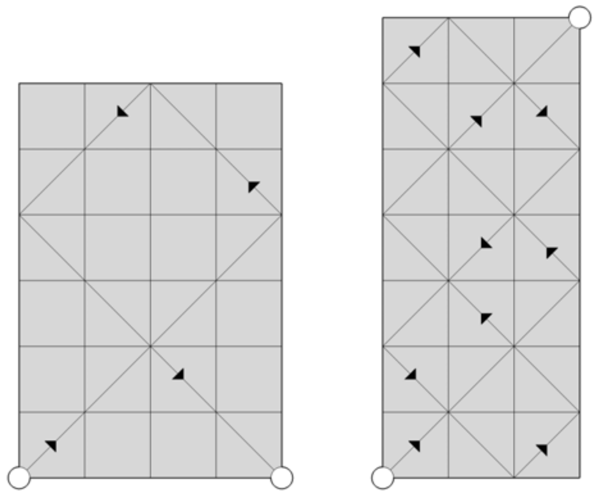


Figure 1.5: Orbits of balls on rectangular, 4-pocket pool tables with balls shot from the lower-left corner. A 4×6 table on the left and a 3×7 table on the right.

8. On a ***global scale*** of the whole path of the ball, what do you notice about the geometry of the individual paths?

9. Will this pattern, or some related pattern, continue when a single ball is shot on a real pool table? Explain.

10. Are there two, or more, tables where the orbit is (mathematically) **similar** (i.e. has the same shape that has been scaled to a different size)? If so, what do you notice about the dimensions of these tables? If not, make two tables on which the orbits are similar.

11. Create several more tables on which orbits are similar.

12. Formulate a statement which describes a pattern that characterizes *all* tables on which the balls' orbits are similar.

13. Are you certain that your statement will hold for all tables of this type? Explain.

In the table in the right of Figure 1.5 the ball crossed through the center of every square while on the table on the left the ball did not pass through the center of every square.

14. Find several tables on which the orbit fails to pass through every square on the table.

15. Find a pattern in the dimensions of these tables and then formulate a precise pattern which characterizes *all* such tables on which the ball fails to pass through every square.

16. Find several tables on which the orbit passes through every square on the table.

17. Find a pattern in the dimensions of these tables and then formulate a precise pattern which characterizes *all* such tables on which the ball passes through every square.

18. Create 10 more tables and draw the orbits of the balls. Use these examples to check your patterns in Investigation **15** and Investigation **17**.

Your statements in Investigation **15** and Investigation **17** are called ***conjectures*** because they are motivated by examples and you do not (yet) have definitive proof that they will hold for all possible pool tables.

Let us now turn to a different problem:

> **19.** INDEPENDENT INVESTIGATION: Can you find patterns which enable you to determine the corner the ball falls into in based on the dimensions of the table?

As an independent investigation, this is a significant task. It may take 30 minutes, an hour, some time away from it so it can percolate in your subconscious, etc.

The patterns you find as part of your independent investigation should enable you to determine the pocket in which a pool ball falls into on any rectangular table whose dimensions are whole numbers. I.e. they should completely answer the independent investigation above. The investigations below are meant to help you be certain that your patterns are complete. If not, please return to the investigations above.

Figure 1.6: Hot sauce pattern.

20. What pocket will the ball fall into on a table that is 3×9? Explain.

21. What pocket will the ball fall into on a table that is 3×17? Explain.

22. What pocket will the ball fall into on a table that is 4×64? Explain.

23. What pocket will the ball fall into on a table that is 17×3? Explain.

24. What pocket will the ball fall into on a table that is 110×120? Explain.

25. What pocket will the ball fall into on a table that is 21×2145? Explain.

26. What pocket will the ball fall into on a table that is 2048×8192? Explain.

27. What pocket will the ball fall into on a table that is 810×4536? Explain.

28. Do you think that these mathematical investigations will make you a better pool player in real life? Explain.

The influential **Henri Poincaré** (French mathematician, physicist, engineer and philosopher; 1854 - 1912) said:

> The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful.

29. Whether or not this exploration of pool tables makes you a better pool player, did you find it interesting or worthy of the words"delight" or "beautiful" used by Poincare? Explain.

1.3 Heavenly Orbits: Kepler, Eclipses and Transits

> Also, when the sun and the moon, they say, and all the stars, so great in number and in size, are moving with so rapid a motion, how should they not produce a sound immensely great? Starting from this argument and from the observation that their speeds, as measured by their distances, are in the same ratios as musical concordances, they assert that the sound given forth by the circular movement of the stars is a harmony.[6]
>
> **Aristotle** (Greek philosopher and scientist; 384 BC - 322 BC)

The vision described by Aristotle of harmony in the movement of heavenly bodies is generally referred to as the ***music of the spheres***. It is a romantic notion of the universe that was begun by **Pythagoras** (Greek philosopher and mathematician; circa 570 BC - circa 495 BC) and his followers, was revered by many until the Scientific Revolution and still has supporters.[7]

The transition to a more scientific approach was sweeping during the Scientific Revolution, but it was slow in the timescale of human generations. Figure 1.7 shows a model of the solar system where the planets orbits are determined by nested *Platonic solids*.[8] This type of whimsical model fits well with those pursuing "theories" that promote the music of the spheres. But they are not compatible with the serious scientific advances that were made during this time - including others by Kepler himself.

In fact, Kepler discovered one of the most important patterns in all of astronomy, **Kepler's laws of planetary motion**:

[6] From Aristotle's On the Heavens.

[7] See e.g. The Music of the Spheres: Music, Science, and the Natural Order of the Universe by Jamie James.

[8] For more on this model and the Platonic solids see the Introduction to Discovering the Art of Mathematics: Geometry.

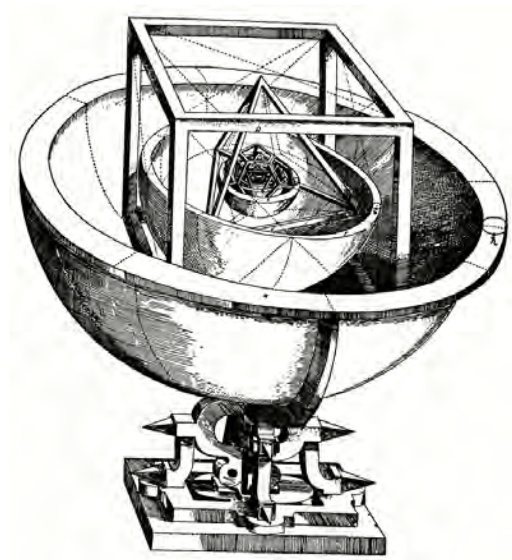


Figure 1.7: Kepler's model of the solar system based on the Platonic solids.

- The orbit of each planet is an *ellipse* with the sun as one of the two *foci*.
- A line segment between a planet and the sun sweeps out equal areas in equal time intervals.
- The square of the ***period*** of the planet's orbit, i.e. the length of the planet's year, is proportional to the cube of the length of the orbit's *semi-major axis*.

These are fundamental laws that hold for all solar systems whose planets orbit a single star.

In retrospect we can look back at efforts to understand the music of the spheres with and discern the real advances from the approaches that now seem rather ad hoc. Kepler's laws and Kepler's model based on the Platonic solids is one striking example. Another is our efforts to discover the distant planets in our solar system. The discovery of Neptune was a remarkable use of deep mathematics. In contrast the Titius-Bode Law was a prime example of searching hard to discover some sort of pattern without any concern for an underlying mechanism that reflected a meaningful physical process at work. (See Further Investigations below for more details on both.)

A beautiful, historically important example of the prime role of mathematics in understanding the music of the spheres is the transits of Venus.

Consider the list of dates below.

- 7 December, 1631
- 4 December, 1639
- 6 June, 1761
- 3 June, 1769

Figure 1.8: 2004 transit of Venus taken at sun-up on the Catawba River near Connelly's Springs, NC by David Cortner.

- 9 December, 1874
- 6 December, 1882
- 8 June, 2004
- 6 June, 2012

30. Find patterns, or at least approximate patterns, in this data - as many or as complete as you can to characterize these dates. Describe what you have found.

31. Use this pattern to predict what the next several dates may be.

32. What is a eclipse?

33. Are there different types of eclipses? Explain.

34. What are the geometric factors that give rise to eclipses?

35. For centuries astronomers have been able to predict future eclipses with great accuracy. While the specific details are quite complex, explain generally how it might be that such predictions can be made.

The dates given above are the dates for the ***transits of Venus***. These are astronomical events akin to an eclipse, only here the intervening body is significantly smaller that the body it "transits" in front of.

36. When looking at the sun from the Earth, what planets can transit in front of the sun?

37. If we were on Jupiter, what planets could transit in front of the sun?

38. From Earth, do you think that transits of Mercury or transits of Venus happen more frequently? Explain why.

No transits of Venus occurred in the twentieth century. There were exactly two in the twenty-first century. These transits are pictured in Figure 1.8 and Figure 1.9. At the time of writing of this book, 2013, most people who have not seen one of the transits of Venus that just occurred will not live to have the opportunity to see the next transit of Venus.

Figure 1.9: 2012 transit of venus taken at dusk near the town green in Westfield, MA by Julian Fleron.

Future dates of transits of Venus are as follow:

- 11 December, 2117
- 8 December, 2125
- 11 June 2247
- 9 June, 2255
- 13 December, 2360
- 10 December, 2368

- 12 June, 2490
- 10 June, 2498
- 16 December, 2603
- 13 December, 2611
- 15 June, 2733
- 13 June, 2741
- 16 December, 2846
- 14 December, 2854
- 16 June, 2976
- 14 June, 2984
- 18 December, 3089

39. Does this new data generally follow the pattern you describe in Investigation **30**?

40. Can you think of a reason why the dates of consecutive transits of Venus separated by less than a decade differ by two days and other times they differ by three? Explain.

41. After that last date on the list above, what do you think the date of the next transit will be?

In fact, there is only a single transit of Venus in the thirty-first century. And there is only one in the thirty-fourth century as well.

In recorded human history the only transits of Venus that are known to have been observed were those in 1639, 1761, 1769, 1874, 1882, 2004, and 2012.

42. The ancient astronomers deeply understood the music of the spheres and were proficient at predicting the dates of eclipses. Why didn't they see transits of Venus and Mercury in their time?

The transits of Venus in the eighteenth century were of profound importance, for they enabled human beings to determine how far the earth was from the sun!

At the outset of the eighteenth century scientists had made great progress in understanding our solar system. However, their knowledge of *all* astronomical distances were all relative measurements. We knew to great certainty that, for example, Mars' average distance from the sun is 1.52 times the distance from the earth to the sun. But because we had no idea how far the earth was from the sun, we could not determine the absolute distance of Mars from the sun either. In 1716 **Edmund Halley** (English astronomer; 1656 - 1742) proposed that measurements of the transits of Venus could be used to determine the distance of the earth from the sun, a distance called the **astronomical unit**. The paper describing this possibility was "a clarion call for scientists everywhere to prepare for the rare opportunity presented by the forthcoming transits of 1761 and 1769."[9]

Scientists all over the world accepted Halley's call to action. They traveled great distances to position themselves in the most hospitable locales to view the transit. Many difficulties hampered their efforts in 1761, including significant measurement error. Great efforts were taken to address these matters so more successful measurements could be taken during the 1769 transit. After the observations from this latter transit were combined and tabulated the distance from the earth to the sun was calculated. The result was within eight-tenths of one percent! I.e. These scientists and mathematicians had determined the distance from the earth to the sun with an error less than 0.008 from the value we can now determine exactly using radar.

These effort had many notable human aspects. Kepler was the first to accurately predict upcoming dates of transits of Mars and Venus. In 1629 he predicted transits of both planets in 1631. Unfortunately he died in 1630 and did not get to see his predictions validated. **Guillaume Le Gentil** (French astronomer; 1725 - 1792) was kept from the transit in 1761 by war and stayed in the Indian Ocean for eight years to make observations of the 1769 transit but was defeated when the transit was obscured by cloudy skies. The most important mathematical details of the use of transit data to determine the astronomical unit were done by **James Short** (Scottish mathematician and optician; 1710 - 1768) who died a year before the 1769 transit!

For more on the history of transits of Venus and the mathematics underlying the calculation of the astronomical unit, see "Transits of Venus and the Astronomical Unit," Donald A. Teets, *Mathematics Magazine*, Vol. 76, No. 5, December 2003, pp. 335-348. There are also two wonderful, full-length books on the topic: Transits of Venus: 1631 to the Present by Nick Lomb and The Transits of Venus by William Sheehan and John Westfall.

1.4 Physical Morphogenesis: Pool Tables, Planetary Orbits, Snowflakes and Rainbows

You have found interesting patterns that allow you to predict which pocket the ball will fall into. But your mathematical exploration is missing a critical aspect - proof. While your many examples, and those of your peers, may feel like sufficient evidence that your theory is correct mathematicians are not content that a conjecture or hypothesis is certain until it has been proven. So there is more work to do on this problem.

Kepler's model of the planets built from Platonic solids is simply an coincidence, there is no real pattern at work here. In contrast, his laws of planetary motion have been extensively tested, providing *empirical validation.* But they have also been mathematically derived from Newton's laws of motion and Newton's law of universal gravitation - so we have a deep understanding of *why* Kepler's laws hold.

From the beginning we have asked you to find lots of patterns. For a proper mathematical study of patterns it is not enough just to *find* patterns, you must understand *why* these patterns work; what the underlying mechanisms are that give rise to these patterns; what has given birth to these patterns. There is a word for such understandings, it is ***morphogenesis***.

The term morphogenesis comes from two Greek roots: *morpho* which means form or structure and *genesis* which means origin, creation or beginning.

[9] P. 337 of the paper by Teets referenced below.

Figure 1.10: School of in St. Croix. Video of this school, dramatically interrupted by the arrival of a barracuda, available online at `https://www.youtube.com/watch?v=oV8rZjwcc_M`.

Figure 1.10 shows a school of fish. This particular school exhibits a wonderful visual pattern. The fact that most fish swim in schools is also a pattern of sorts. While it is a very interesting observation, what is more interesting is to ask the question: "Why do fish swim in schools?" For fish it is a matter of both a matter of efficiency in movement and protection from predators. (See the video for a striking example.) Many types of birds fly in flocks. The patterns exhibited by their flocks are of great interest; often they gain tremendous aerodynamical advantage from flying this way.

The images on the left and right in Figure 1.11 shows beautiful soap bubbles made simply by pouring soapy water out of an old glass bottle. Mathematicians study the geometry of soap bubbles and other minimal surfaces seeking to understand what gives rise to these beautiful patterns. In the center of Figure 1.11 one sees a very similar pattern made by light shining through sea water onto a sandy sea floor. If one watches the video one can begin to understand the mechanism - waves moving past bend the light and the ripples on the sandy sea floor provide a geometrically distorted canvas upon which these refracted light rays shine.

Rainbows are striking physical phenomena. They have captured the attention of artists, writers, mystics, scientists and mathematicians. In their beautiful book The Rainbow Bridge: Rainbows in Art, Myth and Science Raymond Lee jr. and Alistair Fraser give a sweeping history of the rainbow in human culture. They argue that for its early history the rainbow served as a bridge between the arts and sciences, much like the music of the spheres in astronomy, but that "by the close of the seventeenth century... [there was] a growing tension between artistic and scientific images of the rainbow... The rainbow bridge had vanished."[10]

What causes this spectacular pattern of colors in a rainbow? I.e. what is the morphogenesis of a rainbow?

This question occupied many great thinkers beginning with the fundamental work of Aristotle. Many scientific histories are biased toward Western cultures, both for political reasons and because evidence of these histories has been better preserved. However, the case of the rainbow is an example where we have surviving, historical accounts of efforts to understand what makes a rainbow

[10] P. 68.

Figure 1.11: Soap bubbles in thin, glass bottles and light patterns on the sea floor in St. Croix. Video of the light patterns in motion is available online at `http://artofmathematics.org/media/video-571`.

from cultures the world over. It is perhaps the role of the bridge between the scientific question and the arts that helped preserve these histories.

The book Discourse on the Method, and the three essays it is meant to introduce which are generally considered appendices now, is one of the most influential books of all time. Written by **Rene Descartes** (French philosopher, mathematician and scientist; 1596 - 1650), it has had a profound impact on philosophy. But its impact goes well beyond philosophy. The essay “Geometry” introduces ***Cartesian geometry***, the x, y plane from high school geometry, setting the stage for the powerful marriage of geometry and algebra that fundamentally transformed mathematics. The essays “Optics” and “Meterology” consider the nature of light, reflection, refraction, and rainbows! Descartes is one of the first definitive, correct treatments of the morphogenesis of rainbows.

43. Figure 1.13 shows Descartes’ illustration of the morphogenesis of rainbows. Learn about how rainbows are formed and describe this morphogenesis using this image.

As suggested by the title, Descartes was proposing a method by which definitive knowledge could be reached. The rainbow is his archetype to illustrate the method. As he says,

> The rainbow is such a remarkable phenomenon of nature, and it cause has been so meticulously sought after by inquiring minds throughout the ages, that I could not

Figure 1.12: Colored engraving double rainbows; “Niagara” by Harry Fenn.

> choose a more appropriate subject for demonstrating how, with the method I am using, we can arrive at knowledge not possessed at all by those whose writings are available to us.

44. Is it surprising to you that rainbows play such an important role in such an important work?

45. As kids, most of us have drawn rainbows. Did you ever learn much about rainbows in school? Why do you think this is so? In retrospect, should you have? Explain.

46. Make a snowflake by folding up a sheet of paper and then cutting out designs with scissors.

When asked to do this, most people make a very nice looking *rosette*. Many school children do this and their “snowflakes” are hung from windows in their schools in the winter. Unfortunately, the majority of these snowflakes do not reflect an essential property of real snowflakes: snowflakes exhibit *six-fold symmetry*. They have six arms and all of their features are repeatedly regularly around at center point at six equal intervals.

This brings to light two important questions of morphogenesis:

- How does one fold a piece of paper so the *rosette* one obtains by cutting has a desired symmetry type?
- Why do snowflakes exhibit 6-fold symmetry?

The former question is considered in the investigations in the Introduction to Discovering the Art of Mathematics: Geometry. The latter question is at the heart of the wonderful booklet The Six-Cornered Snow Flake written by Kepler in 1611 as a New Year’s gift to his friend **Wacher von Wackenfels** (German diplomat and scholar; 1150 - 1619). Although quite short this books is historically very important. His exploration of this question, his search for the morphogenesis of the snowflake, lead him to ask questions about the morphogenesis of crystals in general. This book is seen as one of the pioneering works in the history of *crystallography*. It is also in this book that

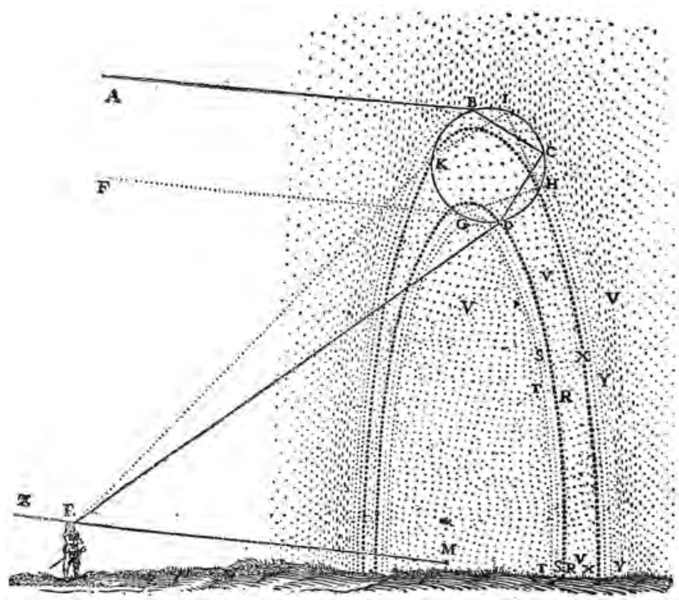


Figure 1.13: Illustration of the optics of rainbows from Descartes' Discourse on the Method.

the first statement of ***Kepler's conjecture*** is made - a conjecture which describes the best way to pack uniformly sized spheres in the smallest volume. This conjecture was not proven correct until 2005, almost 400 years after Kepler made it! Understanding why, the morphogenesis of the *face-centered cubic packing*, was a matter of great mathematical exploration for four centuries.

1.5 Biological Morphogenesis: Phyllotaxis and Plant Growth

Many, many plants exhibit beautiful patterns of spirals in their growth. This includes, for example, the arrangement of leaves along a tree's small branches, the arrangements of petals in a flower and the arrangement of seeds in a pinecone. These arrangements and the patterns they form are called ***phyllotaxis***.

As investigated in the chapter of the same name in Discovering the Art of Mathematics - Number Theory, many of these patterns follow the pattern of Fibonacci numbers. This is a famous sequence of numbers discovered by **Fibonacci** (a contraction of filus Bonaccio, "son of Bonaccio"), the son of a well-known merchant and whose proper name is **Leonardo of Pisa** (Italian mathematician; 1175 - 1250). While Fibonacci's name is attached to this important sequence of numbers, his accomplishments in mathematics were far greater than this sequence. He rejuvenated mathematics as the Dark Ages (circa 450 - 1000 A.D.) closed in Europe. He traveled widely as a student, learning methods of Arabic mathematics when studying in Northern Africa and learning the system of Hindu-Arabic numerals. Fibonacci assembled what he had learned into Liber Abaci (literally "book of the abacus", meaning book of arithmetic), the most comprehensive book of arithmetic of its time. It laid out the benefits of the Hindu-Arabic numeral system and is

Figure 1.14: Photographs of snowflakes by **Wilson Bentley** (American farmer; 1865 - 1931) who was the first known photographer to successfully photograph snowflakes.

partially responsible for its wide acceptance subsequently.

The **Fibonacci numbers** are the numbers $1, 1, 2, 3, 5, \ldots$ where each new Fibonacci number is obtained by adding the previous two Fibonacci numbers.

Figure 1.15: Fruit of the Kousa tree; Westfield, MA.

47. Find the next eight Fibonacci numbers.

48. Google "Fibonacci nature" and browse through pictures using the Image tab of your search engine. Find five images of Fibonacci numbers in nature that are most compelling to you and explain why you find them compelling.

While Fibonacci numbers do occur surprisingly often in natural processes, their ubiquity has been somewhat exaggerated. In discerning whether Fibonacci, or related sequences of, numbers actually arises, it is essential to understand the mechanism that give rise to the patterns. I.e. we must determine the morphogenesis.

Figure 1.16: Sea coral; St. Croix.

Vi Hart (American mathematician; -) is a prolific producer of beautiful, funny, quirky, inspiring and accessible videos about mathematics. A few years ago one of her videos was an open letter to Nickelodeom. In it she rants:

> Dear Nickelodeon, I've gotten over how SpongeBob's pants are not actually square. I can ignore most of the time that Gary's shell is not a logarithmic spiral. But what I cannot forgive is that SpongeBob's pineapple house is a mathematical impossibility... A true Pineapple would have Fibonacci spirals... Pineapples do not have bilateral symmetry... No pineapple could possibly grow this way.[11]

Let us figure out the morphogenesis of pineapples so we can see what we can uncover about this mystery.[12]

Figure 1.18 shows a pineapple and a close-up of pineapple cells. Notice how the cells are very close to being regular hexagons. You will make a model of a pineapple to help you investigate the patterns found in pineapple spirals.

49. Take a sheet of hexagon graph paper. Roll it into a cylinder that you have skewed so the rows of hexagons do not go horizontally around the cylinder but rather form a spiral that goes up and down the whole cylinder as you follow it. (You may cut off the top and bottom corners so your overall object looks more like a cylindrical pineapple if you would like.)

50. Choose any hexagonal cell in your pineapple model. Put a dot at the center of this cell. Draw a straight line starting at the center of your chosen cell and traveling through the center of an edge of this same cell. Continue this straight line[13] all the way up to the top of the pineapple.

51. Return to the center of your starting cell and continue your line in the other direction, all the way to the bottom of the pineapple. This is your first spiral.

[11] Available online at `https://www.youtube.com/watch?v=gBxeju8dMho`.

[12] These investigations are adapted from "Fibonacci or Fairy Tale?" by Burkard Polster and Marty Ross, *Math Horizons*, February 2015.

[13] Although it makes a spiral that is curved in 3-dimensional space, in the geometry of the cylinder this is, in fact, a straight line. After all, there is a constant heading. Compare with great circles on a sphere.

Figure 1.17: Cactus flower; St. Croix.

52. There are many other spirals on your pineapple that start at the center of a different hexagonal cell and are parallel to your first spiral. Draw in several of these parallel spirals and determine the total number of spirals there are in this "first direction" in your pineapple model.

53. Return to the cell your first spiral started from. Using a different color pen/pencil/marker draw a straight line starting at the center of this cell and traveling through the center of a different edge of this same cell than the one that you used before.

54. There are many other spirals on your pineapple that start at the center of a different hexagonal cell and are parallel to this first spiral in the "second direction." Draw in several of these parallel spirals and determine the total number of spirals there are in this "second direction" in your pineapple model.

55. Return to the cell your first spiral started from. Using yet another different color pen/pencil/marker draw a straight line starting at the center of this cell and traveling through the center of a different edge of this same cell than the one that you used before.

56. There are many other spirals on your pineapple that start at the center of a different hexagonal cell and are parallel to this first spiral in the "third direction." Draw in several of these parallel spirals and determine the total number of spirals there are in this "third direction" in your pineapple model.

57. Are the number of spirals in each direction Fibonacci numbers as many websites claim? If not, are they similar to Fibonacci numbers?

58. Why do pineapples have three sets of spirals?

Figure 1.18: Pineapple and close-up of hexagonal cells in pineapple.

59. Team together with friends and construct several more pineapple models, making pineapples of different shapes by changing how much you skew the paper. Determine the number of spirals in each of the different directions. Are these numbers always Finbonacci or Fibonacci-like numbers?

60. We may not know why nature has chosen to make the cells in pineapples hexagons, but do you feel like you understand something about the morphogenesis of number patterns in the spirals in pinecones?

1.6 Chemical Morphogenesis: Animal Coat Patterns

I've got spots, I've got stripes too.[14]

Ani DiFranco (American Poet and Folksinger; 1970 -)

Figure 1.21 shows the coats and surface textures of several animals. A fundamental breakthrough in the understanding of these types of pattern formation was made in 1952 by **Alan Turing** (English mathematician and computer scientist; 1912 - 1954).

Turing's work on patterns was essential in a number of areas. Turing is widely considered to be the father of computer science and artificial intelligence. His *Turing test* remains a fundamental tool to evaluate whether a computer can "think." Turing's work with others at Bletchley Park allowed the Allies to break the German secret encryption codes generated by *enigma machines.* By breaking these secret communication codes, these non-combat contributions played an essential role in turing the tide of the war in the Atlantic in the favor of the Allies. While highly decorated

[14] From "In or out" on the album "Imperfectly."

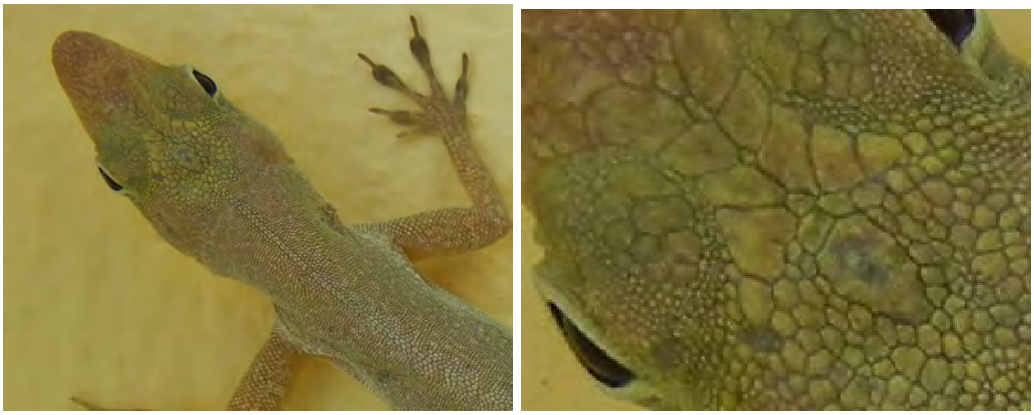


Figure 1.19: Anole and a close-up of her/his skin; St. Croix.

for his work after the war, Turing was later discovered to be homosexual. He was forced to undergo chemical "treatment," was put under house arrest and eventually killed himself by eating an apple laced with cyanide. **Peter Hilton** (; -) tells us,

> I.J. Good, a wartime colleague and friend, has aptly remarked that it is forunate that the authorities did not know during the war that [Alan] Turing was a homosexual; otherwise, the Allies might have lost the war.[15]

Returning to Turing's work in developmental biology, Turing's breakthrough was published as "The Chemical Basis of Morphogenisis,"[16] which used *reaction-diffusion equations* to model the pattern formation. Like his algorithms for machines to learn to play chess, computers appropriate to test these equations did not exist during Turing's life. However, as with chess-playing computers, we can now easily experiment with these models.

We will explore a simple, *two parameter reaction-diffusion model* using the applet at

`https://cgjennings.ca/articles/turing-morph.html`.

The success of these simple models can be seen in Figure 1.22 where the coat and surface patterns of the animals pictured in Figure 1.21 have been modeled.

Begin experimenting with this model by choosing different values for the diffusion constants, the presets and the number of ***iterations*** (steps the model takes). You may record any images you create electronically.[17]

61. Find several animals (different from those pictured above) whose coats or suface patterns you find interesting and you would like to try to model using diffusion-reaction models. (Note: You are not trying to model the geometric shape of structural surface features like the lattice of scales on a snake, these are formed differently.)

[15] "Cryptanalysis in World War II – and Mathematics Education," *Mathematics Teacher*, Oct. 1984.

[16] Philosophical Transactions of the Royal Society of London 237 (641): 3772; available online at `http://www.dna.caltech.edu/courses/cs191/paperscs191/turing.pdf`

[17] Use a computer screen capture or snapshot; "Print Screen" on a PC and Command-Control-Shift-3 on a Mac.

Figure 1.20: Palm tree; St. Croix.

62. For one of your animals, create their coat or surface pattern using the applet. Your final image should be quite close to the original in overall structure. While color is interesting, it is the structure of the pattern that is essential here. (Note: Your pattern should also be from a stable region of the model. If your image is rapidly changing as the model iterates this is not appropriate - animal's coat patterns do not change like this, chameleons notwithstanding.) Capture your image, the settings you used to make it, and identify clearly what animal coat or surface pattern it models.

63. Repeat Investigation **62** for a different animal.

64. Repeat Investigation **62** for a third different animal.

65. Are there some animals or types of coat/surface patterns that you had trouble making using this applet? If so, describe them.

66. Make a "multi-stage pattern," as described on the applet's site, to make a coat/surface pattern that is fundamentally different than others you have already made and which models some animal's coat/surface pattern.

Figure 1.21: A cheetah, a leopard, brain coral and hawksbill turtle.

1.7 Further Investigations

1.7.1 The Titius-Bode Law

The Titius-Bode Law is an empirically discovered law about planetary orbits. It is based on the pattern described numerically in the following table:

n	D
1	0.4
2	$0.4 + 0.3 \times 1$
3	$0.4 + 0.3 \times 2$
3	$0.4 + 0.3 \times 4$

Here the units for D are the **astronomical unit**, the average distance of the earth from the sun.

F1. Simplify each of the decimal values for D in the table.

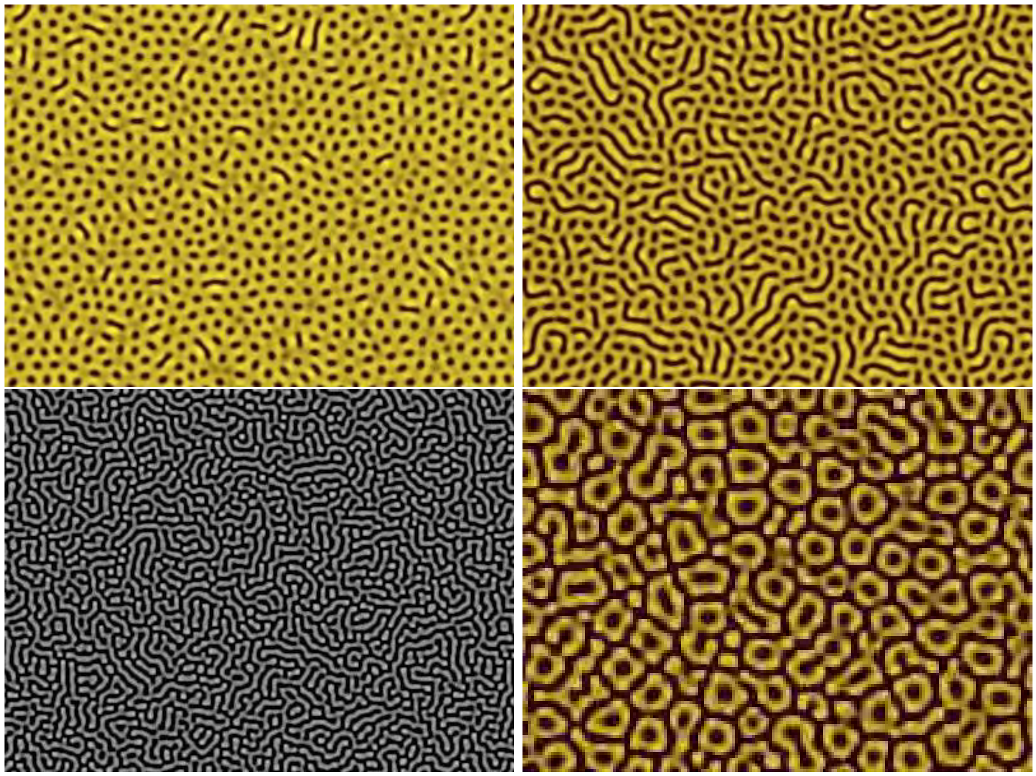


Figure 1.22: Turing reaction diffusion models generated from the applet available at `https://cgjennings.ca/articles/turing-morph.html`; Diffusion constants 3.5 & 16; 3.0 & 22; 1.0 & 22; ?????.

F2. Continue the table several more rows, following the evident pattern and simplifying the decimal values for D.

F3. Find the distances of each of the planets in our solar system from the sun in terms of the astronomical unit.

F4. How do these distances compare to those in the ***Titius-Bode pattern*** in the table above? (Note: At the time this pattern was described, only seven planets were known.)

F5. In 1801 the orbiting body named Ceres was discovered. Its average distance from the sun, correct to two decimal places, is 2.7 astronomical units. How do you think astronomers classified Ceres when it was discovered? How is it classified now?

1.7.2 The Discovery of Neptune

The planet Neptune was first seen on the night of September 23-24, 1846. Unlike the discovery of the other planets, Neptune was not discovered accidentally by having a telescope pointed in

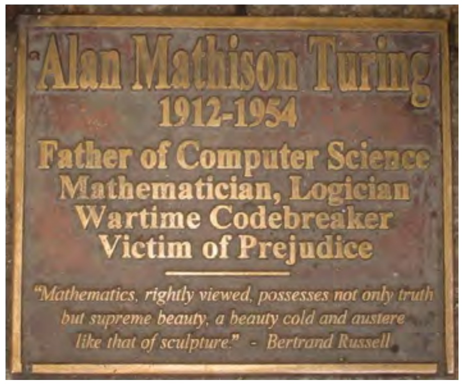


Figure 1.23: Plaque at the Alan Turing Memorial in Manchester, England.

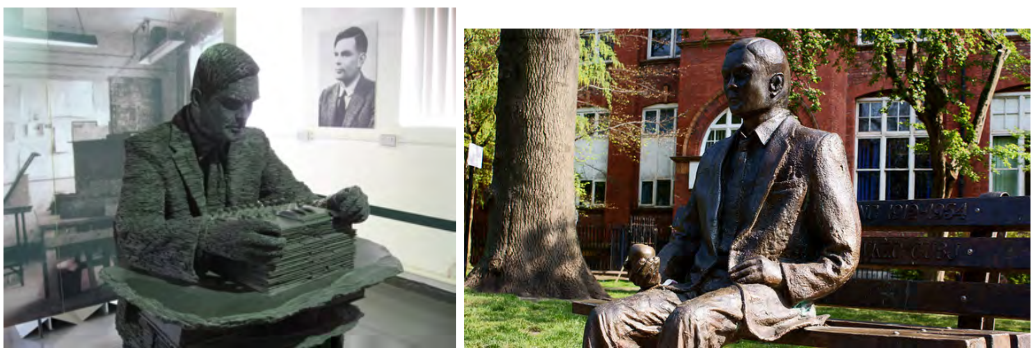


Figure 1.24: Statues of Alan Turing at Bletchley Park and at the Alan Turing Memorial in Manchester, England.

a lucky spot in the sky. Instead, it was discovered through a precise mathematical prediction of where it would be located. Go find out about the mathematical discovery of Neptune.

Chapter 2

Patterns and Proof

> In mathematics, if a pattern occurs, we can go on to ask, Why does it occur? What does it signify? And we can find answers to these questions. In fact, for every pattern that appears, a mathematician feels he ought to know why it appears.
>
> **W. W. Sawyer** (; -)

2.1 Patterns, Impostors and the Limits of Inductive Reasoning

This is a book about patterns. As several earlier quotations suggest, mathematics is very much the science of patterns. Patterns of various forms play essential roles in all of the books in this series, arising in art, dance, music, geometry, games, puzzles, reasoning, etc. While the Introduction suggested that there is not one universal definition of a pattern - we hope that you are becoming more adept at noticing patterns all around you.

And we hope that you are becoming more curious about how and why these patterns work; how their formation is born; the morphogenesis of these patterns.

2.1.1 Regions in a Circle

Figure 2.1 shows the first three stages in a pattern that arises from geometry - a pattern we will call ***Regions in a Circle***.

1. What pattern do you notice that determines what points and lines are added as we move from one stage of the pattern to the next?

2. Use this pattern to draw the next stage, i.e. the fourth stage, in this pattern.

3. Continue this pattern to draw the fifth stage in this pattern.

4. Into how many regions has the circle been decomposed in the first stage?

5. In the second stage? In the third stage?

6. Using the additional stages you have drawn, complete the following table:

Stage	Regions
1	2
2	
3	
4	

7. Do you see a clear pattern formed by the number of regions? Describe it precisely.

8. Use this pattern to predict the number of regions that are formed in the fifth stage in this pattern.

9. Now draw the fifth stage in this pattern and carefully count the number of regions that are formed.

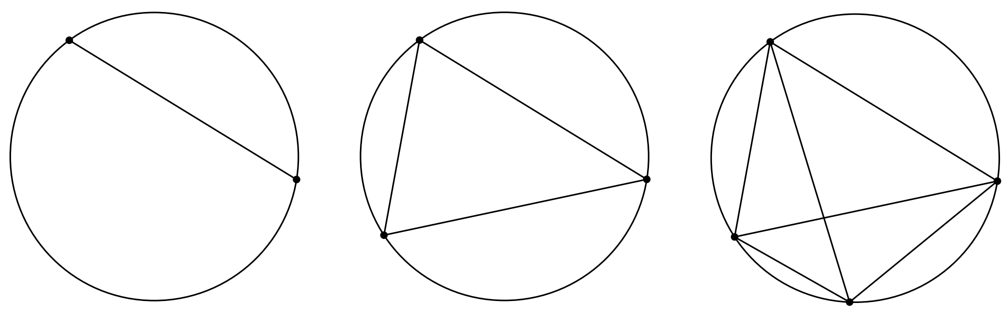


Figure 2.1: Stages 1, 2, and 3 of the Regions in a Circle pattern.

The pattern that you found in Investigation **7** is a beautiful, simple, and important numerical pattern, an examples of *exponential growth*. Unfortunately, as you have seen, this is pattern does *not* match the number of regions in a circle. The pattern fit the data for through the first five stages, but then failed. It is an impostor.

Why did this happen? It happened because human beings naturally use ***inductive reasoning*** - they draw general conclusions from limited evidence. We do this all of the time in our everyday experiences. We also do this in science when we apply the scientific method. If the consequences of our hypotheses fit observed data well enough in repeatable scientific experiments they become known as scientific theories.

2.1.2 Partitions

Let us consider another pattern. *Prime numbers* are important because they are the building blocks of all positive integers via multiplication. The number 42 can be uniquely *factored into primes* as $42 = 2 \times 3 \times 7$. But what if we want to build positive integers using addition? Well, $1 = 1$. And $2 = 2$ but also $2 = 1 + 1$. When we write a positive integer as the sum of positive integers these decompositions are called **partitions**. So the number 1 has only one partition while the number 2 has two partitions.

10. Find all of the different partitions of the number 3. (Note: we do not consider $2+1$ and $1+2$ to be different since they use exactly the same numbers.)

11. Use your previous answer to complete the following table:

Integer, n	Number of Partitions of n
1	1
2	2
3	

12. How many partitions do you think the number 4 will have?

You are making an educated guess in Investigation **12** based on evidence that you have collected. In science this is generally called a hypothesis. In mathematics the word typically used is ***conjecture***.

13. Find all of the partitions of 4.

14. Is you conjecture in Investigation **12** correct?

15. Use your previous answers to complete the following table:

Integer, n	Number of Partitions of n
1	1
2	2
3	
4	

16. How many partitions do you think the number 5 will have?

17. Find all of the partitions of 5.

18. Is your conjecture in Investigation **16** correct?

19. Use your previous answers to complete the following table:

Integer, n	Number of Partitions of n
1	1
2	2
3	
4	
5	

20. How many partitions do you think the number 6 will have?

21. Find all of the partitions of 6.

22. Is your conjecture in Investigation **20** correct?

23. Use your previous answers to complete the following table:

Integer, n	Number of Partitions of n
1	1
2	2
3	
4	
5	
6	

24. How many partitions do you think the number 7 will have?

25. Find all of the partitions of 7.

26. Is your conjecture in Investigation **24** correct?

27. Use your previous answers to complete the following table:

Integer, n	Number of Partitions of n
1	1
2	2
3	
4	
5	
6	
7	

28. How many partitions do you think the number 8 will have?

29. Find all of the partitions of 8.

30. Is your conjecture in Investigation **28** correct?

31. What do you think of inductive reasoning now? How well has it served you here?

Would you like to know the pattern? Here's an algebraic description - as the number n gets larger and larger, the number of partitions of n is better and better approximated by the function

$$\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}},$$

a result which was established by **G.H. Hardy** (English mathematician; 1877 - 1947) and **Srinivas Ramanujan** (Indian mathematician; 1887 - 1920) in 1918. The story behind the collaboration between these mathematicians is great and tragic. The self-taught Ramanujan was miraculously discovered by Hardy but then died at age 32 shortly after having been diagnosed with tuberculosis likely caused by his move from India to Cambridge to study with Hardy. Ramanujan's amazing discoveries of beautiful patterns of *partition congruences* has seen great rejuvenation in recent years with spectacular, surprising breakthroughs.[1]

Back to the pattern. Wow, is that formula complicated enough for you?

[1] For more on these beautiful and important patterns see Discovering the Art of Mathematics - Number Theory.

2.2 Deductive Reasoning

So inductive reasoning has limitations. The world is an inherently uncertain place. For most this is the only reality. But mathematicians yearn for a stronger burden of proof than the fickle inductive reasoning. And they can find it by relying on the use of *logic* to link *premises* to *conclusions* with certainty. This process is called ***deductive reasoning***.

2.2.1 An Existence Proof

32. Excluding people who are bald, do you think that there are (at least) two people on the Earth that have exactly the same number of hairs on their heads?

33. How hard would it be to find an example to guarantee that there were (at least) two such people?

This is a highly impractical task. If we could find two such people we would have an ***explicit proof***. But what if we just wanted to be guaranteed that there were examples, could we use logic to find an ***existence proof***?

34. About how many people are there in the world?

35. Making rudimentary approximations of head size, number of hairs per square inch, etc., can you say with certainty that people have less than 1,000,000 hairs on their heads?

36. Imagine putting signs up, one sign for every number from 1 through 1,000,000. Then imagine having every person in the world line up behind the sign that corresponds to the number of hairs that they have on their head. Could every line have fewer than 2 people? What does that tell you about our problem of finding two people with the same number of hairs on their heads?

Without leaving your chair, without counting hairs (except perhaps to get the upper limit on how many hairs one can have on their head) you have *proven* that there must exist two people on Earth that have the same number of hairs on their heads.

2.2.2 Mathematical Magic - The Human Calculator

In online video at `http://artofmathematics.org/media/video-382` one of the authors of this book is shown as a "Human Calculator." In this trick students volunteers and the mathemagician alternate picking 6-digit numbers until a total of eight have been picked. Then, in a flash, the mathemagician adds the column of eight numbers faster than students armed with calculators. Watch this trick, see if you can discover any patterns that might help you uncover how this trick works.

Here's another example of that same trick:

Student's Number	395014
Mathemagician's Number	604985
Student's Number	759231
Mathemagician's Number	240768
Student's Number	545310
Mathemagician's Number	454689
Student's Number	489132
Mathemagician's Number	200543
	3689672

37. Describe some observations you have made in trying to discover what the magician did to perform this feat of mathematical magic.

38. Find some relationships between these observations that may help discover how to perform this trick.

39. Determine and then describe how this trick is performed, explaining precisely what happens in each step of the trick.

40. Now you become the mathemagician, try this trick on several friends. How'd it go? Were they impressed?

Usually the audience response to this trick is something like "Wow, how'd s/he do that?" There is a real sense of wonder. The audience is curious about the secret that makes this trick work.

This is precisely how mathematicians feel when discover something new, curious, mysterious or magical.

So this trick is a good metaphor for the appeal of mathematics. It is also a good example for the process of mathematical discovery. So far you have figured out what the trick is, when it works, and where to make the appropriate steps. You have some procedural understanding that allows you to perform the trick and have likely conjectured that it will always work. This is generally what happens in the beginning stages of mathematical discovery. But now there is a fundamental second stage - determining why and how the trick works; to establish the trick works with certainty.

Mathematicians do not consider a problem solved until they have proof.

41. Under the conditions you described in Investigation **39**, prove that the trick will always work.

42. Do you have to use 6-digit numbers to do the trick? Does there have to be eight numbers chosen? If so, explain why. If not, describe a few ways the trick can be ***generalized***.

2.2.3 Pennies and Paperclips

Pennies and Paperclips[2] is a two player game played on a board resembling a checkerboard. A beginner board is shown in Figure 2.2. One player, "Penny", gets two pennies as her pieces. The other player, "Clip", gets a pile of paperclips as his pieces. Penny places her two pennies on any two different squares on the board. Once the pennies are placed, Clip attempts to cover the remainder of the board with paperclips - with each paperclip being required to cover two adjacent squares. Paperclips are not allowed to overlap. If the remainder of the board can be covered with paperclips then Clip is declared the winner. If the remainder of the board cannot be covered with paperclips then Penny is the winner.

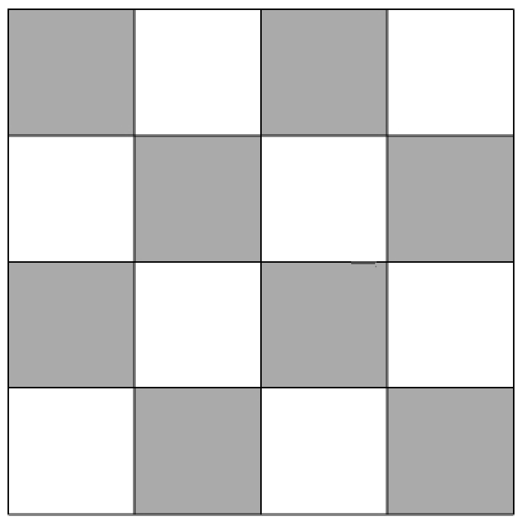


Figure 2.2: Beginner board for Pennies and Paperclips game.

43. With a partner, play this game several different times. Record the results of your games, including the placement of the pennies and paperclips, on the miniature boards in Figure 2.3.

44. Now switch the roles of pennies and paperclips and play several more games, again recording your results.

45. Do you notice any patterns that will enable you to find winning strategies for the players? If so, test them by playing a few more games. If not, continue to play until some pattern appears.

[2]The exact history of this game is not know to the authors. It appears in the 1994 version of Harold R. Jacobs' Mathematics: A Human Endeavor. In the 1970 version of this book the "mutilated checkerboard" appears instead. The latter is from the 1950's and its noteriety is due to Martin Gardner's *Scientific American* columns. It may well be that the translation of the checkerboard problem into a game was done by Jacobs as he created a new edition of his book.

46. State a conjecture which determines precisely when pennies win based on their placement.

47. Prove this conjecture.

48. State a conjecture which determines precisely when paperclips win based on the placement of the pennies.

49. Prove this conjecture.

50. Are there any situations in which neither player wins, or have you *characterized* all possible outcomes? Explain.

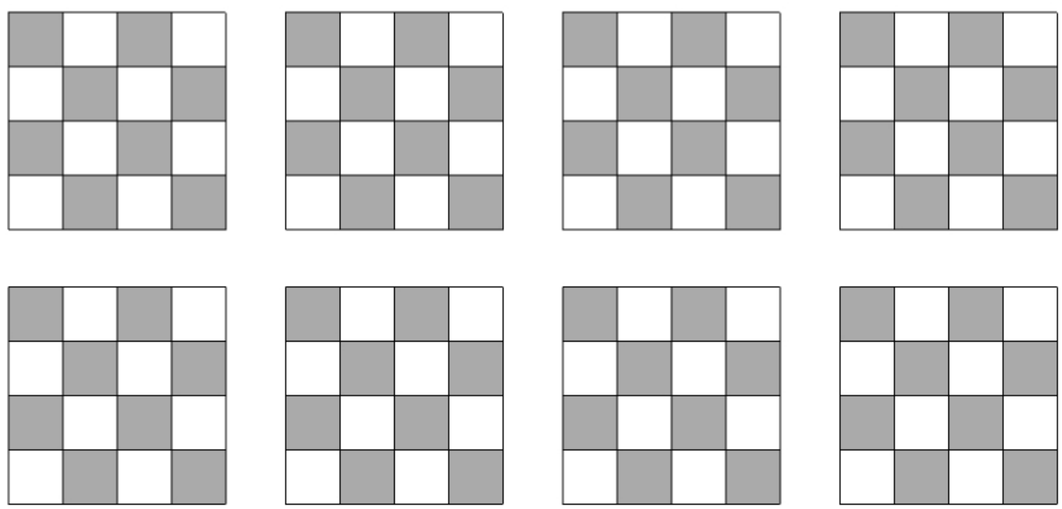


Figure 2.3: Boards for recording Pennies and Paperclips games.

It is important to remember that a proof is a logically complete demonstration that a result must hold. The proof most people originally give for Investigation **49** is merely a counting argument.

51. Play pennies and paperclips a few times on the distorted board in Figure 2.4.

52. Does your conjecture in Investigation **48** hold for this board?

53. Return to your proof in Investigation **49**. Does your proof fundamentally use the geometry of the board or is it simply a counting argument? The distorted board generally shows that the "proofs" in Investigation **49** are not complete.

54. If your proof in Investigation **49** is not complete, the ***Hamiltonian circuit*** shown in Figure 2.5 may help you. This circuit visits every square on the checkerboard and then returns to its starting point. Might it help provide locations for paperclips? Find a correct proof for your conjecture in Investigation **48** using this or some other approach.

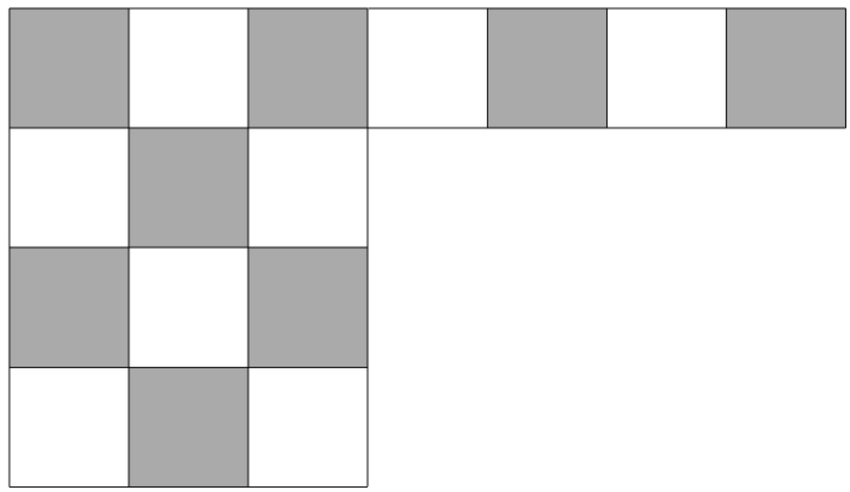


Figure 2.4: Distorted beginner board for Pennies and Paperclips game.

The use of a Hamiltonian path for this problem was first discovered by **Ralph E. Gomory** (American mathematician and executive; 1929 -) in 1973. Working with 8 by 8 checkerboards, mathematicians had sought a clear proof and Gomory's solution to this widely-known problem was so celebrated that it is called ***Gomory's theorem***.

The board you have been playing on was called the "beginner" board. Mathematicians love to generalize. It is a very natural mathematical question to ask whether this game can be extended to other sized boards. See the Further Investigations section for more details.

2.3 What is a Proof?

Proof is an idol before which the mathematician tortures himself.

Sir Arthur Eddington (British astronomer, physicist, mathematician and philosopher of science; 1882 - 1944)

A good proof is one that makes us wiser.

Yuri I. Manin (Russian mathematician; 1937 -)

A elegantly executed proof is a poem in all but the form in which it is written.

Morris Kline (American mathematician and educator; 1908 - 1992)

Proof is fundamental to what defines mathematics. "Proof" is a word that is used commonly and our everyday usage of the term is similar to mathematicians'. However, when used in mathematical settings there are fundamental and deep distinctions from the colloquial usage. Understanding these distinctions are critical to understanding what mathematics truly is. For it is these distinctions which differentiate mathematics from all other fields of knowledge.

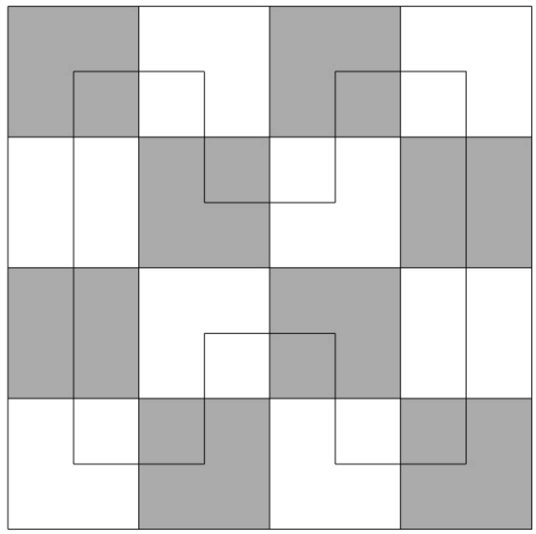


Figure 2.5: Hamiltonian path in a 4 by 4 checkerboard.

Ironically, the nature of proof itself has been the source of deep philosophical debate in mathematics through much of its history. In The Mathematical Experience **Philip J. Davis** (American mathematician; 1923 -) and **Reuben Hersh** (American mathematician; 1927 -) paint a portrait of an ideal mathematician.

> [The ideal mathematician] rests his faith on rigorous proof; he believes that the difference between a correct proof and an incorrect one is an unmistakable and decisive difference. He can think of no condemnation more damning than to say of a student, "He doesn't even know what a proof is." Yet he is able to give no coherent explanation of what is meant by rigor, or what is required to make a proof rigorous. In his own work, the line between complete and incomplete proof is always somewhat fuzzy, and often controversial.

The inability of the ideal mathematician to articulate what is meant by proof is then reinforced in wonderful dialogues between our ideal mathematician and a public information officer, a student, a philosopher, and a classicist.

Since there is no simple mathematical definition of "proof", to help you gain a working understanding we will continue to approach it here incrementally with examples, investigations, and contexts.

In his compelling book A Mathematician's Lament: How School Cheats Us Out of Our Most Fascinating and Imaginative Art Form, **Paul Lockhart** (; -) says this about the central role of proof:

> Math is not about a collection of "truths" (however useful or interesting they may be). Math is about reason and understanding. We want to know *why.* And *not* for any

practical purpose. Here's where the art has to happen. Observation and discovery are one thing, but *explanation* is quite another. What we need is a *proof*, a narrative of some kind that helps us to understand why this pattern is occurring. And the standards for proof in mathematics are pretty damn high. A mathematical proof should be an absolutely clear logical deduction, which, as I said before, needs not only to satisfy, but to satisfy beautifully. That is the goal of the mathematician: to explain in the simplest, most elegant and logically satisfying way possible. To make the mystery melt away and to reveal a simple, crystalline truth.[3]

One context to help understand the distinction between inductive and deductive reasoning is to compare corresponding terms that are used in the different areas:

Inductive Reasoning		Deductive Reasoning
Empirical Evidence	$\leftrightarrow$	Proof
Fits data	$\leftrightarrow$	Matches Cause
Hypothesis	$\leftrightarrow$	Fact
Conjecture	$\leftrightarrow$	Theorem
Expectation	$\leftrightarrow$	Logical Conclusion
Reasonable Certainty	$\leftrightarrow$	Absolute Certainty

2.4 Proof - Pattern Morphogenesis

Proofs aren't there to convince you that something is true - they're there to show you why it is true.

Andrew Gleason (American mathematician; 1921 - 2008)

The quotation by W.W. Sawyer that opens this chapter is so important, we repeat it here:

In mathematics, if a pattern occurs, we can go on to ask, Why does it occur? What does it signify? And we can find answers to these questions. In fact, for every pattern that appears, a mathematician feels he ought to know why it appears.

We have spoken about pattern morphogenesis as one of the themes for this book. At the heart of this is understanding the underlying mechanisms that give rise to the pattern. This is precisely what proof is about - understanding why.

Here are a few examples of how dedicated mathematicians are to this search for the true heart of the pattern; what makes it arise; it's morphogenesis.

2.4.1 The Four Color Theorem

Typically maps are colored so countries sharing borders are colored different colors. In this way it is easy to see the geographical extent of the country. If you gather several uncolored maps and some crayons, by experimenting you will quickly see that four different colored crayons seem to be enough. A fifth color is never required if you pick colors with forethought. On October 23, 1852 **Francis Guthrie** (South African mathematician and botanist; 1831 - 1899) conjectured that four colors were sufficient to color *any* map. He found no proof. For 120 years maps were colored

[3] From A Mathematician's Lament, pp. 110-1.

all over the world and not one single example was found that required more than four colors. Despite this enormous volume of empirical evidence that Guthrie's conjecture was valid, mathematicians continued to search for a proof. In 1976 **Kenneth Appel** (American mathematician; 1932 - 2013) and **Wolgang Haken** (German mathematician; 1928 -) announced that they had a proof. However, their proof relied on a computer and there still remains some hesitancy about the appropriateness of this proof among mathematicians.

2.4.2 Goldbach's Conjecture

On June 7, 1742 **Christian Goldbach** (German mathematician; 1690 - 1764) wrote a letter to **Leonhard Euler** (Swiss mathematician; 1707 - 1783) making the conjecture that every even number greater 4 can be written as the sum of two primes. The beginning cases are easy to check:

$$\begin{aligned} 6 &= 3+3 \\ 8 &= 3+5 \\ 10 &= 5+5 \\ 12 &= 5+7 \\ 14 &= 7+7 \\ 16 &= 3+13 \\ 18 &= 7+11 \end{aligned}$$

It would only take the discovery of one even number that cannot be expressed as the sum of two primes to destroy this conjecture - to break this pattern. Such an example, if it existed, would be called a ***counter-example***. In over 260 years none have been found. Computers have been enlisted to help check for counter-examples and it has been shown that all even numbers through 4,000,000,000,000,000,000 can be expressed as the sum of two primes. But it may not be possible for the very next number that has yet to be checked. And there are infinitely many beyond that which need to be checked as well. So mathematicians' search for proof continues.

2.4.3 The Riemann Hypothesis

The Riemann hypothesis is one of the most important unsolved problems in all of mathematics. Currently a 1$ Million dollar prize for its solution is being offered - it is one of the seven ***Millennium Prize Problems***, each which is worth 1$ Million. Essential to our understanding the behavior of the prime numbers, the **Riemann hypothesis** states that all non-trivial roots of the *Riemann zeta function* lie on the line $x = \frac{1}{2}$. If someone finds just *one* non-trivial root of the Riemann zeta function that is not on this line they will have disproven the Riemann hypothesis, becoming 1$ Million richer and having shocked the mathematical community to its core. Posed by **Bernhard Riemann** (German mathematician; 1826 - 1866) in 1859, mathematicians feel a long way from having hope that a proof is forthcoming. Dedicated to exploring roots of the zeta function, the distributed computing project ZetaGrid was the largest in the world in the early 2000's. Over one billion roots were checked each day and over 10,000,000,000,000 roots were found - all of them lying expectedly along the critical line. The Riemann hypothesis and the *Birch and*

Swinnerton-Dyer conjecture, another of the Millennium Prize Problems, are the focus of a chapter in Discovering the Art of Mathematics - Number Theory.

2.4.4 Fermat's Last Theorem

You likely recognize $3^3 + 4^2 = 5^2$ as an application of the Pythagorean theorem. (See Section 2.4.8 for more details.) In 1637 **Pierre de Fermat** (French lawyer and mathematician; 1601 - 1665) stated that there were no analogues for higher exponents. In other words, Fermat stated that there are no nontrivial integers a, b, c such that $a^n + b^n = c^n$ whenever $n \geq 3$. Remarkably, with the written statement of his conjecture - in the margins of a mathematics book he was reading - he wrote "I have discovered a truly marvelous proof of this, which this margin is too narrow to contain." Nowhere in his papers or other work was this proof found. And so mathematicians searched for their own. They searched 100 years and found no proof. So they search 100 years more and found no general proof. So they searched 100 more. After 350 years some mathematicians had given up, thinking it impossible. But this is when one mathematician secretly took up the charge. On June 23, 1993 **Andrew Wiles** (British mathematician; 1953 -) completed the third of his hour-long lectures at the Isaac Newton Institute for Mathematical Sciences in Cambridge, England. These lectures described his secret work over the past six years, his wife the only person who knew that he had dedicated his professional career to just this one problem. The conclusion of his third lecture - he had proven **Fermat's last theorem**! This tremendous story is explored in detail in Discovering the Art of Mathematics - Number Theory. The story is also beautifully told in the Nova documentary The Proof.

These are some of the remarkable patterns - and mathematicians' search for their morphogenesis - that occupy contemporary mathematics.

Let us return to some take less than centuries to successfully explore.

2.4.5 Finger Man

> We are all concerned about the future of American education. But as I tell my students, you do not enter the future – you create the future. The future is created through hard work.
>
> **Jaime Escalante** (Bolivian Teacher; 1930 - 2010)

Jaime Escalante (Bolivian Teacher; 1930 - 2010) is a well-known and inspiring teacher of mathematics whose work in the East Los Angeles schools received significant attention, most notably in the movie Stand and Deliver. In one scene, Escalante shows a confrontational student how to be a "finger man", computing the multiples of 9 on his fingers.

55. Watch this clip to learn this *algorithm* for computing multiples of 9.[4]

56. Describe this algorithm completely, using precise language that will allow someone who has never seen it to compute multiples of 9 using the algorithm.

57. *Prove* that this algorithm correctly computes each of the first nine multiples of 9.

[4] It appears at minutes 8:30 - 10:00 in the original movie and can often be found online by searching for videos with keywords "Stand and Deliver Finger Man."

Chisenbop is a powerful, twentieth century finger-counting algorithm developed in Korea which for computing multiples of 9 is identical to the method used above. Useful finger-counting methods have existed in many cultures through much of history, as long ago as the Vedic cultures of India over 3,000 years ago.

Figure 2.6: Jaime Escalante

2.4.6 Casting Out Nines

In fact, the number 9 has many remarkable properties in our base-ten number system.

> INDEPENDENT INVESTIGATION - REMAINDER BY NINES PROBLEM
> Find an efficient way to determine what the remainder will be when a whole number is divided by 9.

This is the type of problem that mathematicians typically deal with. There is no obvious answer, no straightforward way to work it out. Instead, it requires experimentation, data collection, intuition, the search for connections to other problems, insight, and the discovery of patterns. All of this is active work. Please get to work on it.

A typical timeline for a problem like this is that it should take an hour or so of active, focussed work. But this work need not come all at once, it can happen over several different sittings - with breaks in between to reflect on the problem a bit less actively, to do something totally different so your subconscious can work on it, or just to take a break.

Note: You should *not* look for a solution from an external resource - book, Internet, or peer. Real mathematical experiences are about problem solving, constructing understanding, developing sense-making and practicing the creation of ideas. You will not have an authentic mathematical experience, nor will you be able to succeed in the remainder of the explorations in this book if you rely on others for all of your significant work. You may not ever need to use this problem again, but strengthening your abilities to problem solve, reason, understand how/why things work, and make sense of new objects, ideas, and settings are critical to real educational growth.

58. Precisely describe how you went about solving this problem.

59. Precisely describe your solution to the Remainder by Nines problem. (Note: Your solution should efficiently enable you to determine the remainder when *any* whole number is divided by 9. If you cannot do this for every number, you only have a partial solution thus far.)

60. Are you confident that your solution is legitimate? Explain what it is that makes you confident.

At present, your result in Investigation **59** is a conjecture, a statement you believe to be true. While you may be confident in your result, it needs to be proven. If proven successfully, your result will no longer be considered a conjecture, but rather a **theorem**, that is, a mathematical certainty which has been established through logical deduction and is therefore eternal.

61. Prove that your solution to the Remainder by Nines problem is correct.

62. There is a rule for determining whether a whole number is divisible by 9. You should be able to determine this rule from your solution to the Remainder by Nines problem. Describe the rule precisely.

63. Explain why this rule follows logically as a direct result of your solution to the Remainder by Nines problem.

When a mathematical Your result in Investigation **62** is called a ***corollary*** because it is a theorem which is established as an immediate logical consequence of your solution to the Remainder by Nines problem.

2.4.7 The $3a + 5b$ Problem - An Infinite Result

It is important to underscore what your Remainder by Nines proof achieved - you established with absolute certainty a result about infinitely many numbers. This is impressive. Here is another result which involves proving something about infinitely many objects.

Consider the expression $3a + 5b$ where $a, b \geq 0$ represent whole numbers.

64. Choose whole number values for $a, b \geq 0$. For these values, evaluate the expression $3a + 5b$.

65. Repeat Investigation **64** for a different pair of values for a, b.

66. Repeat Investigation **64** for another dozen different pairs of values for a, b.

67. Is every positive whole number a possible output of the expression $3a+5b$ where $a, b \geq 0$ are whole numbers? Or are there some numbers that cannot be generated by this expression?

68. Continue investigating until you feel comfortable making a conjecture which identifies exactly which, all infinitely many of them, positive whole numbers are generated by the expression $3a + 5b$ where $a, b \geq 0$ are whole numbers.

69. Prove your conjecture.

We call this problem the **$3a + 5b$ Problem**. Changing the coefficients from 3 and 5 to other integers gives rise to infinitely many other interesting problems of this type. And, thus, an opportunity to find beautiful, infinite patterns of patterns in the search for a general solution. For more see Further Investigation **F2.7.2**.

2.4.8 Pythagorean Theorem

The *Pythagorean theorem* is one of the most well-known results in all of mathematics. It is a beautiful pattern which describes precisely what types of triangles have the special property that the squares built upon their legs have areas which add to the area of the square built on the hypotenuse. While this theorem bears the name of **Pythagoras** (Greek mathematician and philosopher; c. 570 BC - c. 495 BC), it was known to many other cultures. There are *many* proofs of the Pythagorean theorem. **Elisha Scott Lewis** (; -) wrote The Pythagorean Proposition which is a collection of 367 proofs of the Pythagorean theorem! In 1867, when he was a U.S. Representative, future President **James A. Garfield** (American politician; 1831 - 1881) created a proof of the Pythagorean theorem that was published in the *New England Journal of Education*.

Here you will re-discovery a proof attributed to **Bhaskara** (Indian mathematician and astronomer; 1114 - 1185).

70. State the Pythagorean theorem. Be sure to include any assumptions that the theorem requires.

71. The proof uses paper triangles that you should create as follows:

- Fold an $8\frac{1}{2}$ by 11 sheet of paper in half in one direction and then in half again in the opposite direction.
- From the corner which is not along any of the fold lines, measure some distance up along one edge and the same distance along the other edge.
- Draw a line between the two points you have just measured, creating a right, isoceles triangle.
- Cut along this line to create four congruent copies of your triangle.
- Label the legs a and b and label the hypotenuse c.

72. Can you arrange your four triangles so they form the interior of a square whose area is c^2?

73. Now rearrange your four triangles to form two smaller squares.

74. Explain how this proves the Pythagorean theorem for your triangle.

75. Will this same proof work for any right, isoceles triangle? Explain.

Now adapt this method for a general right triangle as follows.

76. Repeat the folding, cutting and labelling process used above to create four congruent, right triangles that are *not* isoceles.

77. Arrange your four triangles so they form the interior of a square whose area is c^2 but which is missing a piece in its center.

78. With the paper scraps left over from your initial construction, make a piece that enables you to fill in the missing component of your c by c square.

79. Now rearrange your triangles and additional piece so they form a single shape that is equal to two smaller squares.

80. In both of the previous steps you should justify how you know that the shapes you have created are indeed squares and have the appropriate dimensions.

81. Explain how this proves the Pythagorean theorem for your triangle.

82. Will this same proof work for any right triangle? Explain.

2.5 Where Do Proofs Come From?

> But how do we do it [discover a proof]? Nobody really knows. You just try and fail and get frustrated and hope for inspiration. For me it's an adventure, a journey. I usually know more or less where I want to go, I just don't know how to get there. The only thing I *do* know is that I'm not going to get there without a lot of pain and frustration and crumpled-up paper... Then, all of a sudden, in one breathless heart-stopping moment, the clouds part and you can finally *see*... The thing I want you especially to understand is this feeling of divine revelation. I feel that this structure was "out there" all along; I just couldn't see it. And now I can! This is really what keeps me in the math game - the chance that I might glimpse some kind of secret underlying truth, some sort of message from the gods.[5]
>
> **Paul Lockhart** (; -)

Where do proofs come from? For those that think of mathematics as a static, procedural subject the typical conception is that the results of mathematics come from some long-dead, God-like authority. But in fact, mathematics is a lively, human subject which is practiced rather like an art. G.H. Hardy tells us:

> A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

Where do these ideas come from? Who knows, from where does other artistic inspiration spring? Can a musician precisely describe the nature of musical composition? An artist their great visions? Novelists the genesis of their works' climaxes? Mathematics is no different.

Paul Halmos (American mathematician; 1916 - 2006) tells us:

> Mathematics - this may surprise or shock some - is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions. He arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof... The deductive stage, writing the result down, and writing its rigorous proof are relatively trivial once the real insight arrives; it is more like the draftsman's work not the architect's.

[5] From A Mathematician's Lament, pp. 113-4.

You worked hard on the problems above. No doubt some of the proofs came easier to you than others. If different proofs were shared with your fellow explorers, you likely saw entirely different approaches than your own. Some are more beautiful, some more straightforward, and some more insightful. Each bears the marks of its creators' efforts and each is appreciated differently by different people.

But each sprung from an effort to make sense of a problem. They came a personal challenge to understand, to crack the mystery, to uncover the secret that explains the inner workings of the problem. Mathematics classes where students are taught to mimic algorithms to repeatedly solve essentially identical problems bear no resemblance to the real work of mathematicians. The only rules that constrain mathematicians are those imposed by their own creations. Mathematicians at work dream of connections, imagine relationships, search for patterns, reinterpret to discover structure, and create entirely new combinations of ideas in order to build bridges of understanding. It is deeply creative work.

> Now where did this idea of mine come from? How did I know to draw that line? How does the painter know where to put his brush? Inspiration, experience, trial and error, dumb luck. That's the art of it, creating these beautiful little poems of thought, these sonnets of pure reason. There is something so wonderfully transformational about this art form. The relationship between the triangle and the rectangle was a mystery, and then that one little line made it obvious. I couldn't see, and then all of a sudden I could. Somehow, I was able to create a profound simple beauty out of nothing, and change myself in the process. Isn't that what art is all about?[6]
>
> **Paul Lockhart** (; -)

2.6 Connections

2.6.1 Stand and Deliver

Stand and Deliver is based on the true-life story of Escalante's notable accomplishments in helping Latino students reach remarkable levels of mathematical success at the East Los Angeles Garfield High School. It recounts the racism and profiling that he and his students faced, including from the Educational Testing Service who accused the students of cheating on the Advanced Placement Calculus exams. Find our more about the accomplishments of Escalante and his students. What was the critical reaction to the movie?

2.6.2 Arithmetic Error Checking

The number nine has many useful properties. One related to the early investigations in this chapter is *Casting Out Nines* which is a useful algorithm for checking errors in arithmetic that used to be widely taught in the United States and is encountered in many other cultures. E.g. it is called ***prova dos noves***, which translates as "proof of nines", in Spanish despite the fact that it cannot be used to prove that the results of an arithmetic calculation is correct. Find out how this method works.

[6] From A Mathematician's Lament, pp. 27.

2.7 Further Investigations

2.7.1 Generalized Pennies and Paperclips

F1. Draw a 5 by 5 checkerboard. Play Pennies and Paperclips on this board. Is this an interesting game?

F2. Find a way to adapt the rules of the game so that meaningful games can be played on 5 by 5 boards.

F3. Do the theorems you proved for the 4 by 4 boards have analogues in this adapted game for the 5 by 5 board?

F4. Draw a 6 by 6 checkerboard. Play Pennies and Paperclips on this board. Is this an interesting game?

F5. State and prove conjectures that give winning strategies for both pennies and paperclips on a 6 by 6 board.

F6. What do you think will happen on 7 by 7 boards? 9 by 9 boards? 11 by 11 boards?

F7. What about 8 by 8 boards? 10 by 10 boards? 12 by 12 boards?

F8. State and prove conjectures that give winning strategies for both pennies and paperclips on *all* even square checkerboards.

The basic ideas upon which successful proofs of winning strategies are based, the parity of the white/black spaces and Hamiltonian circuits, are remarkably robust - able to establish results on boards of all sorts of shapes beyond squares. For more see the paper "Tiling with Dominoes" by N.S. Mendelsohn, *The College Mathematics Journal*, vol. 35, no. 2, March 2004, pp. 115-120.

2.7.2 Generalizing the $3a + 5b$ Problem

There is no particular reason that the equation $3a + 5b$ was chosen. There are infinitely many other *linear equations* of this type that could have been chosen. And each gives rise to its own beautiful patterns.

Once you have discovered many of these patterns you have an opportunity to answer the wonderful question:

Is there a pattern that unites all of these different patterns together?

F9. After collecting data and finding patterns make and then prove exactly which, all infinitely many of them, positive whole numbers are generated by the expression $3a+7b$ where $a, b \geq 0$ are whole numbers.

F10. After collecting data and finding patterns make and then prove exactly which, all infinitely many of them, positive whole numbers are generated by the expression $5a+11b$ where $a, b \geq 0$ are whole numbers.

F11. After collecting data and finding patterns make and then prove exactly which, all infinitely many of them, positive whole numbers are generated by the expression $7a+19b$ where $a, b \geq 0$ are whole numbers.

F12. After collecting data and finding patterns make and then prove exactly which, all infinitely many of them, positive whole numbers are generated by the expression $3a+6b$ where $a, b \geq 0$ are whole numbers.

F13. After collecting data and finding patterns make and then prove exactly which, all infinitely many of them, positive whole numbers are generated by the expression $3a+12b$ where $a, b \geq 0$ are whole numbers.

Can we find patterns among these patterns as the *coefficients* of our linear equation change? Let us describe the general equation as $ma + nb$.

F14. Under what conditions on m and n will there be infinitely many numbers that are not obtained as outputs of $ma + nb$?

F15. In this situation, can you simplify the problem so you can relate it to a different $ma + nb$ family?

F16. For those values of m and n where there are only finitely many numbers that are not obtained as outputs of $ma + nb$, determine exactly how many numbers are not obtained.

F17. In this setting, can you find a pattern that predicts how many numbers are not obtained as a function of m and n? Explain.

F18. In this setting, can you predict what the largest number that is not obtained is as a function of m and n?

F19. Can you prove these results?

Chapter 3

Patterns of the Day

One way to build understanding of the ubiquity of patterns is to begin each class with a "Pattern of the Day." In this chapter we provide a list of patterns that we have found approrpriate for this purpose.

In each there is an indicated pattern and the task is to:

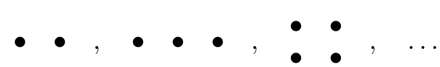
- Guess the next few stages in the pattern.
- Understand why the pattern appears and/or continues.
- Find some context, application, or motivation for the pattern.

Notice how closely these tasks follow the maxim of Sawyer:

> In mathematics, if a pattern occurs, we can go on to ask, Why does it occur? What does it signify? And we can find answers to these questions. In fact, for every pattern that appears, a mathematician feels he ought to know why it appears.
>
> **W.W. Sawyer** (; -)

Occasionally there are investigations that follow the pattern. These involve specific prompts that are useful beyond the typical tasks just described.

3.1 A Prime Pattern



1. Now find a totally different way to continue this pattern.

2. And find another different way to continue this pattern.

3. Exchange your patterns with a peer. Can you understand the patterns the other has found?

4. What is the correct answer?

3.2 Are you positive?

$$\begin{aligned} 5 \times 3 &= 15 \\ 5 \times 2 &= 10 \\ 5 \times 1 &= 5 \\ 5 \times 0 &= 0 \\ \vdots &= \vdots \end{aligned}$$

5. Does this help you to see why the product of a positive and a negative is a negative? Explain.

6. With some rationale for why the product of a positive and a negative is a negative, can you find a pattern which uses this fact to show why the product of two negatives is a positive? Either do so or explain why you cannot.

3.3 PatternProducts

$$\begin{aligned} 1 \times 1 &= 11 \\ 11 \times 11 &= 121 \\ 111 \times 111 &= 12321 \\ 1111 \times 1111 &= 1234321 \\ 11111 \times 11111 &= 123454321 \\ 111111 \times 111111 &= 12345654321 \\ 1111111 \times 1111111 &= 1234567654321 \\ 11111111 \times 11111111 &= 123456787654321 \\ 111111111 \times 111111111 &= 12345678987654321 \end{aligned}$$

$$\begin{aligned} 1 \times 1 + 8 &= 9 \\ 12 \times 2 + 8 &= 98 \\ 123 \times 3 + 8 &= 987 \\ 1234 \times 4 + 8 &= 9876 \\ 12345 \times 5 + 8 &= 98765 \\ 123456 \times 6 + 8 &= 987654 \\ 1234567 \times 7 + 8 &= 9876543 \\ 12345678 \times 8 + 8 &= 98765432 \\ 123456789 \times 9 + 8 &= 987654321 \end{aligned}$$

$$\begin{aligned}
1 \times 9 + 2 &= 11\\
12 \times 9 + 3 &= 111\\
123 \times 9 + 4 &= 1111\\
1234 \times 9 + 5 &= 11111\\
12345 \times 9 + 6 &= 111111\\
123456 \times 9 + 7 &= 1111111\\
1234567 \times 9 + 8 &= 11111111\\
12345678 \times 9 + 9 &= 111111111\\
123456789 \times 9 + 10 &= 111111111
\end{aligned}$$

$$\begin{aligned}
9 \times 9 + 7 &= 88\\
98 \times 9 + 6 &= 88\\
987 \times 9 + 5 &= 88\\
9876 \times 9 + 4 &= 88\\
98765 \times 9 + 3 &= 88\\
987654 \times 9 + 2 &= 88\\
9876543 \times 9 + 1 &= 88\\
98765432 \times 9 + 0 &= 88
\end{aligned}$$

3.4 Arithmetical Savants

TFSSTWTFSMTWFSSMWTFS . . .

There are great stories of savants (??ok word) with tremendous arithmetical abilities and tremendous memories.

Twins brief story.

Born on a Blue Day and pi and Swedish language and being able to describe how he does arithmetic in terms of art.

We still have very little idea how these people are so brilliant in certain ways and so limited in others.

One amazing thing that savants can do is tell you the day of the week...

7. Why do the days of the week follow the pattern that they do from one year to another?? (Reword this?? How to hide this?)

3.5 Continued Fractions

$$
\begin{aligned}
1+\frac{1}{1} &= ? \\
1+\cfrac{1}{1+\frac{1}{1}} &= ? \\
1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}} &= ? \\
1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}} &= ? \\
\vdots &= \vdots
\end{aligned}
$$

For much more see "The Golden Ratio" from Discovering the Art of Mathematics - Number Theory.

It is remarkable to note that

$$\sqrt{2} = 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}.$$

3.6 Imaginary Numbers

The **imaginary unit**, upon which the *complex number field* is built, is defined by

$$i = \sqrt{-1}.$$

$$
\begin{aligned}
i^1 &= i \\
i^2 &= ? \\
i^3 &= ? \\
i^4 &= ? \\
i^5 &= ? \\
\vdots &= \vdots
\end{aligned}
$$

8. So what simplified value represents i^{137}?

9. What about $i^{3,987,243}$?

For much more see "Existence of $\sqrt{-1}$" in Discovering the Art of Mathematics - Reasoning, Proof, Certainty and Truth.

3.7 Galileo Sums

$$\begin{aligned} 1 &= 1 \\ 1+3 &=? \\ 1+3+5 &=? \\ 1+3+5+7 &=? \\ \vdots &= \vdots \\ 1+3+5+7+\ldots+\boxed{\text{algebraic expression}} &= \boxed{\text{algebraic expression}} \\ \vdots &= \vdots \end{aligned}$$

10. Using square tiles or blocks, or graph paper to mimic them, can you arrange one tile of one color and three tiles of another color into an appropriate shape to illustrate the validity of the first equation?

11. Using one square tile of one color, three of another color and five of yet another color, can you arrange these tiles into an appropriate shape to illustrate the validity of the second equation?

12. Show how this process can be continued to provide a *Proof Without Words* for the general result.

This example has critical historical significance. In his inclined plane experiments, **Galileo** (; -) showed successfully that the distance a free falling body falls in successive time intervals is in the ratio of $1:3:5:7:9\cdots$. Over equal time intervals, summing these distances shows that the total distance is proportional to the square of the time.[1] I.e. this pattern is encoded in basic uniform motion.

3.8 Galileo Fractions

$$\begin{aligned} \frac{1}{3} &= \frac{1}{3} \\ \frac{1+3}{5+7} &=? \\ \frac{1+3+5}{7+9+11} &=? \\ \vdots &= \vdots \end{aligned}$$

A beautiful Proof Without Words for Galileo fractions can be developed from the following figure[2]:

[1]See Chapter 1 - "Galileo and Why Things Move" from The Ten Most Beautiful Experiments by George Johnson for a secondary description, "Galileo's Experimental Confirmation of Horizontal Inertia" by Stillwell Drake, *Isis*, vol. 64, no. 3, pp. 290-305 for a primary source description, and Section 4.2 - "The Studies of Galileo" in Basic Calculus: From Archimedes to Newton to Its Role in Science by Alexander J. Hahn for an introductory physics/calclus level description.

[2]From "Proof Without Words: Galileo's Ratios Revisited" by Alfino Flores and Hugh A. Sanders, *College Mathematics Journal*, vol. 36, no. 3, May 2005, p. 198.

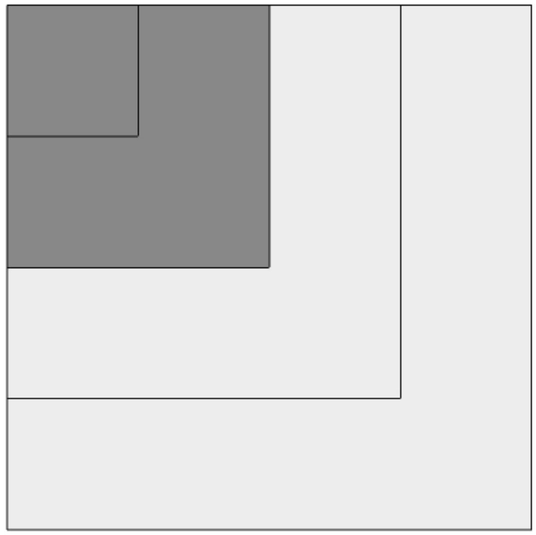


Figure 3.1: Proof Without Words: Galileo fractions.

3.9 Collatz Conjecture

There is a single pattern is consistent across and along all of these lines; the *seed value* which begins the pattern is random.

$$48 \to 24 \to 12 \to 6 \to 3 \to 16 \to 8 \to 4 \to \ldots$$

$$20 \to 10 \to 5 \to 16 \to 8 \to 4 \to \ldots$$

$$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to \ldots$$

[Choose your own seed value] $\to \ldots$

[Choose another seed value] $\to \ldots$

13. What conjecture might you make; i.e. do you see some meta-pattern to these patterns?

14. Try 39. What do you think now?

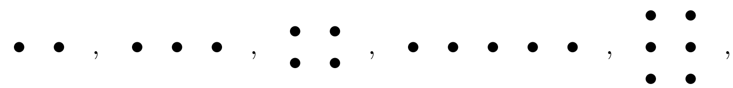
The meta-pattern here, and the fact that it holds for *every* seed value, is called the ***Collatz conjecture*** after **Collatz** (; -). As of this writing this is an open question, nobody knows whether this pattern continues indefinitely.

This might seem like an innocuous difficulty. However, this system is an example of the simplest possible linear dynamical system controlled by two rules. As most of the universe is essentially a dynamical system, from cellular growth which responds to its environment through to the gravitational forces controlling the dances of galaxies through the universe, it is fairly sobering that we cannot even understand how one of the simplest possible examples of such a system behaves.

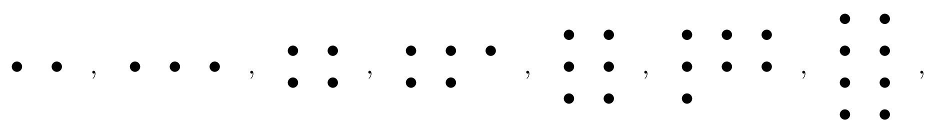
3.10 Teacher Manual

Examples of alternative patterns for 3.1 are:

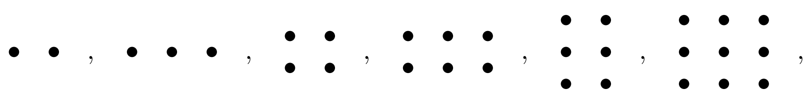
Pattern Number 2



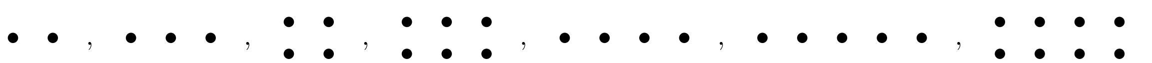
Pattern Number 3



Pattern Number 4



Pattern Number 5:



Note that there is no such thing as a "correct answer" to the pattern.

The pattern that gives rise to the title of this subsection is the representation of the consecutive integers represented in a rectangular array in which the dimensions of the array are as close to one another as possible. For example, 9 is a 3 by 3 array. All primes will have a single row. All composites more than one row. It is interesting to note that when searching for prime factors of a number n it is sufficient to search up to $\sqrt{n}$ as any factorization of n will have to include one factor $\geq \sqrt{n}$ and the other $\leq \sqrt{n}$. This pattern was chosen to show this visually.

The pattern in 3.2 is a way to motivate the fact that the product of a positive number and a negative number is negative. This is something almost all students have simply accepted on faith. They are quite interested to see it arise out of a pattern, where is seems natural.

The continuation, which builds on the original pattern, allows them to see why the product of two negatives naturally is a positive.

In 3.3 none of the patterns continue indefinitely. Calculators are of limited use as they quickly run out of digits *and* they obscure the real reason for the pattern. Computing 1111×1111 by hand shows clearly why the pattern occurs and indicates where it will fail.

For 3.4 the pattern is the day of the week of the 7th of February falls on for the next many years. Of course, any fixed date will give rise to a similar pattern.

The pattern in 3.6 offers a wonderful opportunity to consider rules for exponents. Students generally break things up into simpler pieces. So it is not unusual for them to write:

$$i^{137} = \underbrace{i \times i \times i \times \ldots \times i}_{137} = \underbrace{(i \times i \times i \times i) \ldots (i \times i \times i \times i)}_{39} \times i$$

This can be much more nicely written as

$$(i)^{136} \times i = \left(i^4\right)^{39} \times i = (1)^{39} \times i = i,$$

and we see a nice rationale for utility of the rules for exponents.

Chapter 4

Pick's Theorem

While not a great deal is known about **Georg Alexander Pick** (Austrian Mathematician; 1859 - 1942), we do know he played a nontrivial role in the early career of **Albert Einstein** (German Physicist; 1879 - 1955):

> Pick was the driving force behind the appointment [of Einstein] to a chair of mathematical physics at the German University of Prague in 1911. He held this post until 1913 and during these years the two were close friends. Not only did they share scientific interests, but they also shared a passionate interest in music. Pick, who played in a quartet, introduced Einstein into the scientific and musical societies of Prague. In fact Pick's quartet consisted of four professors from the university including Camillo Krner, the professor of mechanical engineering.[1]

It has been suggested that Pick also played a direct role in the development of Einstein's *general relativity theory* as Pick introduced him to some of the essential work in differential geometry of the time.[2] While Einstein was able to emigrate to the United States through a position at Princeton University, Pick could not avoid the Nazis. He was sent to the Theresienstadt concentration camp where we perished on 26 July, 1942.

Mathematically, Pick is largely remembered for a beautiful geometric result he discovered in 1899, but which was not widely known until it was publicized in the book Mathematical Snapshots by Steinhaus three-quarters of a century later.

In this chapter you will rediscover and prove this striking result.

1. Suppose you happened upon a real, two-dimensional object and you needed to determine its perimeter. What would you do?

2. Suppose now that you had to determine the area of the object. What would you do?

3. Which is easier to determine in real life, the perimeter or the area? Does it matter what type the polygon is or is there a general rule? Explain.

[1] `http://www-history.mcs.st-and.ac.uk/Biographies/Pick.html`
[2] See e.g. `http://en.wikipedia.org/wiki/Georg_Alexander_Pick`.

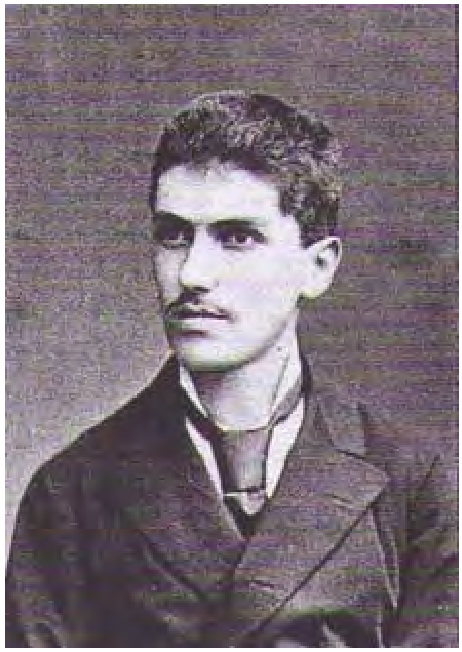


Figure 4.1: Georg Alexander Pick.

A **geoboard** is a board of pegs which are arranged in a regular, rectangular array. One places rubber-bands around the pegs to create geometric polygons, polygons we will call **geoboard polygons**. Several images of Geoboards in action are shown in Figure 4.2. If you have access to one, it may be helpful below. Alternatively, you can use an online version (e.g. `http://nlvm.usu.edu/en/nav/frames_asid_277_g_1_t_3.html?open=activities&from=topic_t_3.html` or `http://www.mathplayground.com/geoboard.html`) or simply use dot-paper or graph-paper.

Focus Question - Is there an easy way to determine the areas and/or perimeters of geoboard polygons?

On a Geoboard the unit length measurement is the distance between adjacent vertical (or horizontal pegs). The unit area measurement is the area of a square all of whose boundary pegs are pairwise adjacent.

4. Make a dozen or so rectangles on a Geoboard. For each determine the area and the perimeter.

5. Do the area measurements share something in common? The perimeter measurements? Is this surprising?

6. For each of these same rectangles, record the number of ***boundary pegs***, i.e. those that are touched by the rubber-band, and the number of ***interior pegs***, i.e. those that are within the confines of the polygon which is bounded by the rubber-band.

7. Do you see a relationship which allows you to predict the area of a geoboard rectangle by knowing the number of boundary and interior pegs? Explain.

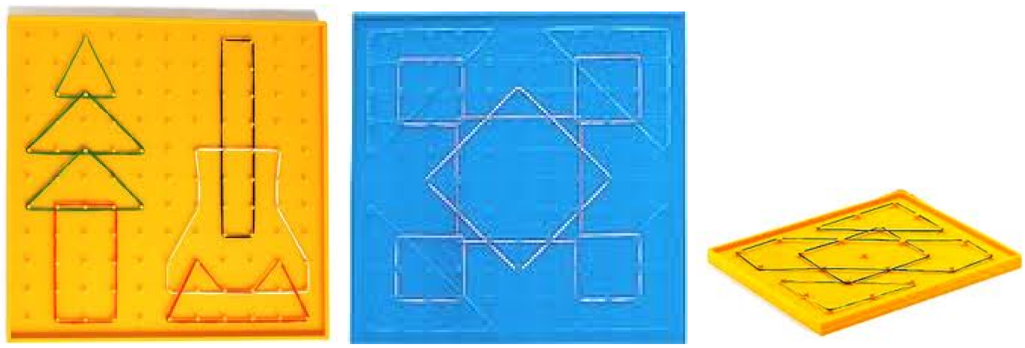


Figure 4.2: Geoboards.

8. Do you see a relationship which allows you to predict the perimeter of a geoboard rectangle by knowing the number of boundary and interior pegs? Explain.

9. Do you think that any pattern you found above will hold for all rectangles? Either prove your result or find a counter-example to your conjecture.

10. If you have found a positive result, can you extend this result to other non-rectangular geoboard polygons? If not, explain. If so, describe carefully the class of objects for which the result holds and any limitations.

11. Now make a dozen or so triangles. For each, determine the area. Carefully describe your method and how you know that it is correct. (If you use a formula, you must justify the formula *and* each application of it.)

12. For each of your triangles, determine the perimeter. (Hint: You may need to remind yourself of the *Pythagorean theorem*.)

13. Do the area measurements of the triangles share something in common? Is this surprising?

14. Do the perimeter measurements of the triangles share something in common? Is this surprising?

15. How do these answers compare to what you reported in Investigation **5**?

16. Is there a way to predict the perimeter of the triangles by knowing the number of boundary and interior pegs? Explain.

17. Is there a way to predict the area of the triangles by knowing the number of boundary and interior pegs? Explain.

4.1 Pick's Formula

18. Choose one of the geoboard polygons whose area you have already determined. Find a way to slightly change the polygon by moving a small part of the (boundary) rubber band so the number of boundary pegs increases by one while the number of interior pegs remains the same. How much did the area increase?

19. Repeat 18 for a different geoboard polygon.

20. For the same geoboard polygon in 18, find a way to slightly change the polygon by moving a small part of the (boundary) rubber band so the number of boundary pegs stays the same while the number of interior pegs increases by one. How much did the area increase?

21. Repeat 20 with the geoboard polygon in 19.

22. Using your investigations with geoboard rectangles, geoboard triangles, and the patterns suggested by Investigation **18** - Investigation **21**, find an algebraic formula that may correctly compute the area of a geoboard polygon which has b boundary pegs and i interior pegs.

The formula in Investigation **22** is known as **Pick's Formula**.

Pick's formula may not seem miraculous. But area, even just triangular area, is not as simple as school mathematics often makes it seem.

If you relied on the area formula for triangles that you learned in high school it may have limited your choice of triangles above. In fact, to find the area of a random triangle is fairly complicated unless the triangle is set up so the base and height are nearly obvious. Given the Cartesian (i.e. (x, y)) coordinates of each of the vertices A, B, C of the triangle the area can be computed using the ***surveyor's formula***

$$A = \frac{1}{2}\left[x_A y_B + x_B y_C + x_C y_A - x_B y_A - x_C y_B - x_A y_C\right],$$

which is also known as the *shoelace formula* or *Gauss' area formula* after **Carl Freidrich Gauss** (German mathematician; -). If the lengths of each side of the triangle are known, the area can be computed using ***Heron's formula***

$$A = \sqrt{\frac{l_A + l_B + l_C}{2}\left(\frac{l_A + l_B + l_C}{2} - l_A\right)\left(\frac{l_A + l_B + l_C}{2} - l_B\right)\left(\frac{l_A + l_B + l_C}{2} - l_C\right)},$$

where l_A, l_B, l_C are the lengths of the triangle's sides, which was discovered by **of Alexandria Heron** (Greek mathematician and engineer; 10 AD - 70AD).

It's not as simple as one-half base times height, is it?

23. Do you think that Pick's formula will hold for other geoboard polygons than rectangles and triangles? Explain.

Let's gather some evidence to see if it supports Pick's theorem.

24. Without using Pick's formula, determine the area of the area of the geoboard polygon on the left of Figure 4.3. Explain your reasoning in detail.

25. Without using Pick's formula, determine the area of the area of the geoboard polygon in the center of Figure 4.3. Explain your reasoning in detail.

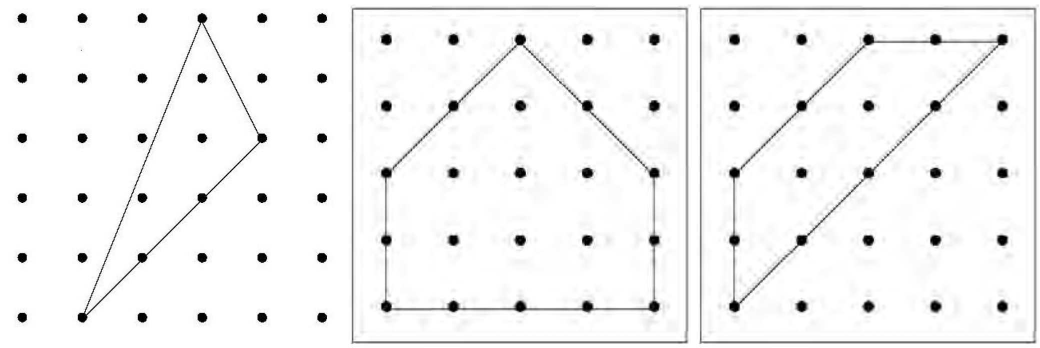


Figure 4.3: Geoboard polygons.

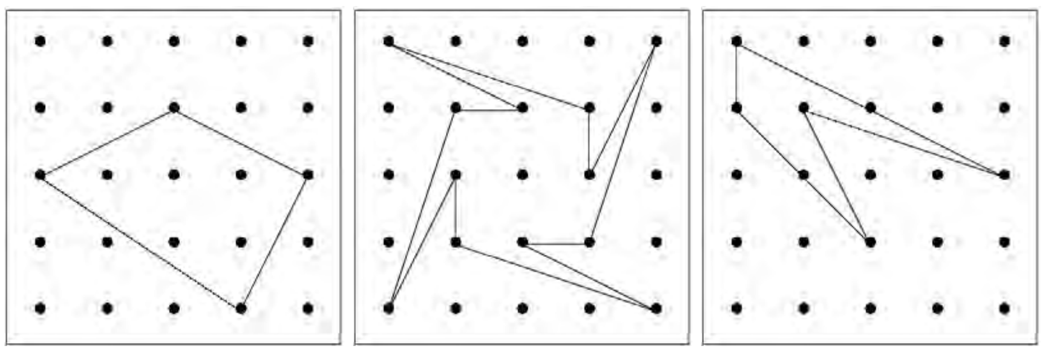


Figure 4.4: Geoboard polygons.

26. Without using Pick's formula, determine the area of the area of the geoboard polygon on the right Figure 4.3. Explain your reasoning in detail.

27. Without using Pick's formula, determine the area of the area of the geoboard polygon on the left of Figure 4.4. Explain your reasoning in detail.

28. Without using Pick's formula, determine the area of the area of the geoboard polygon in the center of Figure 4.4. Explain your reasoning in detail.

29. Without using Pick's formula, determine the area of the area of the geoboard polygon on the right Figure 4.4. Explain your reasoning in detail.

30. Does Pick's formula hold for the polygons in Figure 4.3? Explain.

31. Does Pick's formula hold for the polygons in Figure 4.4? Explain.

32. Use Pick's formula to explain why the observation in Investigation **13** is valid.

33. Do you think that Pick's formula will hold for all geoboard polygons? I.e. might this be Pick's theorem? Explain.

4.2 Applications of Pick's Formula

Whether we and our politicians know it or not, Nature is party to all our deals and decisions, and she has more votes, a longer memory, and a sterner sense of justice than we do.

Wendell Berry (American Poet; -)

2013 was named the year of Mathematics of Planet Earth and with "more than a hundred scientific societies, universities, research institutes, and organizations all over the world have banded together to dedicate 2013 as a special year for the Mathematics of Planet Earth."[3] This focus is essential as "Our planet is the setting for dynamic processes of all sorts, including the geophysical processes in the mantle, the continents, and the oceans, the atmospheric processes that determine our weather and climates, the biological processes involving living species and their interactions, and the human processes of finance, agriculture, water, transportation, and energy. The challenges facing our planet and our civilization are multidisciplinary and multifaceted, and the mathematical sciences play a central role in the scientific effort to understand and to deal with these challenges."

The mission of the Mathematics of Planet Earth project is to:

- Encourage research in identifying and solving fundamental questions about planet earth
- Encourage educators at all levels to communicate the issues related to planet earth
- Inform the public about the essential role of the mathematical sciences in facing the challenges to our planet

Documenting the amount of ice at the Earth's poles is of fundamental importance in our understanding of the causes and impact of global warming. Figure 4.5 shows the recession of Arctic ocean sea ice from 1979 through 2012. Ocean water absorbs much more solar energy than ice, contributing to warming and hence, more ice loss. A vicious cycle.[4]

What does this have to do with Pick's theorem you may be wondering? Well, if we are to understand changing ice masses, it is essential to be able to measure them. And they do not come in shapes where the few area formulas you learned in school will be sufficient. Figure 4.6 shows **Carsten Braun** (Geographer; -) tracking the extent of the Humboldt glacier by walking its bounday and recording GPS data. This glacier is the only remaining glacier in Venezuela. As can be seen, the glacier has shrunk significantly. How much? From 0.15 square kilometers to 0.10 square kilometers in just two years!

And how can one actually determine these areas? The grid on the image acts as the geoboard pegs. Instead of a curved boundary, replace the boundary by straight lines going through corners of the grid. One then gets a very close approximation to the actual shape - one that is a geoboard polygon! Try it! This author's attempt yielded a geoboard polygon with $i = 98$ interior points and $b = 29$ boundary points. Pick's formula gives an area of $A = i + \frac{b}{2} - 1 = 111.5$. The scale in

[3] `http://mpe2013.org/about-mpe2013/`

[4] For a particularly compelling, technical discussion of this process, see the wonderful talk "Applying mathematics to better understand the earth's climate" by Emily Shuckburgh that was given as part of Mathematics of Planet Earth events. It is available online at `http://www.antarctica.ac.uk/staff-profiles/webspace/emsh/talks/AMS-Jan13.pdf`.

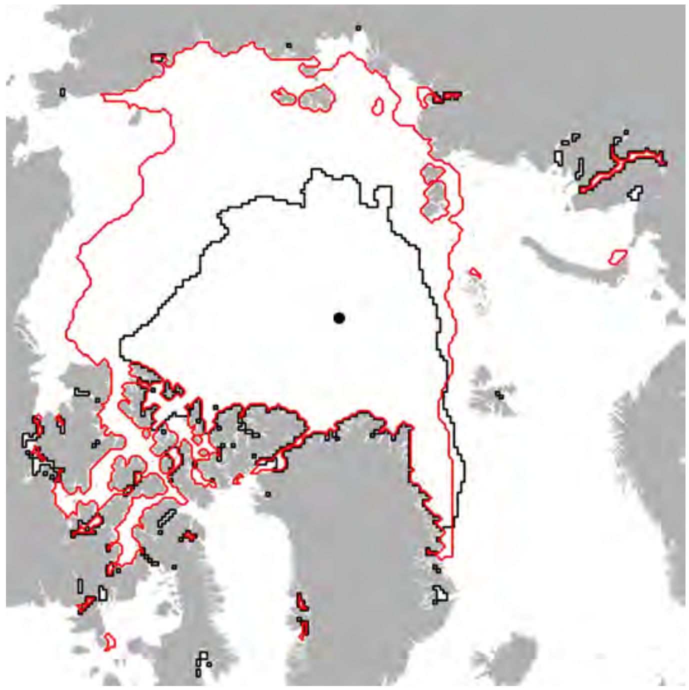


Figure 4.5: Arctic ocean sea ice in September 1979 and September 2012.

the figure is that each grid square is 30 meters on a side. Hence, the approximating area is exactly given by $A = (111.5) \times \left(\frac{30}{1000}\right)^2 = 1.0035$ square kilometers, which is within one-half of a percent of the actual area.

4.3 Generalizing Pick's Theorem

In May, 1991 the National Council of Teachers of Mathematics published an article in its journal *Mathematics Teacher*, a journal for grade 6 - 14 mathematics teachers. The title of the article is "Pick's Theorem Extended and Generalized."

Many geoboards have a square grid on the front and a different grid on the back. Typical alternatives are equilateral triangle lattices and hexagonal (like honeycomb) lattices. The article shows how Pick's formula can be adapted to hold for these other shapes as well.

The abstract of the article reads:

> Our mathematics teacher set us an assignment to investigate Pick's theorem for square lattices. When I finished early, he suggested that I extend the theorem to cover triangular lattics, which I was able to do without too much trouble. The next step was obvious - could I extend the theorem to all lattices? The hexagonal lattice proved difficult, but I worked on it that night and the next day. While I was walking home

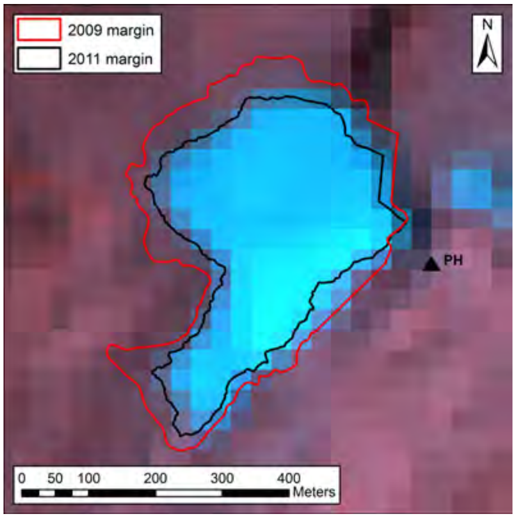


Figure 4.6: Humboldt glacier ice extent in 2009 and 2011; tracking the extent of the Humboldt glacier using GPS.

> from school, I realized that if I introduced a third variable for the type of lattice, I had a formula to cover all lattices. I arrived at the formula through a process of seeing patterns and guessing and checking, and I am not, at this stage, able to give a more mathematical proof.

Typically mathematical articles are not published without containing “a more mathematical proof.” But this was not a typical article. The author here, Christopher Polis, was *an eighth grade student* when he discovered how to extend Pick’s theorem to general geoboard lattices of different shapes!

4.4 Proving Pick’s Theorem

> In mathematics, if a pattern occurs, we can go on to ask, Why does it occur? What does it signify? And we can find answers to these questions. In fact, for every pattern that appears, a mathematician feels [s]he ought to know why it appears.
>
> **W. W. Sawyer** (; -)

Pick’s formula is surprising in its simplicity. But why does it work? Mathematicians are not satisfied that a result is true until there is a proof. When we know definitively for what shapes Pick’s formula holds, we will then have a theorem - Pick’s theorem.

Not compelled that a proof is really necessary after all of the evidence supporting Pick’s formula above?

34. For the geoboard polygon in Figure 4.7, determine the area, number of interior pegs and number of boundary pegs.

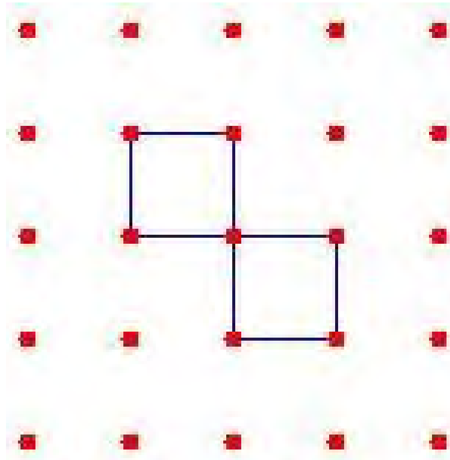


Figure 4.7: Does Pick's formula hold?

35. Does Pick's formula hold for this figure?

36. For the geoboard polygon in Figure 4.8, determine the area, number of interior pegs and number of bounday pegs.

37. Does Pick's formula hold for this figure?

38. Can you explain why these *counterexamples* arise, at least intuitively?

39. Do these counterexamples bother you? Explain.

In mathematics, proofs provide a deductive, logical demonstration that a result always holds. They provide absolute certainty. But proofs also help us to understand why. Since these counterexamples refute our current working version of Pick's theorem - that Pick's theorem holds for every geoboard shape - our search for a proof can help us understand the limitations of Pick's formula as well why it works.

40. On graph [or dot] paper draw a geoboard rectangle. Find the area, number of boundary pegs and number of interior pegs.

41. Draw vertical and horizontal dotted/colored lines exactly between each of the lines [resp. rows and colums of dots] on your graph [resp. dot] paper.

42. Explain why your new figure shows that each interior peg contributes exactly one unit of area to the total.

43. Does your figure show that each boundary peg contributes exactly one-half unit of area? Explain.

44. Use your figure to prove Pick's formula holds for all geoboard rectangles.

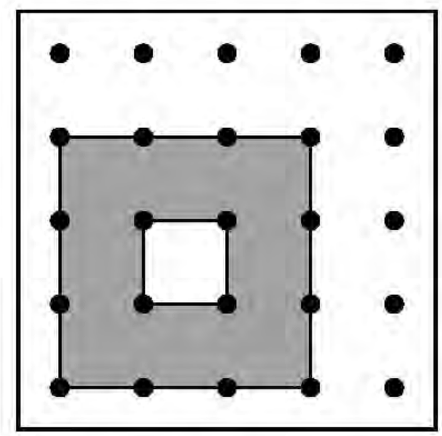


Figure 4.8: Does Pick's formula hold?

4.4.1 Visibility Measures

Can we extend this to general Geoboard polygons? **Dale Varberg** (; -) discovered a wonderful way to do so.[5]

For each peg, boundary and interior, of a geoboard polygon, define the **visibility measure** at that point to be the relative percentage you can see into the interior of the Geoboard figure from that point. For example, for the corner of a geoboard rectangle the visibility measure is $\frac{1}{4}$. For any non-corner peg on the boundary of a rectangle the visibility measure is $\frac{1}{2}$.

45. Find the visibility measure at each interior and boundary peg of each of the geoboard polygons in Figure 4.3.

46. For each geoboard polygon just considered, add up all of the figure's visibility measures. What do you notice?

47. Find the visibility measure at each interior and boundary peg of each of the geoboard polygons in Figure 4.4.

48. For each geoboard polygon just considered, add up all of the figure's visibility measures. What do you notice?

49. Find the visibility measure at each interior and boundary peg in Figure 4.9.

50. Find the area of the geoboard polygon in Figure 4.9. How does the area compare to the sum of the visibility measures?

51. Find the visibility measure at each interior and boundary peg in Figure 4.10.

[5]This approach is from "Pick's Theorem Revisited" by Dale E. Varberg, *American Mathematical Monthly*, vol. 92, no. 8, October 1985, pp. 584-7.

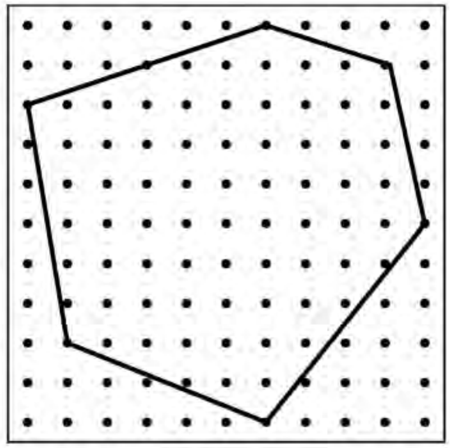


Figure 4.9: Geoboard hexagon.

52. Find the area of the geoboard polygon in Figure 4.10. How does the area compare to the sum of the visibility measures?

53. You should notice that certain types of interior and boundary pegs have predictable visibility measures. What are they?

54. Do you think that the relationship between the area and the sum of the visilibity measures will hold for all geoboard figures? Why?

4.4.2 Proving $A(S) = M(S)$

For any geoboard polygon, S, define the **visibility measure** $M(S)$ to be the sum of the visibility measures at all of the interior and boundary pegs of S.

55. You've already proven that $M(S) = A(S)$ whenever S is a rectangle. Explain.

56. Suppose the Geoboard polygon S is composed of two other Geoboard polygons T and R who may share boundaries but are otherwise non-overlapping. Prove that $M(S) = M(T) + M(R)$.[6]

57. Explain why, in the context of the previous problem, $A(S) = A(T) + A(R)$.

58. Explain why $M(S) = A(S)$ must be true for right, geoboard triangles.

59. Explain why it now follows that $M(S) = A(S)$ for all geoboard triangles.

60. Can all geoboard polygons be decomposed into non-overlapping geoboard triangles? Explain.

61. Combine these results to explain why it follows that $M(S) = A(S)$ for all geoboard polygons S.

[6]This property is called ***subadditivity***.

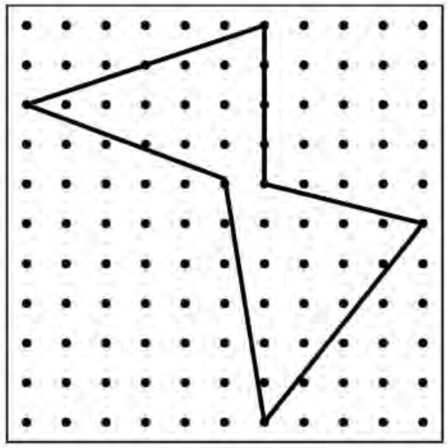


Figure 4.10: Geoboard hexagon.

4.4.3 Simplifying $M(S)$

We've now found an alternative way to measure areas of geoboard polygons, the measure $M(S)$. How's this help us prove Pick's theorem? We've already seen that certain types of pegs have predictible visibility measures. Can we use this to simplify?

62. For a given geoboard polygon, how many points have visibility measure equal to 1? Explain.

63. If you add together all of the visibility measures that equal 1, what is the sum equal to?

64. For a given geoboard polygon, how many points have visibility measure equal to $\frac{1}{2}$? Explain.

65. If you add together all of the visibility measures that equal $\frac{1}{2}$, what is the sum equal to?

66. How close are you to proving Pick's theorem? What remains to be analyzed?

4.4.4 Interior Angle Sums

The angle formed on the inside of a geoboard polygon where two edges come together is called the ***interior angle*** of the polygon.

67. For several triangles you have already considered, determine each of their interior angles and the sum of their interior angles. What do you notice?

68. For several quadrilaterals you have already considred, determine each of their interior angles and the sum of their interior angles. What do you notice?

69. For several pentagons you have already considered, determine each of their interior angles and the sum of their interior angles. What do you notice?

70. For several hexagons you have already considered, determine each of their interior angles and the sum of their interior angles. What do you notice?

71. Make a conjecture about the sum of the interior angles of a polygon with n sides.

72. Does your formula for the sum of the interior angles apply to the geoboard polygon in Figure 4.7? So is this a counterexample to our angle sum conjecture?

73. Does your formula for the sum of the interior angles apply to the geoboard polygon in Figure 4.8? So is this a counterexample to our angle sum conjecture?

A ***simple polygon*** is a polygon whose sides are connected to each other one after another until *closed* without *intersecting* other than when the figure becomes closed.

74. Does this new category of polygons help you ammend your conjecture about angle sums? Explain.

75. Figure 4.11 provides a "Proof Without Words" that the sum of the exterior angles in a simple polygon with 6 sides agrees with your answer above. Explain this proof.

76. Illustrate how it works with several other geoboard polygons you have considered above.

77. Do you find this proof to be compelling? Explain.

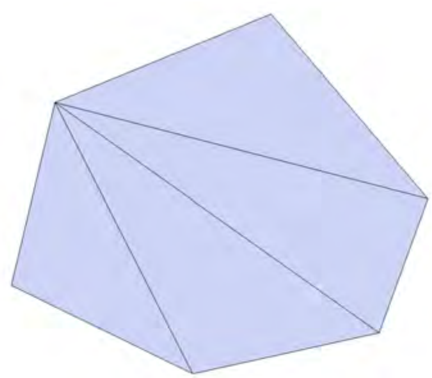


Figure 4.11: Geoboard hexagon.

4.4.5 Finishing the Proof

You have proven $A(S) = M(S)$. And in Investigation **63** and Investigation **65** you have the sums of most of the visibility measures that add to $M(S)$.

78. Determine the sum of the visibility measures at the points whose visibility measures differ from 1 and $\frac{1}{2}$.

79. Show how this allows you to complete the proof of Pick's theorem.

Chapter 5

The Use of Patterns and Language in the Creation of Powerful Number Systems

> Science is the attempt to make the chaotic diversity of our sense-experiments correspond to a logically uniform system of thought.
>
> **Albert Einstein** (; -)

> We encounter patterns all the time, every day: in the spoken and written word, in musical forms and video images, in ornamental design and natural geometry, in traffic patterns, and in objects we build. Our ability to recognize, interpret, and create patterns is the key to dealing with the world around us.
>
> **Margorie Senechal** (; -)

5.1 Introduction

About the number $\sqrt{2}$ **Erwin Schrödingier** (German physicist; 1887 - 1961) said:

> We are told such a number as square root of 2 worried Pythagoras and his school almost to exhaustion. Being used to such queer numbers from early childhood, we must be careful not to form a low idea of the mathematical intuition of these ancient sages, their worry was highly credible.

Forget about $\sqrt{2}$ for now, what about numbers in general? Since childhood we are used to numbers - their names, how they are written in our positional number system, and later perhaps in scientific notation - that these things seem timeless.

Number systems, how we name and denote numbers, are not timeless. In human history they are relatively new.

And they are very human inventions.

And they lie at a wonderful intersection where mathematics meets the world language. An essential part of the bridge between mathematics and language is the role that patterns play.

So what is the history of number systems? In the Western world this development came very late.

The *base-ten, Hindu-Arabic number system* that we use today - largely without thought of how amazing it is - is a positional number system developed by Hindu mathematicians in the first few centuries A.D. They were adopted and adapted by Persian and then Arabic mathematicians by the ninth century. It was only much later that these systems made their way into European culture, their first significant treatment in Liber Abaci by **Leonardo Fibonacci** (Italian mathematician; 1170 - 1250) in 1202 and was not widely adopted until well after that.

So what happened before this?[1]

- Our counting was similar to the other animals - "one, two, three, four, five, many."

- While a very early system, the ***Babylonian number system*** was a very sophisticated number system - as was their astronomy. It was a positional number system with base 60 and this is why we still divide hours into 60 minutes and minutes into 60 seconds. Roughly, the symbol $\langle$ stands for 10, the symbol T stands for 1. Spaces are left between the different positional places. So,
$$\langle^{T} \quad \langle\langle^{TTT} = 11 \times 60 + 23 = 683.$$

- The ***Greek number system*** uses the 24 Greek letters and three additional symbols to represent each of the numbers 1 - 9, each group of 10 between 10 and 90, and each group of 100 between 100 and 900. In this system $\chi\pi\gamma = 600 + 80 + 3 = 683$.

- The ***Roman numerals*** used different letters for different numbers: $I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, M = 1000$. Numbers can be repeated and when a smaller number precedes a larger one it is a sign of subtraction; e.g. $IX = 10 - 1 = 9$. In this system $DCLXXXIII = 683$.

- While not a numeral system, the difficulty in how we represent numbers can also be illustrated via the mathematical subject in Schrödinger's quote. In the time of the *Pythagoreans* the ancient Greeks discovered that not all numbers were fractions. The *incommensurability* of $\sqrt{2}$ was enormously troubling and initiated a search for a new number system. One approach was the use of ***continued fractions***. The continued fraction representation for the offending incommensurable is
$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}}.$$

Try to do arithmetic with these systems!

In The Sand Reckoner, **Archimedes** (Greek mathematician, physicist, engineer and astronomer; 287 B.C. - 212 B.C.), one of the greatest mathematicians of all time, determined a way to name numbers so large they could describe the number of grains of sand in the observable

[1] For more on the history of numbers there is a nice, brief introduction in The book of Numbers by Conway and Guy. Focussed on zero, both The Nothing that Is: A Natural History of Zero by Robert Kaplan and Ellen Kaplan and Zero: The Biography of a Dangerous Idea by Charles Seife contained detailed, accessible histories of the major number systems of humanity.

universe. He considered this feat so great that he dedicated this work to King Gelon! With today's number systems we can easily use *scientific notation* to determine[2] such a bound - a bound that is

$$3.95 \times 10^{107} \text{ molecules.}$$

We don't need the brilliance of Archimedes to extend this further and further. You want to make larger and larger numbers? Just add more digits - one at a time, a hundred at a time, a million at a time, etc.

5.2 Powers of Ten

For each of the numbers whose name in English is given below, write the number named using base-ten notation:

1. Nineteen thousand, four hundred sixty-five.

2. Three hundred fifty two thousand, eight hundred nineteen.

3. Seventeen million, forty three thousand, five hundred eighty-two.

When writing numbers in English one uses commas only to separate words as you would when writing the digits - only in groups of three. The word "and" is used to indicate where a decimal point goes. The numbers twenty-one through ninety-nine are hyphenated.

For each of the numbers written below in base-ten notation, give their English name:

4. 784.

5. 562,978.

6. 6,587,581.

7. 27,000,000.

8. 5,914,490,937.

9. 56,320,000,000.

We can write large numbers, in fact as large as we desire, because the base-ten Hindu-Arabic number system that we use is positional. We do not need to adapt this system in any way, we just fill in appropriate digits to build larger and larger numbers.

Our positional number system is an amazing human invention. Let's review how it works...

5,327 is five thousands, three hundreds, two tens, and seven ones. Written in ***expanded notation*** this number is:

$$5 \times 1000 + 3 \times 100 + 2 \times 10 + 7 \times 1.$$

The *digits* are always 0 - 9 and the critical bases are $\ldots, 1000, 100, 10, \text{ and} 1$.

For positive integers a and n we define a to the **power** n, denoted by a^n, by

[2]See the chapter "The Very Large" in Discovering the Art of Mathematics - Infinities for this determination.

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n}.$$

The number a is called the **base** and the number n is called the **exponent**. It is then natural to call the numbers $a, a^2, a^3, \ldots$ the **powers of** a.

10. Is 100 a power of 10? Explain.

11. Is 1000 a power of 10? Explain.

12. Is 10000 a power of 10. Explain.

13. You should see a pattern forming in Investigations **10-12**. Use this pattern to write the number $1\underbrace{000\ldots000}_{n \text{ zeroes}}$ as a power of 10.

We can now simplify the expanded notation using what we have learned about powers of ten. So, for example, we can write $5,327$ as $5\times10^3+3\times10^2+2\times10^1+7\times10^0$ where, for the moment, we have defined $10^0 = 1$. (Shortly we will see that this really is a natural consequence of an important mathematical/linguistic pattern.)

14. Write the number in Investigation **4** in expanded notation using powers of ten.

15. Write the number in Investigation **5** in expanded notation using powers of ten.

16. Write the number in Investigation **6** in expanded notation using powers of ten.

17. Write the number in Investigation **7** in expanded notation using powers of ten.

18. Write the number in Investigation **8** in expanded notation using powers of ten.

19. Write the number in Investigation **9** in expanded notation using powers of ten.

20. Use the investigations above to explain why it is appropriate to call our number system the "base-ten" number system.

Because successive powers of ten are a factor of ten larger than the power which precedes it we have a powerful tool to study things that exist on massive scales - like our universe. In their wonderful book Powers of Ten authors **Philip Morrison** (American Physicist and Author; 1915 - 2005)[3] and **Phylis Morrison** (American Teacher, Educator, and Author; - 2002) provide a tour of the universe by starting at the edge of our local cluster of galaxies and with each successive page moving our viewpoint a factor of ten closer. We see galaxy clusters, then several galaxies, then a single galaxy, then part of its arm, then a solar system, then a planet, etc. After some 25 pages we see we have been focusing on a couple lying on a blanket at a city park in Chicago, Illinois. Not stopping there, the photos continue to move ten times closer, eventually reaching the sub-atomic particles that make up the DNA of one of these people. Subsequent movies, flip-books, screen savers, and interactive Internet sites immortalize this powerful idea. One wonderful interactive, online tour is at:

[3] Morrison is a fabulously interesting character. Not nearly as well-known as some of his contemporaries like Carl Sagan, he was very important. Among other things he was the driving force behind the development of ***SETI*** - the Search for Extratterestrial Intelligence.

`http://micro.magnet.fsu.edu/primer/java/scienceopticsu/powersof10/index.html`.

The American Museum of Natural History in New York, New York integrates these ideas into a spectacular installation called "Scales of the Universe." At the center of this installation housed in the Rose Center for Earth and Space is the Hayden Planetarium - a 150 foot tall sphere which houses a full IMax theatre in the top half and an interactive tour on the bottom half. Spiraling around the Hayden Sphere is "Scales of the Universe" - a walkway through the sizes and scales of the universe. Instead of using visual images like Powers of Ten, it uses physical models which are successively compared to the massive Hayden Sphere which is suspended right in front of your view to help you understand the awesome scale of the universe through the subatomic workings of each little piece of the universe.

Earlier it was mentioned that lthe chapter "The Very Large..." in Discovering the Art of Mathematics - Infinities considers many truly, amazingly large numbers. In addition to the number of molecules in the observable universe, this includes:

- A googolplex.
- The total number of different 410 page books in the "Library of Babel."
- The total number of different possible human DNA strands.
- The 22, 338, 618-digit prime number $2^{74207281} - 1$.
- The staggeringly large $2^{2^{2^{2^{2^{2}}}}}$.

Here we won't be concerned about finding "real" applications of these enormous number but about determining how give really large numbers English names.

21. Choose a number which is a four digit number when written in base-ten. Write this number in base-ten and then write the English name for this number.

22. Now add a digit to the front and write the English name for this number.

23. Continue adding digits one at a time to the front of your number and writing the English names for these numbers until you have a 14-digit number. (When possible, you need only provide the added components of the name if other parts of the name remain the same.)

24. Suppose you were asked to add one more digit. Would you need any additional information to know how to provide the English name of the number? Explain.

25. Suppose you were asked to add four more digits. Would you need any additional information to know how to provide the English name of the number? Explain.

5.3 Millinillitrillion and the Other Illions

While its easy to add digits to make larger numbers, naming larger and larger numbers requires an increasing repertoire of names for these numbers.

26. If you only knew the large number names thousand and million, what is the largest number you could name in English?

27. If you only knew the large number names thousand, million and billion, what is the largest number you could name in English?

The terms million, billion, trillion, quadrillion, quintillion, sextillion, septillion, octillion, and nonillion were introduced by **Nicholas Chuquet** (French mathematician; 1445 - 1488) in 1484 and appeared in print in a 1520 book by **Emile de la Roche** (; -). The meanings of these words were subsequently changed and there continue to be linguistic debates and discrepancies about the names of large numbers.

The fascination with naming large numbers led **Allan Wechsler** (; -), **John Horton Conway** (British mathematician; 1937 -), and **Richard Guy** (British mathematician; 1916 -) to proposed a system that can be extended indefinitely to provide an English names for *any* base-ten number, no matter how large! We will consider this naming scheme now.

28. Complete the first three columns in Table 5.1, using your knowledge of prefixes and patterns to help you establish a pattern that gives meaning to the terms you are not certain of.

English	Base-Ten	Power of Ten	Prefix	Ordinal illion
thousand	1,000	10^3	N.A.	zeroeth
million	1,000,000		mi	first
billion			bi	second
trillion				
quadrillion				
quintillion				
sextillion				
septillion				
octillion				
nonillion				

Table 5.1: Classical names of the first illions.

29. Do these names give you what you needed to positively answer Investigation **24** and Investigation **25**? Explain.

30. Using these names, describe the largest number you can name in English.

Our goal is to use patterns in this chart to expand our linguistic ability to name increasingly larger numbers.

31. Complete the two columns on the right in Table 5.1.

32. Find patterns in the table that allow you to determine the 10^{th}, 13^{th}, and 21^{st} illion as powers of ten.

33. What is the 237^{th} illion as a power of ten?

34. What is the 649^{th} illion as a power of ten?

35. Extend Investigation **32** to find a algebraic expression for the n^{th} illion as a power of ten.

36. Now reverse this process, what illion is 10^{57}? 10^{219}? 10^{399}?

37. Does every power of ten correspond to a given illion? If so, describe how you find it. If not, describe the limitations.

Our powers of ten and ***ordinal illions*** can continue indefinitely. What we need to extend the English names of the numbers is more prefixes. Wechsler, Conway and Guy provided a way to extend these prefixes indefinitely. Their scheme relies on the following prefixes:

	Units	Tens	Hundreds
1	mi	deci^{n}	centi^{nx}
2	bi	viginti^{ms}	ducenti^{n}
3	tre*	triginta^{ns}	trecenti^{ns}
4	quad	quadraginta^{ns}	quadringenti^{ns}
5	quint	quinquaginta^{ns}	quingenti^{ns}
6	se*	sexaginta^{n}	sescenti^{n}
7	sept*	septuaginta^{n}	septingenti^{n}
8	oct	octoginta^{mx}	octingenti^{mx}
9	non*	nonaginta	nonagenti

Table 5.2: Wechsler/Conway/Guy extension of illion names.

Notes On Using Illion Table 5.2:

- The prefixes are attached in the order of units, tens, and then hundreds.
- If the final prefix in the English name ends in an "a" or an "i", this letter is dropped before adding "illion" to complete the name so there are not two vowels in a row. If "se" is the final prefix, it becomes "sex" before adding "illion".
- Like many prefixes, slight modifications are necessary depending on the context in which they are used. Modifications are needed for the four units marked with $\star$ above. These modifications are as:
  - "tre" becomes "tres" when used directly before a component marked with an s *or* an x.
  - "se" becomes "sex" when used directly before a component marked with an x and becomes "ses" when used directly before a component marked with an s.
  - "sept" becomes "septem" when used directly before a component marked with an m and becomes "septen" when used directly before a component marked with an n.
  - "non" becomes "novem" when used directly before a component marked with an m and becomes "noven" when used directly before a component marked with an n.

Examples:

- $\underbrace{\text{quint}}_{5}\underbrace{\text{deci}}_{10}\underbrace{\text{sescenti}}_{600}$ is 615 so quintdecisescentillion is the 615$^{\text{th}}$ illion. This is 10^{1848} as a power of ten.

- $\underbrace{\text{septem}}_{7}\underbrace{\text{octigenti}}_{800}$ is 807 so septemoctigentillion is the 807$^{\text{th}}$ illion. This is 10^{2424} as a power of ten.

38. In Investigation **33** you translated the 237$^{\text{th}}$ illion into a power of ten. What is the English name of this number?

39. In Investigation **34** you translated the 649$^{\text{th}}$ illion into a power of ten. What is the English name of this number?

40. Determine the English names of the numbers 10^{57}, 10^{219}, and 10^{399} that you considered in Investigation **36**.

41. What is the largest number that you can name using the naming scheme above? Explain.

Our goal was to be able to name arbitrarily large numbers. It may seem like any naming scheme will run out of names just as those in Table 5.1 and Table 5.2 have. What Wachsler, Conway, and Guy did to surpass the limit in Investigation **41** was to extend their numbering scheme by concatenating the English names together in blocks representing three digits in the base-ten notation. Where the base-ten numbers use commas to separate the blocks of digits, this new naming scheme uses "illi" as a separator. When a whole block of three consecutive digits is empty, e.g. 72,000,317, they indentify these place-holding digits in the English name by inserting "nilli".

Examples:

- $\underbrace{\text{bi}}_{2}\underbrace{\text{cent}}_{100}\underbrace{\text{illi}}_{,}\underbrace{\text{oct}}_{8}\underbrace{\text{deci}}_{10}\underbrace{\text{septingent}}_{700}$illion is the $102,718^{\text{th}}$ illion.
- $\underbrace{\text{trecent}}_{300}\underbrace{\text{illi}}_{,}\underbrace{\text{nilli}}_{000,}\underbrace{\text{oct}}_{008}\underbrace{\text{illi}}_{,}\underbrace{\text{nonagint}}_{900}$illion is the $300,000,008,900^{\text{th}}$ illion.

42. What illion has the English name millinillitrillion?

43. Express millinillitrillion as a power of ten.

44. What illion has the English name novemsexagintatrecentillitresquadragintasescentillion?

45. Express novemsexagintatrecentillitresquadragintasescentillion as a power of ten.

46. What illion has the English name trestrigintillinillimicentillion?

47. Express trestrigintillinillimicentillion as a power of ten.

48. Express the $42,903,271^{\text{st}}$ illion as a power of ten.

49. What is the English name of the $42,903,271^{\text{st}}$ illion?

50. What illion is the number $10^{24,921,846}$?

51. What is the English name of the number $10^{24,921,846}$?

So far the numbers we have named have all been powers of ten - with 1 as the only non-zero digit - so we have been jumping from one illion to the next skipping over all the intermediary numbers. But it is easy to fill in this gap if we simply use our everyday knowledge of naming numbers as illustrated in Investigation **4** - Investigation **9**.

52. Name the number $27,\underbrace{000,\ldots,000}_{\text{36 zeroes}}$.

53. Name the number $400,\underbrace{000,\ldots,000}_{\text{150 zeroes}}$.

54. Name the number $56,320,\underbrace{000,000,\ldots,000,000}_{\text{318,174,639 zeroes}}$.

5.4 0.999999 . . . and 1

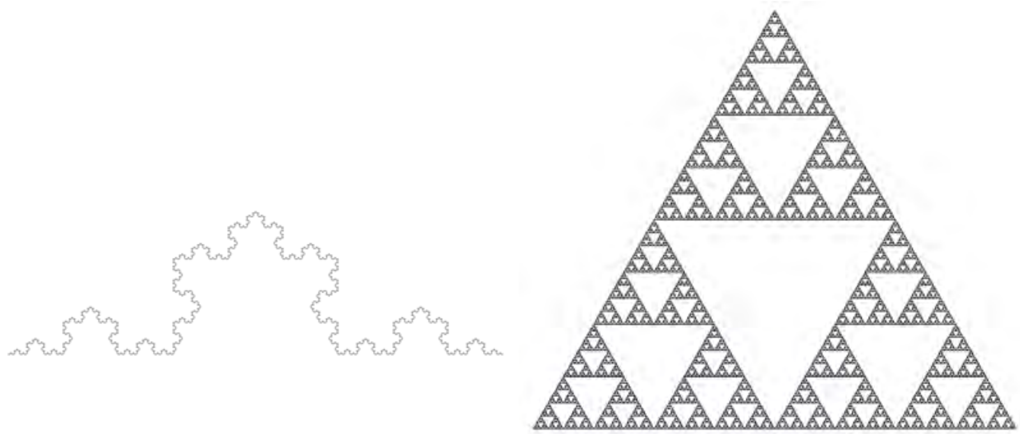


Figure 5.1: Famous geometric *fractals* - the ***Koch curve*** on the left and the ***Sierpinski gasket*** on the right.

55. Do you believe the infinitely repeating decimal 0.999999 . . . is equal to 1?

56. Write down reasons for your belief(s) about the relationship(s) between the numbers 0.999999 . . . and 1. Be as precise as you can in articulating each of your reasons.

57. Describe what the number $\frac{1}{3}$ represents.

58. Convert the number $\frac{1}{3}$ into a decimal, writing your result as an equation:

$$\frac{1}{3} = \underline{\qquad\qquad}.$$

59. Prove that your representation of $\frac{1}{3}$ as a decimal in Investigation **58** is correct using long division.

60. Do you have any concerns or objections about your result in Investigation **58**? Do your reasons in Investigation **56** apply to 0.333333 . . .? Are your beliefs being consistently applied?

61. Multiply both sides of your equation from Investigation **58** by 3. What does this suggest about the value of 0.999999 . . .? Surprised?

People often object to the result in Investigation **61**. One objection is that the decimal representations 0.999999 . . . and 1 appear so different. But remember, the two expressions 0.999999 . . . and 1 are simply symbolic representations of *real numbers*.

62. What number is represented by $\frac{6}{2}$? By $\frac{21}{7}$? How many other fractions are there that represent this same number?

63. What number is represented by $\sqrt{9}$?

64. What number is represented by the Roman numeral III?

65. In *binary arithmetic*, which is the arithmetic that is used by computers, the number 11_2 has a digit "1" in the "twos digit" and a "1" in the "ones" digit. So the number 11_2 represents what number?

66. All of these mathematical/linguistic representations look very different, does it bother you that they all represent the same number? So why can't $0.999999\ldots = 1$?

67. Give several real-life examples of objects that we commonly represent in different ways.

Let's try converting some different fractions to decimals. Use a calculator for the next several investigations.

68. Convert $\frac{1}{9}$ to a decimal using your calculator, describing the process that you used.

69. Convert $\frac{2}{9}$, $\frac{3}{9}$, and $\frac{4}{9}$ into decimals using your calculator.

70. Describe the pattern you see and use it to predict the decimal values your calculator will provide for $\frac{5}{9}$, $\frac{6}{9}$, $\frac{7}{9}$, and $\frac{8}{9}$.

71. Now use your calculator to convert $\frac{5}{9}$, $\frac{6}{9}$, $\frac{7}{9}$, and $\frac{8}{9}$ into decimals. Do the results agree with your predictions? Why or why not?

72. Use long division to determine the exact, decimal identity of $\frac{1}{9}$, writing the result as an equation:

$$\frac{1}{9} = \underline{\hspace{3cm}}.$$

73. Explain how you can use your result in Investigation **72** to determine the exact, decimal identities of $\frac{2}{9}, \ldots, \frac{8}{9}$.

74. Use your results to compute $\frac{1}{9} + \frac{8}{9}$ as a decimal and a fraction. Surprised?

75. Multiply both sides of your equation in Investigation **72** by 9. Surprised?

76. Has your opinion about whether $0.999999\ldots{=}1$ changed?

Above you considered $0.999999\ldots$ numerically. We can also consider this object algebraically, treating it is something whose identity we do not know and using algebra to help us find it.

When we do not know the value of a quantity and we are working algebraically we denote the quantity by a variable. So, let's set

$$x = 0.999999\ldots$$

77. Determine an equation for $10x$ as a decimal.

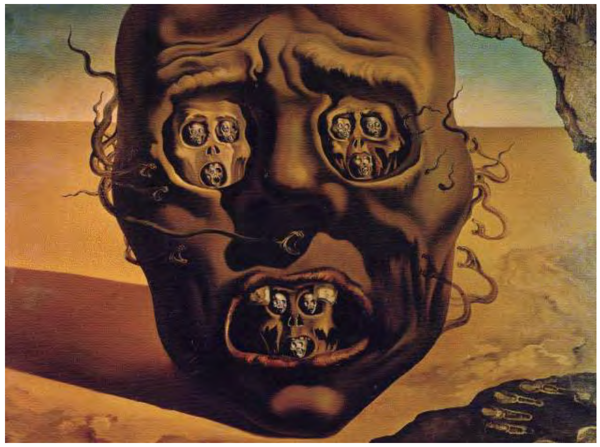


Figure 5.2: "The Face of War" by Salvidore Dali.

78. Using your equation for $10x$ in the previous investigation, complete the following subtraction:

$$\begin{array}{r} 10x = \\ -x = 0.9999999\ldots \\ \hline = \end{array}$$

79. Solve the resulting equation in Investigation **78** for x. Surprised?

Seventh Grader Makes Amazing Discovery

New discoveries and solutions to open questions in mathematics are not always made by professional mathematicians. Throughout history mathematics has also progressed in important ways by the work of "amateurs." Our discussion of $0.9999999\ldots$ provides a perfect opportunity to see one of these examples.

As a seventh grader **Anna Mills** (American Writer and English Teacher; 1975 -) was encouraged to make discoveries like you have above about the number $0.999999\ldots$ Afterwards Anna began experimenting with related numbers on her own. When she considered the (infinitely) large number $\ldots 999999.0$ she was surprised when her analysis "proved" that $\ldots 999999.0 = -1$! She even checked that this was "true" by showing that this number $\ldots 999999.0$ "solves" the algebraic equations $x + 1 = 0$ and $2x = x - 1$, just like the number -1 does.

Encouraged by her teacher and her father to pursue this matter, Anna contacted **Paul Fjelstad** (American Mathematician; 1929 -). Fjelstad was able to determine that Anna's seemingly absurd discovery that $\ldots 999999.0 = -1$ is, in fact, true as long as one thinks of these numbers in the settings of *modular arithmetic* and *p-adic numbers*.

You can see more about this discovery in Discovering the Art of Mathematics - The Infinite or in Fjelstad's paper "The repeating integer paradox" in *The College Mathematics Journal*, vol. 26, no. 1, January 1995, pp. 11-15.

80. Have your beliefs, articulated in Investigation **55**, about the relationship between 0.999999 . . . and 1 changed? Do you remain attached to your rationales from Investigation **56** or have they changed? Explain.

The algebraic method in Investigation **77** - Investigation **79** can be used for more general infinitely repeating decimals.

Suppose you wished to determine a non-decimal identity of the number

$$0.121212\ldots$$

81. Denote the infinitely repeating decimal by $x = 0.121212\ldots$ Determine an equation for $100x$ as a decimal.

82. Following Investigation **78**, complete the subtraction $100x - x$ and solve to express $0.121212\ldots$ as a fraction.

83. Using a calculator, check that the fractional value you obtained for $0.121212\ldots$ is correct.

84. Find a non-decimal identity for $0.272727\ldots.$

85. Find a non-decimal identity for $0.264264264\ldots$.

86. Find a non-decimal identity for $0.695695695\ldots.$

87. Find a non-decimal identity for $0.123451234512345\ldots$

88. You should be becoming aware of a pattern. Precisely describe a general method, or ***algorithm***, for quickly converting any infinitely repeating decimal (with a *leading zero*) into a fraction. If the examples above are not enough, try a few more to uncover the appropriate morophogenesis.

Throughout this chapter there are many images and representations of ***self-similar fractals***. These are objects in which the object is made up of a *motif* which is repeated over and over again on smaller and smaller scales. The objects in this chapter which are self-similar fractals include Figure **??**, the continued fraction in Equation **??**, Figure 5.1, Figure 5.2, and Figure 5.3.

89. Explain why infinitely repeating decimals are self-similar fractals by describing the motif and the scale on which they repeat for several different examples.

The pattern/process you described in Investigation **88** is a very interesting. It shows how an object that is a self-similar fractal in one representation is really a very simple object in a different representation. Yet it is the same object, just conceptualized in a different way.

A natural mathematical question is whether their is a corresponding reverse process. Namely, if you start with a fraction must its decimal representation be an infinitely repeating, and thereby self-similar fractal, decimal?

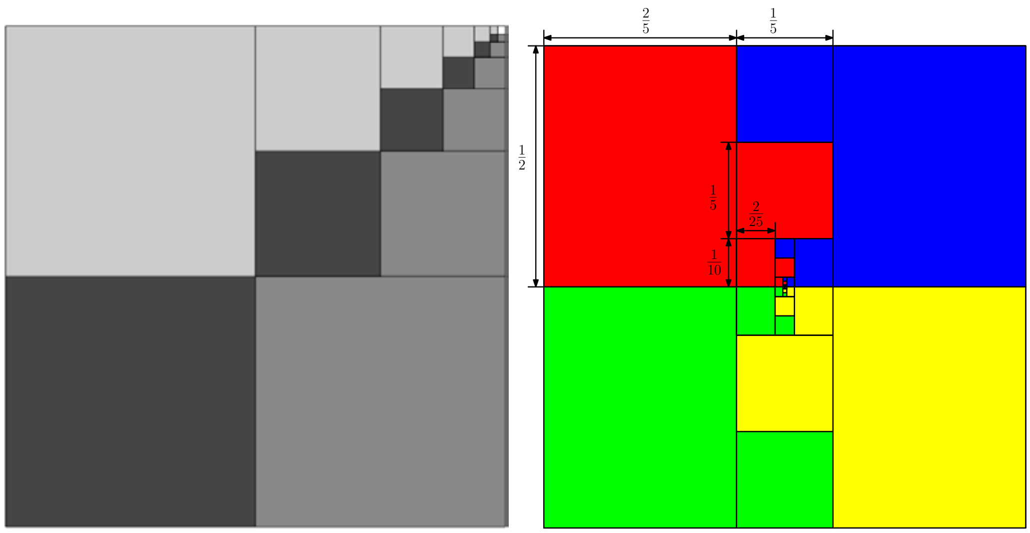


Figure 5.3: Proof without words that $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1}{3}$ and $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} = \ldots = \frac{1}{4}$.

90. Collect some data to test out this idea; i.e. choose a number of fractions and determine if their decimals are all infinitely repeating. Describe what you found.

91. Using long division, determine the decimal representation for $\frac{2}{7}$.

92. Using long division, determine the decimal representation for $\frac{3}{41}$.

93. Using long division, determine the decimal representation for $\frac{11}{73}$.

94. Exactly what is it in the long division process in Investigation **91** - Investigation **93** that caused the repeating behavior to commence?

95. Do you see an infinitely repeating, self-similar process at work here? Explain.

For an infinitely repeating decimal, the **period** is the number of terms in the repeated portion. For example, the period of 0.121212... is 2.

96. How many different possible remainders could appear in the long division to find the decimal for $\frac{5}{17}$?

97. What does this tell you about the maximum length of the period it could have?

98. How many different possible remainders could appear in the long division to find the decimal for $\frac{3}{31}$?

99. What does this tell you about the maximum length of the period it could have?

100. How many different possible remainders could appear in the long division to find the decimal for $\frac{7}{53}$?

101. What does this tell you about the maximum length of the period it could have?

102. In general, if you wished to convert $\frac{m}{n}$ into a decimal, how many different possible remainders could appear as part of the long division process?

103. What does this tell you about the maximum length of the period it could have?[4]

104. Summarize your findings to characterize the types of decimals that arise when one converts fractions to decimals.

105. If you're really liking playing with the infinite, or you are particularly fond of fractals, or you like your results nice a tidy, you might be a little disappointed by the situation you have just summarized. But the situation can be salvaged by the observation. Since $\frac{1}{4} = 0.25$ it *must* follow that

$$0.25 = 0.24999\ldots999\ldots$$

Explain why.[5]

106. Explain why any ***terminating decimal*** can be converted into an infinitely repeating decimal.

[4]If you are interested in learning more about the rich patterns that arise in the periods of infinitely repeating decimals based on their fractional equivalents see Chapter 6 - Further Fruitfulness of Fractions in The Book of Numbers by John H. Conway and Richard K. Guy.

[5]Throughout we have used ..., called **ellipsis**, to denote the repeating part of the decimal. This is because it is often helpful to visually see the digits repeating. However, in this particular problem the situation becomes a bit muddied since there are digits that do not repeat. So the use of a bar over the repeating digits, called a **vinculum**, is helpful. In this case the number can be written $0.24\overline{999}$ without any ambiguity.

5.5 Exponents and Exponential Functions

5.5.1 Britney Gallivan and the Paper-Folding Myth

107. Take a blank sheet of 8 1/2” by 11” paper and fold it in half. Now fold it in half again. And again. See how many such folds you can make, describing what limitations you face.

Urban legend has it that the maximum number of times you can fold a piece of paper in half is seven. Like many things, myths become “fact” when they hit the Internet - see the PBS entry `http://pbskids.org/zoom/activities/phenom/paperfold.html`. Like many such “truths”, they are false.

In December, 2001 **Britney Gallivan** (American Student; 1985 -) challenged this legend by analyzing paper-folding mathematically and finding limits on the number of folds based on the length, L, and thickness, t, of the material used. What she found is that in the limiting case the number of folds, n, requires a material whose minimum length is

$$L = \frac{\pi \cdot t}{6}(2^n + 4)(2^n - 1)].$$

Using this result she was able to fold paper with 11 folds and later with 12 folds.[6] Subsequently, Gallivan received quite a bit of noteriety: her story was mentioned in an episode of the CBS show Numb3rs, was included on an episode of The Discovery Channel’s Myth Busters, and resulted in her being invited to present the keynote address at the regional meeting of the National Council of Teachers of Mathematics in Chicago in September, 2006.

Figure 5.4: Britney Gallivan and paper folded 11 times.

[6]See `http://pomonahistorical.org/12times.htm` for more information.

108. After your first fold, how many layers thick was your folded paper?

109. After your second fold, how many layers thick was your folded paper?

110. After your third fold?

111. Continue a few more times until you see a clear pattern forming. Describe this pattern.

112. Suppose you folded your paper 20 times; how many layers thick would it be?

113. Suppose you folded your paper 30 times; how many layers thick would it be?

114. Use Investigations **108-113** to determine a *closed-term algebriac expression* for the number of layers thick your folded paper will be after n folds.

115. How tall, in an appropriate unit of measure, is a pile of paper that has as many layers as the folded paper you have described in Investigation **112**? (Hint: a ream of paper, which is 500 sheets, is about 2 1/2" tall.)

116. How tall, in an appropriate unit of measure, is a pile of paper that has as many layers as the folded paper you have described in Investigation **113**?

117. How many folds are necessary for the number of layers to exceed a googol? How tall would such a pile of paper be? Explain.

118. If Brittany Gallivan used paper that was the same thickness as standard copier paper, how long would her "sheet" of paper need to have been to fold it 11 times - as pictured in Figure 5.4?

This paper folding example illustrates what is known as *exponential growth* because a quantity grows by a fixed rate at each stage; i.e. we repeatedly multiply at each stage. We see it yields gigantic numbers. If we want, we can built fantastically larger numbers by repeatedly exponentiating at each stage.

5.5.2 Exponents

119. Make a table of values for 2^1 - 2^{12}. Explain how you determined these values and what the meaning of the exponents are.

120. Complete the following table by filling in the missing entries.

m	2^m	2^m as base-ten #	n	2^n	2^n as base-ten #	$2^n \times 2^m$ as base-ten #	Is $2^n \times 2^m$ a power of 2?
3	2^3	8	5	2^5	32	256	Yes. $256 = 2^8$.
2			3				
1			4				
3			4				
2			5				
3			6				
5			6				

121. In your table you should see that $2^n \times 2^m$ is always a power of 2. Find a pattern that relates the exponents n and m of the factors to the exponent in the product $2^n \times 2^m$.

122. Find an algebraic formula describing the pattern in Investigation **121**; i.e. express $2^n \times 2^m$ as a function of n and m.

123. Return to the definition of powers and prove that the result in Investigation **122** follows from the definition of powers.

124. Generalizing the rule you found in Investigation **122**, state and prove a multiplication rule for exponents for any base $a > 0$.

Having determined patterns in the multiplication of powers with a common base, a natural question is whether there is a corresponding rule for division.

125. Complete the following table by filling in the missing entries.

m	2^m	2^m as base-ten #	n	2^n	2^n as base-ten #	$2^n \div 2^m$ as base-ten #	Is $2^n \div 2^m$ a power of 2?
3	2^3	8	5	2^5	32	4	Yes. $4 = 2^2$.
2			3				
1			4				
3			4				
2			5				
3			6				
5			6				

126. In your table you should see that for positive integer values of m and n with $m < n$ the quotient $2^n \div 2^m$ always a power of 2. Find a pattern that relates the exponents n and m to the exponent in the quotient $2^n \div 2^m$.

127. Find an algebraic formula describing the pattern in Investigation **126**; i.e. express $2^n \div 2^m$ as a single power of 2 whenever $m < n$.

128. Return to the definition of powers and prove that the result in Investigation **127** follows from the definition of powers.

129. Generalizing the rule you found in Investigation **127**, state and prove a division rule for exponents for any base $a > 0$.

130. The type of patterns and reasoning you used for $2^n \times 2^m$ and $2^n \div 2^m$ can naturally be extended to provide an analogous result for $(2^n)^m$ where m and n are positive integers. Compute $(2^n)^m$ for several different values of m and n. Find a pattern and use it to express $(2^n)^m$ as a power of 2.

131. Prove that your formula in Investigation **130** follows from the definition of powers.

132. Generalizing the rule you found in Investigation **130**, state and prove a power to power rule for any base $a > 0$.

Row	Power Notation	Definition	Numerical Value
5	2^5	$2 \times 2 \times 2 \times 2 \times 2$	32
4			
3			
2			
1			
0			
-1			
-2			
-3			
-4			
-5			

Table 5.3: Powers of 2

The results you have found in Investigation **124**, Investigation **129**, and Investigation **132** are the classical **rules for exponents** that most of us were told in middle or high school.

133. Do these rules make sense to you now? Did they when you were first taught them? Compare and contrast both your understanding of these rules and the learning experiences that went along with them then and now.

134. Apply your division rule from Investigation **127** to each of the expressions $2^5 \div 2^5, 2^3 \div 2^5$, and $2^4 \div 2^7$. Is this curious?

In Investigation **134** the division rule gave rise to an exponent which are 0 or even negative numbers. The definition of 2^4 should seem like second nature, we're used to "2 times 2 times 2 times 2." But 2^{-3}? You certainly "can't have 2 times itself negative three times," this doesn't make any sense.

Exponents and a suitable notation to express them is a human construct. It is part of a language - the language of algebra. Intuitive ideas and numerical patterns give rise to precise definitions. Yet this whole process would be of small value if the use of exponents in mathematics was limited to the narrow cases considered above.

Like any other language, mathematics grows to accommodate new needs. Here we look to extend the notion of exponents to include 0 and negative numbers. How do we do this? Patterns.

135. Complete rows 1 - 5 in Table 5.3.

136. As you move from Row 1 of the table to Row 2 of the table, describe what happens to the entries in each column.

137. What happens to the entries in each of the columns as you move from Row 2 to Row 3? Row 3 to Row 4? Row 4 to Row 5? Describe the patterns you see precisely.

138. Now describe what happens to the entries in each column as you move from Row 5 to Row 4.

139. What happens to the entries in each of the columns as you move from Row 4 to Row 3? Row 3 to Row 2? Row 2 to Row 1? What patterns do you see now? How do these patterns compare with those when you move up rows?

140. Following the pattern in Investigation **139**, you should be able to extend the table down another row, to build *meaning* for 2^0! Note: Leave the definition column blank since we have no formal definition (yet) and no intuitive idea what should appear there.

141. Repeat Investigation **140** to extend the table to have five more rows, building *meanings* for the powers $2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}$, and 2^{-5}. Compute in a way that allows your numerical values to be expressed as fractions rather than decimals.

142. If they aren't already, convert the numerical values in each of the last five rows to fractions with numerators 1 and denominators a power of 2. Use this to provide a definition for powers with negative exponents:

Definition For exponents $m > 0$, the power 2^{-m} is defined to mean $2^{-m} =$.

143. Posit a definition for powers with **negative exponents** that builds meaning for all of these strange creatures in *any* base:

Definition For bases $a > 0$ and exponents $m > 0$, the power a^{-m} is defined to mean $a^{-m} =$.

144. Return to Investigation **134** and make sense of these computations.

145. More generally, with this new definition, prove that your division rule in Investigation **129** applies for exponents $m \geq n$ as well.

You've built meanings for exponents for *any* integer exponent. But the language can be extended much further.

And it is essential to do this. The *exponential functions*, among the most important functions in all of mathematics, require exponents that change continuously - exponents that take *every* value. Below we will see exponents such as π. This is an irrational exponent!

So we can proceed concretely for a while, let us switch bases to base 16.

We would like to build meaning for *fractional exponents*.

146. Use your multiplication rule for exponents to simplify $(16)^{\frac{1}{2}} \times (16)^{\frac{1}{2}}$ to a power of 16.

147. Determine what the value of $(16)^{\frac{1}{2}}$ must be.

148. How is this value related to the initial base of 16?

149. Similarly, use your multiplication rule for exponents to simplify $(16)^{\frac{1}{4}} \times (16)^{\frac{1}{4}} \times (16)^{\frac{1}{4}} \times (16)^{\frac{1}{4}}$ to a power of 16.

150. Determine what the value of $(16)^{\frac{1}{4}}$ must be.

151. How is this value related to the initial base of 16?

152. Repeat this process to give meaning to $(27)^{\frac{1}{3}}$.

153. Explain why $\sqrt{9} = 3$.

154. Explain why $\sqrt[3]{64} = 4$.

155. Explain why $\sqrt[5]{32} = 2$.

Each of the expressions $\sqrt{9}$, $\sqrt[3]{64}$ and $\sqrt[5]{32}$ are linguistic or notational expressions. They do not have meaning until we build/create meaning for them. Just as with 2^{-6} this meaning is not arbitrarily established by mathematicians or some higher power. Instead, these meanings followed from powerful patterns that were discovered by mathematicians like Chuquet who then used these patterns to extend these languages in powerful ways.

156. Use the examples in Investigation **153** - Investigation **155** to build meaning for n^{th} **roots**:
Definition For bases $a > 0$ and positive integers n we say that x is the n^{th} root of a, and we write $x = \sqrt[n]{a}$, precisely when .

157. How does this definition connect with our notion of powers above? I.e. compare expressions of the form $\sqrt[n]{a}$ with those of the form $b^{\frac{1}{m}}$.

158. Posit a definition for powers with **unit fraction exponents** that builds meaning for all of these strange creatures in *any* base:

Definition For bases $a > 0$ and exponents $\frac{1}{n} > 0$ with n an integer, the power $a^{\frac{1}{n}}$ is defined to mean $a^{\frac{1}{n}} =$.

For this to be a language of value, all of these different meanings must play nicely together. And when they do play together they must do so consistently - we cannot have different answers as the meanings intermingle. Let's see if we check that.

159. Use your rules for exponents to explain why $\left(3^{13}\right)^{\frac{1}{17}} = 3^{\frac{13}{17}}$.

160. Use your rules for exponents to explain why $\left(3^{\frac{1}{17}}\right)^{13} = 3^{\frac{13}{17}}$.

The two expressions you have for $3^{\frac{13}{17}}$ are *very* different. One of a root of a very large power. The other is a power of a root. These must be equal if our language is to be consistent.

161. Prove that $\left(3^{\frac{1}{17}}\right)^{13}$ is a 17^{th} root of 3^{13}.

162. Explain why this proves the consistency of our language for this base and these exponents.

163. How hard would it be to prove consistency for other bases and exponents? Explain.

Above you followed a path to extend meaning to negative exponents that was much like Chuquet and the other early explorers of these ideas. Mathematicians have since been able to extend our basic intuitive understanding of powers and exponents to a surprising and beautiful diversity of numbers. Perhaps the most remarkable illustrations of how far we can extend the basic notion of exponents involve the *imaginary unit* $i = \sqrt{-1}$.

In elementary, middle and high school most mathematics teachers warn their students about taking square roots of negative numbers - that such roots cannot exist. But if we rely on our imagination, if we find a new way to conceptualize numbers, and if we are flexible in how we extend the language of mathematics, we find that we can give meaning to square roots of negative numbers. When we do we create the field of *complex numbers*. Not only is this new creation remarkably beautiful, it unifies different areas of mathematics in profound ways. It also has, surprisingly, enormous practical application:

> There can be very little of present-day science and technology that is not dependent on complex numbers in one way or another. [7]
>
> **Keith Devlin** (; -)

The creation and exploration of the imaginary unit and complex numbers is the subject of the chapter "Existence of $sqrt{-1}$: A Case Study" in the book Discovering the Art of Mathematics - Truth, Reasoning, Certainty & Proof.

Leonhard Euler (; -) is one of the greatest mathematicians of all time and the results attributed to him are enormous. For all of his notoriety, the formula that is universally known as **Euler's formula** is the one he discovered in 1740:

$$e^{i\cdot\pi} + 1 = 0.$$

Here π is the ubiquitous numerical constant related to circles and spheres, a transcendental irrational number which is approximated by the base-ten decimal $3.1415\ldots$ e is another fundamental mathematical constant, also named after Euler, which arises in fundamental growth problems in biology, economics, and many other areas. It is also a transcendental irrational number which is approximated by the base-ten decimal $2.718\ldots$ The most elementary proof of the validity of this formula involves ideas from calculus relying on infinite series and periodic functions closely related to harmonics in music. You can navigate this proof with the help of the "Further Investigations" section from chapter noted above.

164. We've just noted that π, e, and i are fundamentally important numbers. How about 0 and 1?

165. What are the fundamental mathematical operations are involved in Euler's formula? Are there fundamental operations that are not part of this formula? Explain.

166. In light of your answers to Investigation **164** and Investigation **165**, how remarkable is it that these five numbers and these four mathematical operations are expressed so concisely by this one formula? Explain, perhaps by attempting to create a simpler analogue or comparing with some other unifying statement from some other area of intellectual thought.

A formula similarly curious to Euler's formula is $i^i = \frac{1}{\sqrt{e^\pi}}$. Regarding this formula **Benjamin Pierce** (American Mathematician; 1809 - 1880), the "Father of American mathematics", said, "We have not the slightest idea of what this equation means, but we may be sure that it means something very important."

[7] From Mathematics: The New Golden Age.

5.6 Concluding Reflections

Albert Einstein (; -) remarked,

> It is not so very important for a person to learn facts. For that he does not really need a college. He can learn them from books. The value of an education in a liberal arts college is not the learning of many facts but the training of the mind to think something that cannot be learned from textbooks.

Essay 1: Compare and contrast the approach above for learning about exponents to that which you experienced in middle and/or high school. Relate this comparison/contrast to Einstein's quote, providing either supporting evidence for or dissenting views against Einstein's claim.

Noted mathematical author **Ian Stewart** (; -) once noted:

> One of the biggest problems of mathematics is to explain to everyone else what it is all about. The technical trappings of the subject, its symbolism and formality, its baffling terminology, its apparent delight in lengthy calculations: these tend to obscure its real nature. A musician would be horrified if his art were to be summed up as 'a lot of tadpoles drawn on a row of lines'; but that's all that the untrained eye can see in a page of sheet music.

Essay 2: In this chapter we have described a few of the ways that common notions in mathematics are extended far beyond their original intuitive meaning. In this sense, is the language of the mathematician that much different in its history, development and accessibility much different than spoken languages? Much different from other formalized languages such as musical notation? Explain.

Chapter 6

Patterns Linear and Arithmetic

> In 1953 I realized that the straight line leads to the downfall of mankind. But the straight line has become an absolute tyranny. The straight line is something cowardly drawn with a rule, without thought or feeling; it is the line which does not exist in nature... Any design undertaken with the straight line will be stillborn. Today we are witnessing the triumph of rationalist knowhow and yet, at the same time, we find ourselves confronted with emptiness. An esthetic void, desert of uniformity, criminal sterility, loss of creative power. Even creativity is prefabricated. We have become impotent. We are no longer able to create. That is our real illiteracy.
>
> **Friedensreich Regentag Dunkelbunt Hundertwasser** (Austrian Artist and Architect; 1928 - 2000)

> The whole science of geometry may be said to owe its being to the exorbitant interest which the human mind takes in lines. We cut up space in every direction in order to manufacture them.
>
> **William James** (American Psychologist; 1842 - 1910)

6.1 Introduction: The Line

Certainly there are significant limitations to a world populated only by the lowly line. Art would certainly be relatively crippled if it could employ only lines, limiting us to *line art* and *string art* like those pieces shown in Figures 6.1, 6.2, and 6.17. Perhaps that is what Hundertwasser meant in his lengthy quote above, or when he called the straight line "ungodly". Sections of Discovering the Art of Geometry in this series show us how *fractals* are one way mathematics has freed itself from it's most basic objects: lines, circles, and spheres.

Appreciating the freedoms that we can find in mathematics as well as art when we are freed from lines, we think Hundertwasser's condemnation is a bit too harsh. We believe there is significant merit in William James' view - the line has natural resonances with the human mind. Throughout this chapter we provide short asides that illustrate the power and beauty of mathematical objects that are *linear* or *arithmetic*. We hope this removes some of the line's stigma.

There is also a practical side to our approach. Students of mathematics as well as of art are often well-served by starting with a limited slate of objects to work with as they begin to explore

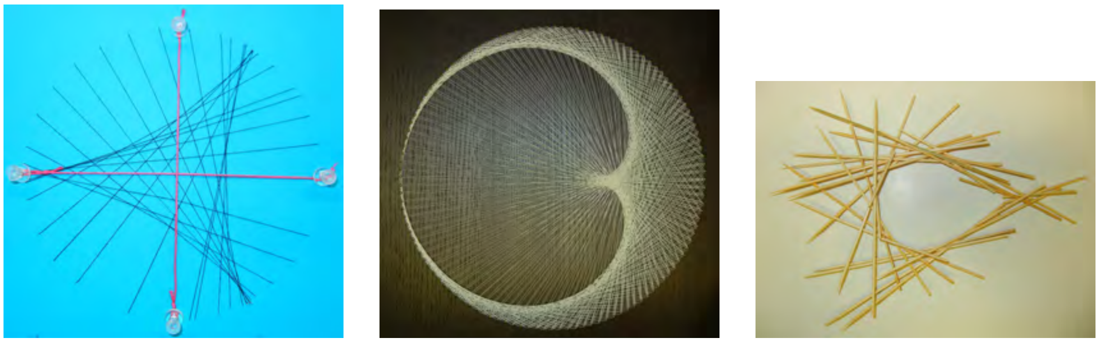


Figure 6.1: Original student string art.

Figure 6.2: Line Art

these subjects. Once they have some familiarity with the general principles, ideas, and methods of these arts then the palets can be expanded fruitfully.

6.2 Chronophotography

[Stuff here about chronophotography. Horse issue. This time and motion stuff has important sociological connections (Frank and Lillian Gilbreth). Sports pysiology.]

We get a *sequence* of *equally timed* images. If we run through these images in *linear* time we get - a movie, television images, cartoons and animation, video! A basic way to see this is using flip books. [There were a number of precusors to video - ???give examples here. Yet all were based on a sequence of images being replaced linearly in time.]

The most important early work on chronophotgraphy was done by Marey and Muybridge. These "time and motion"studies are still fundamental to artists and animators - as well as many connections to other areas. Human locomotion was one of the first areas studied. To determine the movements of a human walking Marey dressed a man in a black velvet suit and had reflective *lines* along his upper spine, arm, and leg, as shown on the left in Figure **??**. The result is the striking image on the right in Figure **??**. This image and similar ones due to Maybridge were the impetus for Marcel Duchamp's Nude Descending Staircase (1912), one of the more important works of the early *Modernist* movement in art. The relationship is clear, as is the fundamental role that lines play.

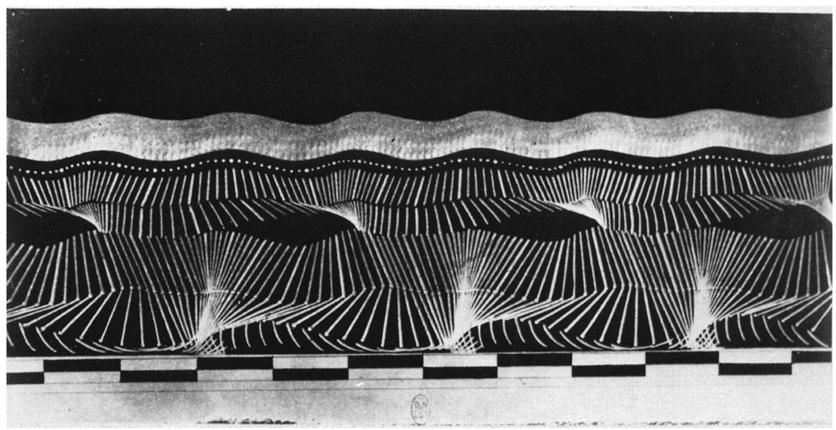


Figure 6.3: Man in black velvet and **tienne-Jules Marey** (French Scientist and Photographer; 1830 - 1904)

6.3 Representing Information

How we display and represent data is fundamental to what impact it has on us. Edward Tufte's reknown books are a testament to this. Because we have different purposes at different times in mathematics it is no less important here.

> To envision information - and what bright and splendid visions can result - is to work at the intersection of image, word, number, and art.[1]
>
> **Edward Tufte** (American Statistician; 1942 -)

[Brief introduction into multiple representation as foreshadowing.]

[1] From the "Introduction" to Envisioning Information

Figure 6.4: Nude Descending Staircase by **Marcel Duchamp** (French Artist; 1887 - 1968)

6.4 A First Arithmetic Pattern - The Darbi-i Imam Frieze

In Section 6.3 we talked about different ways to represent data. Figure 6.5 are models which represent the first three stages in the construction of a tile ***frieze*** (a pattern which extends periodically in one direction). Made out of ceramic tiles like those which are found in mosaics throughout the world, the particular frieze being modeled is the entry portal of the Darbi-i Imam Shrine in Isfahan, Iran. The original frieze is pictured in Figure 6.9. These models provide ***physical representation*** of an underlying pattern.

1. If the frieze pattern in Figure 6.5 keeps growing in the *evident*[2] way, draw major features of the next three stages in this pattern.

Suppose that you were building this tilework frieze. It is of interest to know how many tiles of each type will be needed.

[2]As noted in the Student Toolbox, any finite number of terms create infinitely many patterns. E.g. given the four terms $1, 3, 5, 7,$ this pattern can be extended as $1, 3, 5, 7, 9, 11, 13, \ldots$ as odds, or as $1, 3, 5, 7, 11, 13, 17, \ldots$ as primes, or as $1, 3, 5, 7, 1, 3, 5, 7, 1, 3, 5, 7, \ldots$ just because, or ... So, when you see the word *evident* way it is just a caveat that we're hoping you might see the pattern the way we intended and not some unique way of your own. We'll continue to emphasize the term as a reminder.

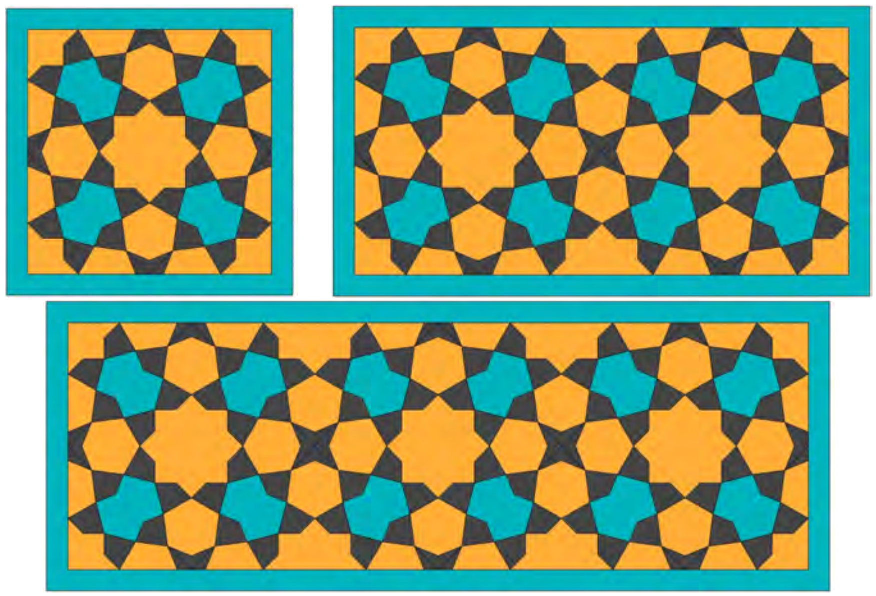


Figure 6.5: First three stages from a model of the *frieze pattern* on the Darbi-i Imam shrine.

One ***numerical representation*** for the number of turquoise octagons required to complete the different stages of this pattern is the ***sequence***:

$$4, 8, 12, \ldots$$

Each entry in a sequence is called a ***term*** in the sequence.

2. Assuming the frieze pattern continues in the evident way, what are the next four terms in the turquoise octagon sequence?

The terms in a sequence come in a specific order and we specify their location using ***subscripts***; if we use the letter t to represent the turquoise sequence then t_1 represents the first term, t_2 the second term, etc. I.e. $t_1 = 4, t_2 = 8, t_3 = 12, \ldots$

3. What is the tenth term in the turquoise octagon sequence? I.e. $t_{10} =?$

4. What is the fiftieth term in the turquoise octagon sequence? I.e. $t_{50} =?$

Another numerical representation for the number of turquoise octagons that make up this pattern is as a ***table of values***:

n	t
1	4
2	8
3	12
⋮	⋮

5. Fill in the indicated values in the table for the turquoise octagon:

n	t
1	4
2	8
3	12
4	$\boxed{}$
5	$\boxed{}$
⋮	⋮
25	$\boxed{}$
⋮	⋮
75	$\boxed{}$
⋮	⋮
1000	$\boxed{}$
⋮	⋮
n	$\boxed{}$

6. By filling in the line representing the number of tiles needed in the n^{th} stage you have found an ***algebraic representation*** for the pattern. Write this representation as an algebraic formula: $t_n = \boxed{}$.

One way to look for patterns in numerical data is to compare successive terms in a sequence or table by considering their ***first differences*** as illustrated in Figure Figure 6.4. We use the capital Greek letter Δ to represent the first differences.

n	t		Δ
1	4		
		$>$	4
2	8		
		$>$	4
3	12		
		$>$	4
4	16		
⋮	⋮		⋮

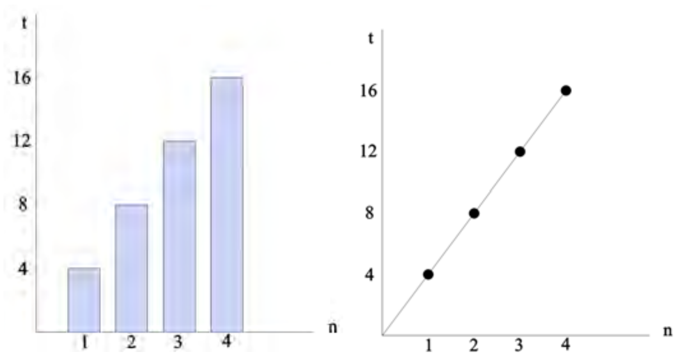


Figure 6.6: ***Cartesian graph*** and ***Bar graph*** of the turquoise octagon pattern.

The growth of a numerical pattern is called **arithmetic** whenever the first differences in a sequence or table of values is constant.

Graphical representations are also important. There are many ways to represent our turquoise octagon data as a graph, as illustrated in Figure **??**.
Notice that the variables t and n continue to represent the number of turquoise octagons and the stage number respectively. The standard convention is to have the *independent variable* on the horizontal axis and the *dependent variable* on the vertical axis. Typically the Cartesian graph more regularly than the bar graph.

7. Extend the graph of the turquoise data to include all values of $n \leq 8$.

8. The graph of the turquoise data in Figure 6.6 is called ***linear***. Explain why, providing a working definition of the term linear for use hereafter.

9. You should already know about the *slope* of a linear graph. Give a working definition of what the slope of a linear graph is. Then determine the slope of the graph in Figure 6.6.

10. You should know about the *vertical intercept* of a graph.[3] Give a working definition of what the vertical intercept of a graph is. Then determine the vertical intercept of the graph in Figure 6.6.

Functions of the form $f = m \cdot n + b$ where m and b are fixed numbers are called **linear functions**.

Now consider the beige hexagon tiles in the Darbi-i Imam frieze. Let's use the dependent variable b to represent the number of beige hexagons at each stage, counting only the number of whole hexagons that appear.

11. Represent the pattern of beige hexagons numerically as both a sequence and as a table of values. Provide six or eight terms of each.

12. Compute the first differences of both of the numerical representations in 11. Describe these first differences.

[3]When we denote the vertical axis by the dependent variable y the intercept is generally known as the y-intercept.

13. Is the pattern of beige hexagons in the frieze arithmetic?

14. Represent the pattern of beige hexagons in the frieze pattern graphically.

15. Is the graph in 14 linear? If not, how can it be described? If so, what are its slope and vertical intercept?

16. Represent the pattern of beige hexagons in the frieze pattern algebraically.

17. Is the function that describes the pattern of beige hexagons in the frieze pattern linear?

6.5 Paradigm Shift - The Darbi-i Imam Tessellation

In the study of ***crystals*** (e.g. diamonds, salt, ice, snowflakes, and quartz), five-fold symmetry was not seen and it was long thought that such symmetry could not exist in a natural crystal. This occurs because crystals that are created by nicely ordered structures all appear to have ***translational symmetry***, that is, they repeat periodically in a natural way.

Analogously, it was thought that in two-dimensional tilings, any collection of tiles that could *tessellate* could also be made to tessellate in a periodic way.

Mathematicians, physical scientists, artists, and craftspeople have thought about crystals and tilings throughout human history. So it was a paradigm shift when in the 1960's it became clear that ***aperiodic tiles***, tiles that would tessellate but could *never* be made to tessellate periodically, existed. In 1973 **Roger Penrose** (English physicist and mathematician; -) discovered a remarkably simple set of two tiles that were aperiodic. Shown in Figure 6.7, are Penrose's ***Kites and Darts*** which can be put together in many different ways to tessellate the plane, but cannot tessellate the plane in a periodic way.

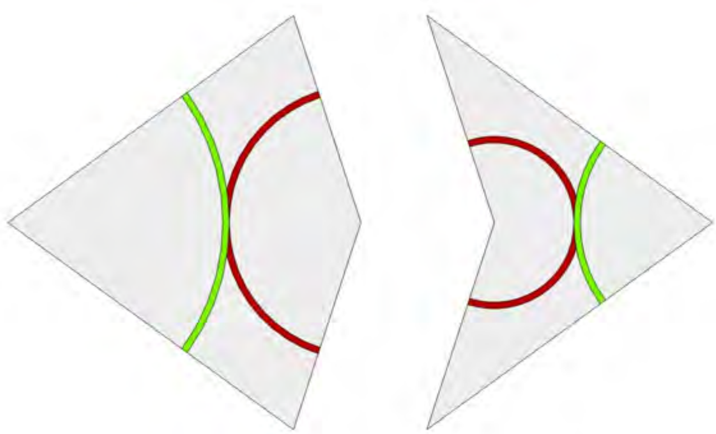


Figure 6.7: Penrose's kites and darts. (Note: Colors along edges must match for adjacent tiles.)

This celebrated discovery ushered in new urgency in the attempt to find three-dimensional analogues; naturally occurring ***quasicrystals***. They were found in 1982 by **Dan Shechtman**

(Israeli physicist; -). Unfortunately, they were so revolutionary to accepted scientific doctrine that Shechtman was, in his own words, "a subject of ridicule and lectured about the basics of crystallography. The leader of the opposition to my findings was the two-time Nobel Laureate Linus Pauling, the idol of the American Chemical Society and one of the most famous scientists in the world." Pauling went so far to say, in reference to Shechtman, "There is no such thing as quasicrystals, only quasi-scientists." Shechtman perservered, saying, "For years, 'til his last day, he [Pauling] fought against quasi-periodicity in crystals. He was wrong, and after a while, I enjoyed every moment of this scientific battle, knowing that he was wrong."[4]

Shechtman's perserverence paid off. On 5 October, 2011, he was awarded the Nobel Prize in Chemistry "for the discovery of quasi-crystals". The Press Release by the Nobel Committee is particularly interesting:

> **A remarkable mosaic of atoms**
>
> In quasicrystals, we find the fascinating mosaics of the Arabic world reproduced at the level of atoms: regular patterns that never repeat themselves. However, the configuration found in quasicrystals was considered impossible, and Dan Shechtman had to fight a fierce battle against established science. The Nobel Prize in Chemistry 2011 has fundamentally altered how chemists conceive of solid matter.
>
> On the morning of 8 April 1982, an image counter to the laws of nature appeared in Dan Shechtman's electron microscope. In all solid matter, atoms were believed to be packed inside crystals in symmetrical patterns that were repeated periodically over and over again. For scientists, this repetition was required in order to obtain a crystal.
>
> Shechtman's image, however, showed that the atoms in his crystal were packed in a pattern that could not be repeated. Such a pattern was considered just as impossible as creating a football using only six-cornered polygons, when a sphere needs both five- and six-cornered polygons. His discovery was extremely controversial. In the course of defending his findings, he was asked to leave his research group. However, his battle eventually forced scientists to reconsider their conception of the very nature of matter.
>
> Aperiodic mosaics, such as those found in the medieval Islamic mosaics of the Alhambra Palace in Spain and the Darb-i Imam Shrine in Iran, have helped scientists understand what quasicrystals look like at the atomic level. In those mosaics, as in quasicrystals, the patterns are regular - they follow mathematical rules - but they never repeat themselves.
>
> When scientists describe Shechtman's quasicrystals, they use a concept that comes from mathematics and art: the *golden ratio*. This number had already caught the interest of mathematicians in Ancient Greece, as it often appeared in geometry. In quasicrystals, for instance, the ratio of various distances between atoms is related to the golden mean.
>
> Following Shechtman's discovery, scientists have produced other kinds of quasicrystals in the lab and discovered naturally occurring quasicrystals in mineral samples from a Russian river. A Swedish company has also found quasicrystals in a certain form of steel, where the crystals reinforce the material like armor. Scientists are currently

[4] From "Ridiculed crystal work wins Nobel for Israeli" by Patrick Lannin and Veronica Ek, Reuters, 8/5/2011.

> experimenting with using quasicrystals in different products such as frying pans and diesel engines.

You may be surprised to see mention of the Darbi-i Imam shrine in this citation, aperiodic tilings and quasi-crystals as we have described them are recent discoveries.

Once again, as with the geocentric model of the solar systems and a non-flat earth, this is largely a function of our self-importance and our lack of respect for the brilliance of the ancient scholars, artists and craftspeople.

In 2005, while visiting the Middle East, graduate student **Peter Lu** (; -) became very interested in the tile-work on the Darbi-i Imam shrine. When he returned to Harvard he set to work studying these tilings. What he discovered was shocking:

> The asymptotic ratio of hexagons to bowties approaches the golden ratio τ (the same ratio as kits to darts in a Penrose tiling), an irrational ratio that shows explicitly that the pattern is quasi-periodic. Moreover, the Darbi-i Imam tile pattern can be mapped directly into Penrose tiles.[5]

Iranian craftspeople had predated the discoveries of Penrose by over 500 years!

The (combined) mathematical and artistic study of medieval tilings such as these is rich and beautiful.[6]

18. What are some other examples in science and/or mathematics where there were particularly personal attacks/disagreements between scientists?

19. What would you have done had you found yourself in Shechtman's place in this controversy?

6.6 Multiple Representations of Patterns

The frieze patterns above appeared as physical patterns. But we can just as well start with a function represented algebraically. For example,

$$f = 3 \cdot n + 7.$$

20. Is the function f linear?

21. Represent the function f numerically as both a sequence and as a table of values. Provide six or eight terms of each.

22. Compute the first differences of both of the numerical representations in Investigation **21**. Describe these first differences.

23. Is the numerical data formed by f an arithmetic pattern?

[5] From "Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture" by Peter J. Lu and Paul J. Steinhardt, *Science*, Vol. 314, 23 February, 2007, pp. 1106-1110.

[6] One fun place to start is building with these tiles. See `http://www.3dvinci.net/mathforum/GirihTiles_StudentVersion.pdf` where Google SketchUp is used to help create tilings with the tiles that are found in the Darbi-i Imam.

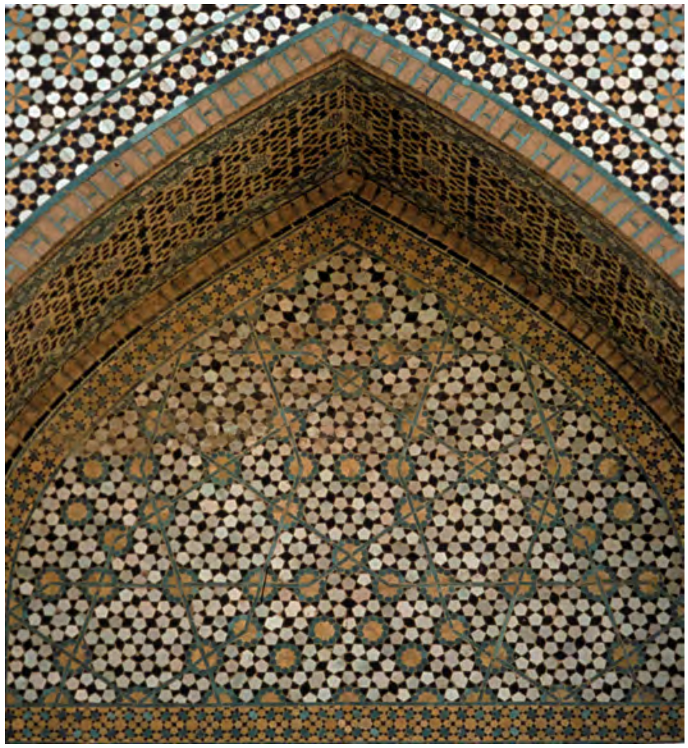


Figure 6.8: Entry portal from Dabri-i Imam in Isfahan, Iran. Photo courtesy of Peter Lu.

24. Represent the function f graphically.

25. Is the graph in 24 linear? If not, how can it be described? If so, what are its slope and vertical intercept?

26. What about physically? Suppose you worked with tiles. Is there a natural way to show how we can represent the function f physically as a growing pattern of tiles?

Now consider new data, which is given graphically as in Figure 6.11. Assume that this graph continues in the *evident* way.

27. Is the graph in 6.11 linear? If not, how can it be described? If so, what are its slope and vertical intercept?

28. Represent the function g numerically as both a sequence and as a table of values. Provide six or eight terms of each.

29. Compute the first differences of both of the numerical representations in 28. Describe these first differences.

30. Is the numerical data formed by g an arithmetic pattern?

Figure 6.9: Darbi-i Imam frieze detail.

Figure 6.10: Peter Lu.

31. Represent the function g algebraically.

32. Is the function g linear?

33. Represent the function g physically.

Consider the sequence s given by $7, 11, 15, 19, \ldots$

34. Represent the function s numerically as a table of values. Provide six or eight terms.

35. Compute the first differences of both of the numerical representations in 34. Describe these first differences.

36. Is the numerical data formed by s an arithmetic pattern?

37. Represent the function s graphically.

38. Is the graph in 37 linear? If not, how can it be described? If so, what are its slope and vertical intercept?

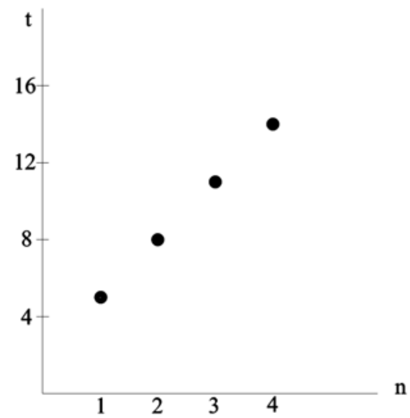


Figure 6.11: Graph of the data from a function g.

39. Can you find a way to represent the function s physically.

40. Represent the function s algebraically.

41. Is the function s linear?

Now it's time to make your own arithmetic pattern as a growing frieze. Figure 6.12 below shows a growing arithmetic pattern that is constructed using ***pattern blocks***, a collection of six different, brightly colored tile blocks that are often used in elementary school mathematics classrooms.

42. Use pattern blocks or one of the online pattern block applets (e.g. `http://nlvm.usu.edu/en/nav/frames_asid_170_g_2_t_3.html`) to build a growing frieze pattern where at least one of the types of blocks illustrates arithmetic growth.

43. Graph your arithmetic pattern.

44. Find the algebraic representation of your pattern.

6.7 Meta Patterns

We have been considering single functions/patterns to see if they were linear/arithmetic. Now we would like to see if there are patterns that unite what we have learned about these patterns. Such a pattern could be called a *meta pattern*.

45. If you have a linear function $f = m \cdot n + b$ what can you say about its graphical representation? Its numerical representations? Its physical representation? Explain.

46. If you have a function whose graph is linear, what can you say about its algebraic representation? Its numerical representation? Its physical representation? Explain

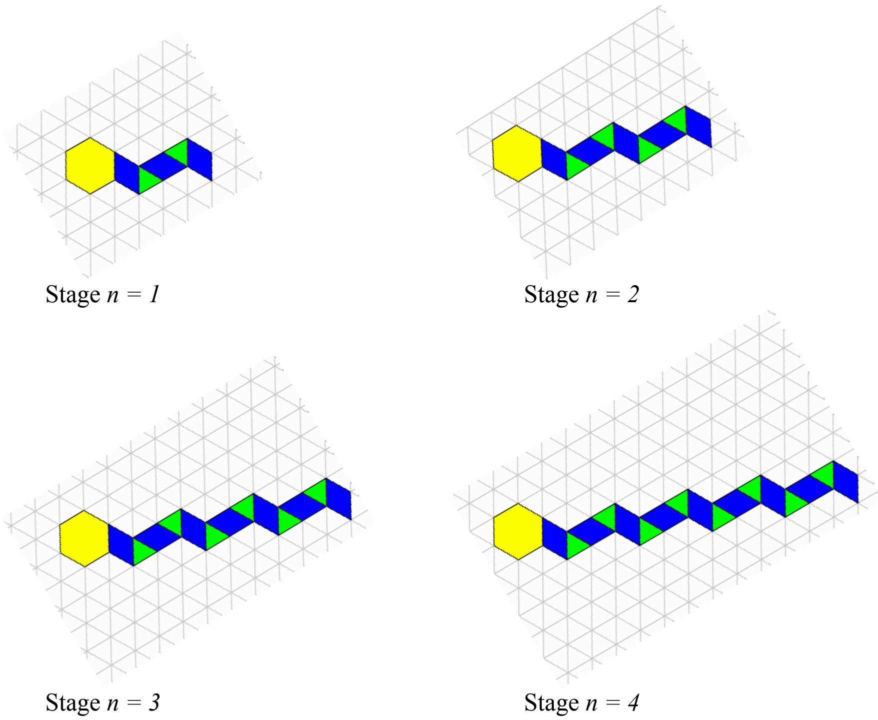


Figure 6.12: Growing caterpillar pattern constructed from pattern blocks.

47. If you have a function whose numerical data is arithmetic, what can you say about its algebraic representation? Its graphical representation? Its physical representation? Explain.

6.8 Rascals' Triangle - Another Meta Pattern

A famous pattern that you will investigate in an upcoming chapter is called *Pascal's Triangle*. The first 4 rows Pascal's triangle are given in Figure 6.13.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

Figure 6.13: The first 4 rows of Pascal's Triangle.

48. See if you can find patterns that would enable you to predict several more rows of Pascal's triangle. Write down the entries in these rows and explain your pattern.

Of course, without more information there are lots of different patterns that can start from this same starting point. This was one of the main lessons of the section "Patterns, Impostors and the Limits of Inductive Reasoning" in the chapter "Patterns and Proof."

In 2010 the question in Investigation **48** was asked on an I.Q. test. A student suggested that the pattern would be that shown in Figure 6.14. Thus began a wonderful, long-distance collaboration between two seventh graders and an eighth grader: **Alif Anggoro** (Indonesian student; circa 1998 -), **Eddy Liu** (American student; circa 1997 -), and **Angus Tulloch** (Canadian student; circa 1997 -) who are pictured in Figure 6.15.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 5 & & 4 & & 1 & \\
1 & & 5 & & 7 & & 7 & & 5 & & 1
\end{array}$$

Figure 6.14: The first 6 rows of Rascals' Triangle.

They invented the wonderful pattern called **Rascals' triangle**.

49. Find the next five rows in Rascal's triangle.

50. Anggoro, Liu and Tulloch found a beautiful "diamond pattern" that described this pattern. Using their terminology relating nearby entries

$$\begin{array}{ccc}
 & \text{North} & \\
\text{West} & & \text{East} \\
 & \text{South} & ,
\end{array}$$

find a pattern and then formula that relates North and South to East and West.

51. Solve your equation for South.

To form the next row in Rascals' triangle you need to create the next South entries. Because the equation relating West and East to North and South involves division, there is a risk that the division wouldn't result in a whole number, wrecking Rascals' triangle. Challenged by the teacher, Anggoro, Liu and Tulloch went to work trying to justify their triangle.

The formula for the "diamond pattern" is a ***recursive rule*** which gives new entires only in terms of previous entries. There is no way to figure out what happens far down in Rascal's triangle without working out all of the earlier entires. The boys set to work to find an *explicit, closed-term, algebraic formula* so they could determine any entry in Rascals' triangle just by knowing its location.

Figure 6.15: The Rascals - Alif Anggoro, Eddy Liu, and Angus Tulloch.

To do this we need a method to describe the locations in Rascals' triangle. The **rows** are the horizontal arrangements of entries, starting with Row 0 which has only the single entry 1. This means that Row 3 consists of the entries 1 3 3 1. This is part of the required information, now we need to specify where in each row entries are.

If one rotates Rascals' triangle 45 degrees counter-clockwise one sees very clear columns, as shown in Figure 6.16. We start counting with column $m = 0$ on the left.

1	1	1	1	1	1
1	2	3	4	5	6
1	3	5	7	9	11
1	4	7	10	13	16
1	5	9	13	17	21
1	6	11	16	21	26

Figure 6.16: Rotated version of Rascals' triangle.

Together with the information about rows, this gives sufficient information to describe all location in Rascals', and Pascal's, triangle. And this is how mathematicians typically describe these locations. But now that the triangle is rotated, it is easy to use the columns and location up and down in this array. For example in column $m = 3$ the $n = 0$ term is 1, the $n = 1$ term is 4, the $n = 2$ term is 7, the $n = 3$ term is 10, ...

WARNING: Here and below we will use the following conventions - we will denote the columns in Rascals' (resp. Pascal's) triangle by m, the general term in this column by n, when needed, the row number of the unrotated triangle by r. Mathematicians have good reason to give some prominence to the rows of the unrotated triangle. It's unfortunate that a fixed value of n also describes a row - a row of the rotated triangle.

52. The left-most 9, what column is it in (i.e. $m =$?) and what number entry (i.e. $n =$?) in this column is it?

53. The 17 in Figure 6.16, what column is it in (i.e. $m =$?) and what number entry (i.e. $n =$?) in this column is it?

54. What is the value of the entry in column $m = 6$, entry number $n = 4$ in Rascals' triangle?

55. What is the value of the entry in column $m = 3$, entry number $n = 5$ in Rascals' triangle?

56. What is the pattern in Column 1? Find an explicit, closed-term, algebraic formula which describes this pattern (denoted by f as above) as a function of n.

57. What is the pattern in Column 2? Find an explicit, closed-term, algebraic formula which describes this pattern (denoted by f as above) as a function of n.

58. What is the pattern in Column 3? Find an explicit, closed-term, algebraic formula which describes this pattern (denoted by f as above) as a function of n.

59. What is the pattern in Column 4? Find an explicit, closed-term, algebraic formula which describes this pattern (denoted by f as above) as a function of n.

60. What is the pattern in Column m? Find an explicit, closed-term, algebraic formula which describes this pattern (denoted by f as above) as a function of n.

This is precisely the explicit, closed-term, algebraic formula that is needed to determine any entry in Rascals' triangle directly from its location.

Using this formula we can now do what the Rascals did - prove that their triangle continues indefinitely without the specter of non-integer divisions creeping in.

Let $South$ be entry n in column m.

61. What is the value of the entry in $South$?

62. In what location is $East$?

63. What is the value of the entry in $East$?

64. In what location is $West$?

65. What is the value of the entry in $West$?

66. Compute $West \times East + 1$ and simplify as much as possible.

67. In what location is $North$?

68. What is the value of the entry in $North$?

69. Compute $North \times South$ and simplify as much as possible.

70. Explain why this shows your equation in Investigation **51** is valid and will always result in a whole-number solution for $South$.

The Rascals, and their free-spirited thinking, are now vindicated.

6.9 Connections

[Some of these maybe should be included above to break up the other sections.]

6.9.1 Pattern Block Patterns

It is difficult to overemphasize the power of simple manipulatives like pattern blocks to nurture the creative spirit. By all means, try to find the opportunity to create your own mosaic.

For those interested in teacher, there are many wonderful resources which describe or model the use of such manipulatives in elementary teaching. For examples, “Case 19: Growing Worms 1” and “Case 20: Growing Worms 2” in Discovering Mathematical Ideas: Patterns, Functions and Change Casebook by Deborah Schifte, Virginia Bastable and Susan Joe Russell.

6.9.2 Linear Programming

> The linear-programming was – and is – perhaps the single most important real-life problem.[7]
>
> **Keith Devlin** (; -)

Linear Programming Not sure where this is going to go in general, but there certainly needs to be a big hook here.

Put in a simple example? The lemonade example from way back in the day?

6.9.3 Origami

Origami EVERY one of these shapes is made by folding lines!!! There is nothing else.

6.9.4 Shadows

Light/CAT scans/Shadows All of these things are just the lines made by light. CAT scans are just lines and look what they tell us about our 3D bodies!!

6.9.5 Linear Regression

Linear Regression A fundamental application. That it is applied so much gives us some sense of how many things exhibit approximately linear growth.

6.9.6 Art

Perspective Drawing Give links to the appropriate sections in the Geometry book.

6.9.7 Calculus

Rates and Calculus Anything that is a rate is linear by implication. In other words, lines are what tell us all about Calculus. So there needs to be big hooks to this book.

> Points. Have no parts or joints. How can they combine. To form a line?
>
> **J.A. Lindon** (; -)

[7] From Mathematics: The New Golden Age, p. 605.

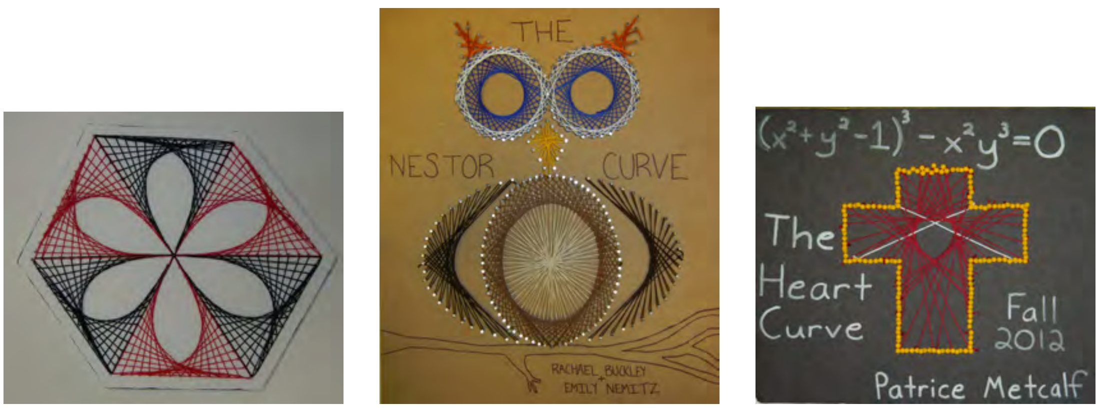


Figure 6.17: Original student string art.

Chapter 7

Higher Order Patterns and Discrete Calculus

In the two-dimensional plane there are three fundamental types of symmetries - *rotation symmetries*, *reflection symmetries*, and *translation symmetries.* In Chapter 10 the beautiful *rosettes* studied can have rotational and reflective symmetries, but no translation symmetries. In contrast, the *frieze patterns* like the Darbi-i Imam pattern in Chapter 6 can exhibit rotational symmetries, reflective symmetries and translation symmetries. Importantly, the growth of these patterns is linear precisely because they repeat periodically in just one direction. I.e. the translational symmetry appears just in one direction.

Just as artists, artisans, and builders have employed rosettes and frieze patterns, they have also utilized patterns that exhibit translational symmetries in two directions.

7.1 First Higher Order Patterns - Tilings and Crystals

Figure 7.1 shows the first three stages in a tiling that is made by square tiles, each square with a single red circle at its center. This tiling will grow and grow, filling every finite portion of the plane and is therefore called a ***wallpaper pattern***.

1. If the pattern in Figure 7.1 keeps growing in the evident way, draw or precisely describe the major features of the next four stages in this pattern. Explain carefully how you have determined the major features.

2. Using your description, compile data for the following table that gives the number of tiles,

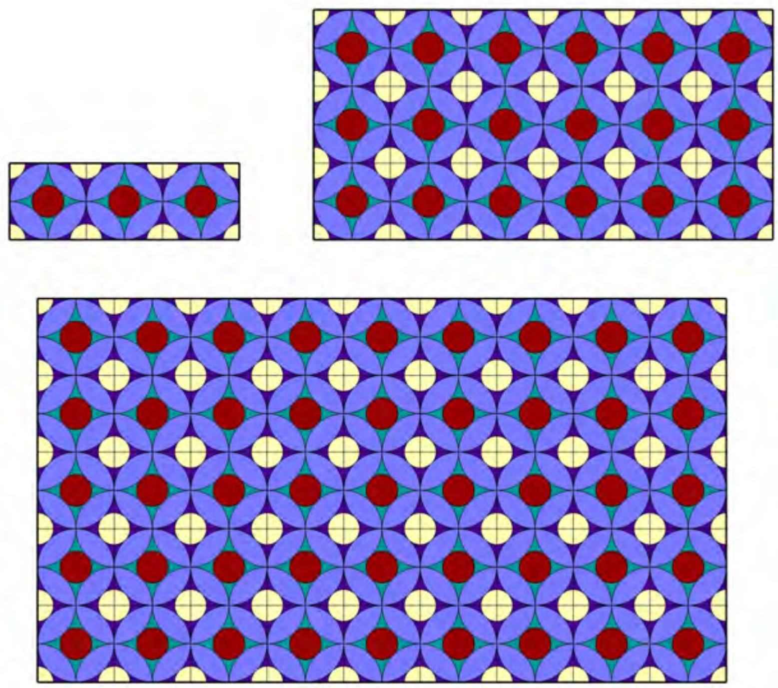


Figure 7.1: The first three stages in a wallpaper pattern.

t, needed in each stage, n:

n	t
1	3
2	18
3	45
4	
5	
6	
7	

3. In Chapter 6 you learned how to describe linear growth algebraically. This gives you the tools to create an algebraic formula for the number of tiles in each stage of this tiling pattern. Do so.

4. Plugging in several values of n, show that your formula agrees with the data in your table.

Similarly, Figure **??** shows the first three stages in a crystal that is made by cubes. This crystal will grow and grow, filling every finite portion of space.

5. If the pattern in Figure **??** keeps growing in the evident way, draw or precisely describe the major features of the next four stages in this pattern. Explain carefully how you have determined the major features.

6. Using your description, compile data for the following table that gives the number of cubes, c, needed in each stage, n:

n	c
1	15
2	108
3	315
4	$\square$
5	$\square$
6	$\square$
7	$\square$

7. In Chapter 6 you learned how to describe linear growth algebraically. This gives you the tools to create an algebraic formula for the number of cubes in each stage of this tiling pattern. Do so.

8. Plugging in several values of n, show that your formula agrees with the data in your table.

7.2 Uncovering Patterns by Factoring

The patterns in the previous section were given as physical representations. From these physical representations you found patterns that you could describe numerically and algebraically. The physical characteristics of the pattern held important clues for finding these other representations. As it is important to be able to move from one representation to another, the question is whether the information necessary to find an algebraic representation can be recovered without the clues of the physical representation.

9. Show how the data from the tiling from the previous section can be nicely factored into two terms, showing two clear patterns:

n	t	t factored
1	3	1×3
2	18	3×6
3	45	5×9
4	$\square$	$\square \times \square$
5	$\square$	$\square \times \square$
6	$\square$	$\square \times \square$
7	$\square$	$\square \times \square$

10. Consider the numerical sequence $15, 35, 63, 99, 143, \ldots$ Make a table of values for this data.

11. Factor this data nicely into pairs of factors.

12. Find patterns in this data that allows you to find a formula for the smaller factor and a formula for the larger factor.

13. Combine these formula to give an algebraic representation for this pattern. Plug in several values of n to check your algebraic representation.

14. Suppose you were only given the data in table of values in Investigation **??**. Can you factor this data in a way that would enable you to find an algebraic representation of this data? Explain in detail.

15. Factor the values f in the table of values below and then find patterns in these factors:

n	f	f factored
0	0	
1	2	
2	6	
3	12	
4	30	
5	42	

16. Find an algebraic representation for this pattern. Plug in several values to check your algebraic representation.

17. Factor into three factors the values f in the table of values below and then find patterns in these three factors

n	f	f factored
0	0	
1	2	
2	6	
3	12	
4	30	
5	42	

18. Find an algebraic representation for this pattern. Plug in several values to check your algebraic representation.

7.3 Characterizing Symmetry Groups

There are *only* 17 wallpaper groups.

There are *only* 7 frieze patterns.

There are infinitely many different symmetry groups for rosettes, but there are only two different types - cyclic and dihedral. (Are these terms actually used in the spirograph chapter? They need to be.)

Really interesting study. Related to flexagons. Abstract algebra.

Is there already a note somewhere in the patterns book about this? Maybe early in the Teacher Manual. Maybe put something here for students.

7.4 JF Notes

Something about crystal growth that would give a physical representation of a cubic? It would also link back to asymmetry stuff from the previous chapter. So we have a connection due to multiple representations but also for the historical/physical part?

> The tantalizing and compelling pursuit of mathematical problems offers mental absorption, peace of mind amid endless challenges, repose in activity, battle without conflict, refuge from the goading urgency of contingent happenings, and the sort of beauty changeless mountains present to senses tried by the present-day kaleidoscope of events.
>
> **Morris Kline** (American Mathematician; ??? - 1992)

> It is the duty of all teachers, and of teachers of mathematics in particular, to expose their students to problems much more than to facts.
>
> **Paul Halmos** (??? Mathematician; ??? - ???)

> Almost every American who has a degree, however ignorant, can live better than even competent people in much poorer countries around the world... But this cannot last long in the situation when “competence” and a diploma tautologically mean each other. The advantages enjoyed by Americans are the results of real competence and real efforts of previous generations... And someday ignorant people with degrees and diplomas may want power according to their papers rather than real competence. We Russians have some experience of this sort... It is clear to me right now that the winners in the modern world will be those countries which will really teach their students to think and solve problems. I sincerely wish America to be among these.
>
> **Andre Toom** (Russian Mathematician; 1942 - ???)

> The problem is not that there are problems. The problem is expecting otherwise and thinking that having problems is a problem.
>
> **Theodore Rubin** (American Psychiatrist; 1923 - ???)

> The best way to escape from a problem is to solve it.
>
> **Alan Saporta** (???; ??? - ???)

> We only think when confronted by a problem.
>
> **John Dewey** (American Educator; 1859 - 1952)

> When I am working on a problem, I never think about beauty. I only think of how to solve the problem. But when I am finished, if the solution is not beautiful, I know it is wrong.
>
> **Buckminster Fuller** (American Architect; 1895 - 1983)

7.5 Introduction

We all face problems each and every day. What an amazing thing that brains have developed to help us solve some of these problems. It is a miraculous thing. Give stories about amazing things that animals can do to help them solve problems, showing sorts of higher order thinking skills.

We take such "routine" problem solving as routine. But it is not. Think of all of the problems that have been solved to make life as "simple" as it is. 5,000 years ago there was no metal. Now for a few days wages you can afford to wear your iPod onto an airplane that will fly at 50,000' at 600 miles per hour and take you to New York City, a city where over 8 million people live in an area of just over 300 square miles. The problem solving that has enable this to happen is immense.

(COOL!! Historical, philosophical, and intellectual context!)

Because solving problems is so important, humankind has thought a great deal about how we solve problems. Meno, etc. Links to the philosophy stuff here.

Socrates/Plato

Meno?? (See p. 99 of Hersch; knowledge is innate. Links to our pedagogical style.)

Descartes Discourse on Method, Optics, Geometry, and Meterology. His four simple rules were philosophical analogues of the revolutionary views that were fanning the Protestant revolution in parallel. Philosophy tied with deep mathematics and optics. Great quote by D'Alembert on p. 111 of Hersch.

The rainbow is such a remarkable phenomenon of nature, and its cause has been so meticulously sought after by inquiring minds throughout the ages, that I could not choose a more appropriate subject for demonstrating how, with the method I am using, we can arrive at knowledge not possessed at all by those whose writings are available to us. Rene Descartes, attributed in The Rainbow Bridge (p. 182) to De l'arc-en-ciel from Discours de la methode. Wonderful that this was part of the Discourse.

Bacon - Scientific Revolution

Polya - How to Solve It.

Kuhn - Structure of Scientific Revolutions

Lakatos - Proofs and Refutations.

Must include links to deeper questions about the way you reason limiting what you can reason about. This latter stuff deserves more complete airing in the Proof book. This is a great way to delineate the split here. Write much of this as one piece. Then dole it out in smaller measure more tailored to problem solving for the Patterns book and it broader measure more philosophically/epistemologically in Proof book.

Prominent Philosophers Links to Mathematics

Husserl (phenomenology) wrote his Ph.D. on the Calculus of Variations under Weierstrass; Bertrand Russell; Descartes - Cogito and Cartesian Geometry; Newton - A philopher? Leibniz - How to describe this in an accessible way? Frege, Hilbert, and Godel - Do many philosophers or philosophy students even read about them? Kant - Much of his discussion of synthetic a priori is geometrical. (Links to Meno?) Plato; Pythagoreans; C.S. Pierce - Father of Pragmatism; important mathematician; Wittgenstein, Godel and the Vienna Circle; Lakatos first worked with Renyi in 1954 at the Hungarian Mathematical Research Institute as a translator. One of these books was Polya's How to Solve It. In 1958 he met Polya who later becam his dissertation advisor and thus Proofs and Refutations was born. Spent more time on Philosophy of Science afterward, including arguing with Kuhn. (See Hersch, pp. 208-216.)

So maybe you don't want to be somebody who thinks about how we solve problems: a philosopher, a psychiatrist, or an educational theorist. About now you might be bringing out the oft-used contemporary mantra, "When will I need to know this?" Show me the money! We hope that your work through the material in these guides will help you develop an understanding of why this is an unfortunate and limiting attitude. Whether you have arrived at this point yet, will arrive there, or even differ with us - regarding learning in general and mathematics in particular - nobody can deny that they will always need to know how to solve problems. Each day brings forth a wealth of problems to solve. And there are many benefits to solving problems:

POLYA!!!!!!!!!!!!!!!!!

> The value of a problem is not so much coming up with the answer as in the ideas and attempted ideas it forces on the would be solver.
>
> **I.N. Herstein** (??? Mathematician; ??? - ???)

> Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime.
>
> **George Polya** (??? Mathematician; ??? - ???)

> Solving problems is a practical art, like swimming, or skiing, or playing the piano if you wish to become a problem solver you have to solve problems.
>
> **George Polya** (; -)

> The problems that exist in the world today cannot be solved by the level of thinking that created them.
>
> **Albert Einstein** (German Physicist; 1879 - 1955)

7.6 Three Motivating Problems

Our work in this lesson will be motivated by three problems:

The Handshake Problem: A number of people are to be introduced to each other by shaking hands. If each person shakes hands with every person (excluding themselves) exactly once, what is the total number of handshakes that must be are made?

The Circle Problems: On a circle n points are drawn. Lines are drawn which connect each of the points to all of the other points on the circle. Circle Problem 1: How many lines must be drawn to connect all of the points? Circle Problem 2: Into how many regions is the circle decomposed by the lines?

Example:

Four points yielding six lines and eight regions.

???Image here

The Line Problem: On a plane n lines are drawn. Each line intersects every line exactly once and no more than two lines intersect at a single point. Line Problem 1: How many points of intersection are formed by the lines? Line Problem 2: Into how many regions is the plane divided by the lines?

Example:

Four lines yielding six points of intersection and eleven regions.

???Image here

These are your problems to solve. You should spend significant time working on these problems before you move on. As wrestle with these problems, be reminded of what John Dewey told us:

> No thought, no idea, can possibly be conveyed as an idea from one person to another. When it is told it is to the one to whom it is told another fact, not an idea... Only by wrestling with the conditions of the problem at first hand, seeking and finding his own way out, does he think.

Indeed, the whole purpose of this series is to get you working on problems, to get you thinking and for you to be mathematically active. Here the problems are both clear enough and meaty enough that we can give them to you without much guidance. We are confident that you can make significant headway on these problems on your own.

7.7 Problem Solving Strategies

Here we provide some strategies that can be used to help solve ??? and related problems. These strategies are presented in the same guided discovery framework that typical investigations are. In how much detail you decide to consider these sections depends on your success in finding patterns in the original problems, your success in completing the investigations, and the directions/requirements of your teacher.

Did you spend an hour working on the problems above? Have you solved them to your satisfaction? If not, go back and do this. This chapter will not be successful if you have not done this. And then what? It depends on you, your teacher, and the nature of your solutions to these problems. Our hope is that in your investigation of these problems you have found sufficiently robust ideas, strategies, relationships, and patterns that you can solve the problems in the penultimate section "Using what you have discovered/learned." For now skip ahead to this section and try out some of these investigations. Yes, we said, SKIP AHEAD TO THE END. You'll know when and if you need to come back

So you're back. Yeah, we expected you might be. There is a reason for the topics and investigations in between. Your idea of "solving" these problems may not have been robust enough to help you solve all of the investigations at the end. Your strategies might be limited, allowing you to solve only certain problems in the final section. Etc. The intervening sections provide guided prompts for a variety of different strategies that are both typical and effective means of solving the three problems you've been working on. Maybe you need some helping solving one or more of the three problems - work through a section that may seem to be related to your strategy if you are stuck. Try one of these if you need a push. Work through some of these sections if your strategies did not apply to some of the investigations at the end and you need a new approach. Or just work through these intervening sections to see what you can discover there. It's up to you. (And perhaps your teacher.) Additionally, there are many ways that you can choose to work on these problems cooperatively with peers, including the wonderful Jigsaw Classroom method. ???Footnote here.

We learned all of these things from students!!!!!!!!!!! These are not ideas handed down, they did it. Their diversity reflects the diversity of our experiences, our learning styles, and our ideas.

Big Idea - The diversity of ways that you can solve a single problem. Moreover, how very much you can learn if you look at things from a different perspective. Some strategies give rise to whole classes of problems that can be readily solved by these means while not being useful to other whole classes of problems.

Teacher notes: The premise of the Problem Solving Strategies section is that students will have many of their own ideas and strategies. But these subsections should demonstrate that there is a great deal of important mathematics that can be accessed via these problems. As a teacher it will be your job to adjust pacing, emphasis, assignments, classroom dynamics, and the like. Here are some suggestions for you: Jigsaw Classroom Assign specific sections of interest, leave the rest to students as necessary. Extensive work only on the problems at the outset and leave this sections to students as necessary. Assessment on some sections but not others. Assessment only on the final section.

7.7.1 Problem Solving Strategy: Collecting Data

In each of the ??? problems above there is a variable - the number of people, points, and lines - which is indeterminate. You need to solve the problem no matter how many of each they are. A natural response to this is to collect some data and see what this quantitative data might tell you.

Let's do that here.

Below are circles with two, three and four points on the circle. These points are connected by straight lines as shown.

With n, P, R, and L, representing the stage number, number of points, number of regions, and number of lines, the properties of the figures can be tabulated as follows:

n 1 2 3 4 5 P 2 3 4 L 1 3 R 2 4

19. Determine the values of R and L for the 3rd stage.

20. Assuming the pattern continues in the indicated way, draw the 4th stage in this pattern.

21. 3. Use 2) to determine the values of P, R, and L for the 4th stage.

22. 4. Now draw the 5th stage in the pattern.

23. 5) Use 4) to determine the values of P, R, and L for the 5th stage.

24. 6) Make a conjecture which describes how we can predict the value of P as an explicit function of the stage number n.

25. 7) Make a conjecture which describes how we can predict the value of R as an explicit function of the stage number n.

26. 8) Find and describe a pattern in the values of L as a function of the stage number n. Is it easy to find an explicit function which describes L as a function of n?

Now let's switch to the lines problem.

27. 9) Draw two lines, extending indefinitely, that are neither concurrent nor parallel.

28. 10) How many points of intersection are there in the pair of lines you drew in 9)?

29. 11) Draw three lines, extending indefinitely, so that no pair are concurrent, no pair are parallel, and only two lines cross at any point of intersection.

30. 12) How many points of intersection (where at least two lines meet) are there among the three lines you drew in 11)?

31. 13) Repeat 11) for four lines.

32. 14) How many points of intersection (where at least two lines meet) are there among the four lines you drew in 13)?

33. 15) There is a pattern in your answers to 10), 12), and 14). Describe it.

16) Use 15) to make a conjecture about the number of points of intersection if you draw five lines, six lines, and seven lines.

Now let's try the handshake problem where you might want to have a small group of people nearby to test things out.

How many handshakes will there be when there are

17) two people in the room?

18) three people in the room?

19) four people in the room?

20) ... five people in the room?

21) Complete the table of values below. You should see a pattern forming. Describe the pattern in detail.

n	h
2	
3	
4	
5	
6	
7	

22) How is this problem related to the problems considered in Sections I and II? Explain in detail. In particular, if you were to make tables for the data of these earlier problems, how would they compare?

7.7.2 Problem Solving Strategy: Gauss's Epiphany

Carl Friedrich Gauss German Mathematician (1777; 1855 - i)s, without doubt, one of the greatest mathematicians that ever lived. His work is explored in detail in Discovering the Art of Number Theory in this series. This strategy and group of explorations is named after an epiphany widely attributed to him:

From The Joy of Mathematics: Discovering Mathematics All Around You by Theoni Pappas, Wide World Publishing/Tetra, 1986.)

23) Use this method to determine how many handshakes there are if there are 27 people in a room.

24) Use this method to determine how many points of intersection there are if 53 lines in the plane are drawn according to the approach in 7.7.2.

25) Use this method to determine how many lines are needed to connect 85 points around a circle according to the approach in Section 7.7.1.

Each of the problems 23) - 26) involve determining the value of an arithmetic series of the form:

$$1 + 2 + 3 + \quad + (n - 1) + n.$$

26) Use Gauss's method to determine a formula for the series $1 + 2 + 3 + \quad + (n - 1) + n$ as a function of the variable n.

27) Check that the formula for in 26) provides the correct answers for investigations 23) - 26).

28) Each of the investigations in 23) - 26) involved odd numbers so the value of n used in 27) are all even. What potential difficulty is there when n is odd? Resolve this difficulty and precisely describe your resolution.

29) What is the sum of the series

$$1 + 2 + 3 + \quad + 1,345,217 + 1,345,218 + 1,345,219?$$

30) What do you think of this method?

7.7.3 Problem Solving Strategy: Blocks and Other Manipulatives

In the illustration below, a student determines the value of 1+2+3+4+5+6 using Multilink cubes - plastic cubes that join together. By creating two identical staircases out of 1+2+3+4+5+6 blocks each and joining them together to form a 7 by 6 rectangle, she shows that $1 + 2 + 3 + 4 + 5 + 6 = 42/2 = 21$. [1]

31) Use this method to determine how many handshakes there are if there are 27 people in a room.

32) Use this method to determine how many points of intersection there are if 53 lines in the plane are drawn according to the approach in Section 7.7.2.

33) Use this method to determine how many lines are needed to connect 85 points around a circle according to the approach in Section 7.7.1.

Each of the problems 31) - 33) involve determining the value of an arithmetic series of the form:

$$1 + 2 + 3 + \quad + (n - 1) + n.$$

34) Use Gauss's method to determine a formula for the series $1 + 2 + 3 + \quad + (n - 1) + n$ as a function of the variable n.

35) Check that the formula for in 34) provides the correct answers for investigations 31) - 33).

36) What is the sum of the series

$$1 + 2 + 3 + \quad + 1,345,217 + 1,345,218 + 1,345,219?$$

37) What do you think of this method?

[1] From Essentials of Mathematics: Introduction to Theory, Proof, and the Professional Culture by Margie Hale, Mathematical Association of America, 2003.

7.7.4 Problem Solving Strategy: Combinatorics - The Art of Counting

The important online mathematical encyclopedia mathworld.wolfram.com defines combinatorics as follows: Combinatorics is the branch of mathematics studying the enumeration, combination, and permutation of sets of elements and the mathematical relations that characterize their properties. More colloquially, combinatorics is the mathematical art and science of counting. It is a critical tool in many areas of mathematics and relies heavily on patterns. Hmmm..., it must be exactly what we need to solve the ??? handshake problem since it is all about counting.

Here's an example of combinatorial reasoning. A town sports league has each team play every other team exactly twice, once as the home team and once as the visiting team. How many games must they schedule? With 2 teams there are clearly only 2 games. With 3 teams you can check that there are 6 games. And with 4 teams there are 12 games. What if there were 24 teams? It seems complicated. But we can reason as follows. As a home team, each team must play 23 games, one with each of the teams in the league. Since this is true for each team and there are 24 teams, there are 23 24 = 552 games.

38) Describe precisely what's wrong with the following argument for determining the number of handshakes with 12 people in the room: Each person must shake hands with 11 other people. Since there are 12 people, there are 12 11 = 132 handshakes.

39) Despite it's incorrectness, the method in 38) can be adapted to provide the appropriate result. Explain.

40) Use 39) to determine the number of handshakes if there are 27 people in a room.

41) Use 39) to determine the number of handshakes if there are 532 people in a room.

42) Use 40) to determine to determine a formula for the number of handshakes as a function of the number n of people in a room.

43) Suppose that 53 lines in the plane are drawn according to the approach in Section II. Translate this problem into a handshake problem and determine the total number of points of intersection.

44) Suppose 85 points around a circle are connected with lines according to the approach in Section I. Translate this problem into a handshake problem and determine the total number of lines.

45) What do you think of this method?

7.7.5 Problem Solving Strategy: Discrete Calculus

In Topic ?? we saw that the first differences of the terms in a numerical sequence are constant precisely when the sequence is generated by a linear equation of the form (i.e. a function of the form $f = mn + b$.) In Discovering the Art of Calculus in this series we extend this result to a much more general pattern at the heart of a discrete calculus:

Fundamental Theorem of Discrete Calculus Let S be a sequence. The sequence of k^{th} differences is the lowest degree of constant first differences if and only if the original sequence is generated by a k^{th} degree polynomial function.

We can use this profitably to solve problems like those considered in Sections I - III.

46) Copy the table of data from Investigation 21), denoting the dependent data by f instead of h. Fill in the first differences.

47) Does this data belong to a linear equation? Explain.

48) On the table in 46) fill in the second differences.

49) What does 48) tell you about the type of function that generates the handshake data?

You should remember the quadratic formula from high school algebra. It is used to calculate the roots of the general quadratic function . This function is also known as the general 2nd degree polynomial function, which you can now use. You must only determine the values of a, b, and c.

50) Use results from earlier sections to determine appropriate values of a, b, and c.

Here's similar data that also has constant second differences:

n	f		1st Δ		2nd Δ
0	2				
		>	4		
1	6			>	2
		>	6		
2	12			>	2
		>	8		
3	20			>	2
		>	10		
4	30			>	2
		>	12		
5	42			>	2
		>	14		
6	56				

This data must also be described by a quadratic function .

51) Substitute $n = 0$ into the quadratic and solve for c.

52) Substitute $n = 1$ into the quadratic to generate an equation containing only the variables a and b.

53) Repeat 52) with $n = 2$.

54) Solve the equations in 52) and 53) simultaneously to determine the values of a and b.

55) Write out the quadratic explicitly and show that it correctly generates the appropriate values for $n = 0, 1, 2, , 6$.

56) What do you think of this method?

7.7.6 Problem Solving Strategy: Factoring

Often we cannot see patterns simply because there are several intertwined processes at work that can only be understood once they are isolated. For number sequences, factoring can often help us untangle these processes so we can describe the underlying processes that give rise to the pattern.

57) Let's return to the pattern in Section VI:

n f Factors of f 0 2 1 6 2 12 112 or 26 or 34 3 20 4 30 5 42 6 56

Fill in the remaining factors of f that have not been filled in.

58) Choosing appropriate factors from each row, there is a very regular pattern to the factors. Highlight these factors and describe the pattern precisely.

59) Describe the pattern of smaller factors from 58) as a function of n.

60) Describe the pattern of larger factors from 58) as a function of n.

61) Suitably combine 59) and 60) to determine an explicit formula for the pattern f as a function of n. Check that this function provides the appropriate data for $n = 0, 1, 2, , 6$.

We'd like to use this approach for the handshake problem as well.

62) Use 21) to fill in the table below and then determine factors of the data for f in the middle column: n h Factors of h 2 3 4 23 5 6 7

63) You should see a clear pattern among some subset of the factors. Describe it precisely.

The pattern in 63) is not easy to describe directly. It is easier if we force a factor of into each term. In other words, instead of $6 = 23$ we write . 64) Rewrite the table in 62) forcing a factor of into each of the factors you have. Precisely describe the pattern in the factors that you see.

65) Following the examples of 59) - 61), determine an explicit formula for h as a function of n.

66) Compare your expression for h in 65) with the results of previous sections.

7.7.7 Problem Solving Strategy: Algebra

Note that this builds on the Factoring section.

7.7.8 Problem Solving Strategy: Recursion

One meaning of the word recur is to happen again. While we have not defined patterns, there is something inherent in our understanding of this term that in a pattern something happens again, and again, and again,... Mathematicians have adopted this root and use the word **recursion** to name a process in which objects are defined relative to prior objects in the same process. They also use the term ***recursive*** as the associated adjective.

While you may not have heard the term before, the importance of recursion in the world around us cannot be understated. Populations, weather, account balances, and many other real phenomena we might study are dependent at any stage on their size, behavior, distribution, and makeup at prior stages. Anybody who has used a spreadsheet has used recursion when they define a new cell using information in other cells. Recursion underlies the develoment of fractal and chaos described in Discovering the Art of Geometry.

In an important sense, to discover a recursive relationship is to really see what it is that makes the object being studied a pattern. They can also be helpful in problem solving.

At each stage, add... Do a table with first differences shown. Do it in English. Do it using recursive notation. Multiple representations!!! It is done physically already. Factorial as a choosing problem. Links with the fundamental counting principle. Need to say something about how one can solve recursive relationships. This is huge and there is an entire branch of mathematics devoted to this. ???Some ILL books are coming on this. Find a good linear growth pattern among Frieze patterns in a famous place.

???Need a note that the triangular numbers are the additive analogue of factorials. Not sure if this needs to be here or elsewhere.

???Teachers Notes - Note about the most basic recurrence relations subsuming a tremendous amount of mathematics. $r_n = n \cdot r_{n-1}$ gives rise to factorial,$r_n = n + r_{n-1}$ gives rise to triangular numbers, $r_n = k \cdot r_{n-1}$ gives rise to exponential growth, and $r_n = k + r_{n-1}$ to arithmetic growth.

34. A joke among mathematicians is that a dictionary had the following definition:

recursion \ ri-'k ər-zh ən\ n See "recursion".

Explain this joke. To be funny jokes generally have some kernel of truth in them. What underlying truth does this joke point out about languages and/or dictionaries?

In 7.7.5 we represented the data in our pattern numerically in a table and considered the associated differences. In this case we have

n	f		1st Δ
1	8		
		$>$	5
2	13		
		$>$	5
3	18		
		$>$	5
4	23		
		$>$	5
5	28		
		$>$	5
6	33		

7.8 Trying Out What You Have Discovered

Now comes your chance to utilize the strategies you have discovered and/or learned. You should be able to answer each of the questions in this section. If not, you should go back and work to find a method that you can use. Also, please note that most problems can be solved using several different methods.

100) How many handshakes are there if there are 27 people in a room?

101) How many handshakes are there if there are 532 people in a room?

102) Determine a closed-term, algebraic expression for the number of handshakes (represented by the dependent variable h) there are if there are n people in a room.

103) Determine a recursive relationship for the number of handshakes when there are n people in the room (represented functionally by) as a function of the numbers of handshakes with fewer people in the room (denoted by respectively).

104) How many points of intersection are there if 53 lines as described in the Line Problem?

105) How many points of intersection are there if 264 lines as described in the Line Problem?

106) Determine a closed-term, algebraic expression for the number of points of intersection (represented by the dependent variable p) there are if there are n lines drawn as described in the Line Problem.

107) Determine a recursive relationship for the number of points of intersection (represented functionally by) as a function of the number of points of intersection when there are fewer lines (denoted by respectively).

108) How many lines are needed to connect 85 points around a circle as described in the Circle Problem?

109) How many lines are needed to connection 388 points around a circle as described in the Circle Problem?

110) Determine a closed-term, algebraic expression for the number of lines needed (represented by the dependent variable l) if there are n points connected as described in the Circle Problem.

111) Determine a recursive relationship for the number of lines needed (represented functionally by) as a function of the numbers of lines needed with fewer points around a circle (denoted by respectively).

112) Determine the sum of the series 5 + 10 + 15 + 20 + + 7,895.

113) Determine the value of the sum of the following series: 1 + 3 + 5 ++ 2101 + 2103 + 2105.

114) Determine the value of the sum of the following series: 1 + 4 + 7 ++ 158509 + 158512 + 158515.

115) Determine the value of the sum of the following series: 8 + 10 + 12 ++ 11212 + 11214 + 11216.

116) Determine the value of the sum of the following series: 284 + 291 +298 + 305 + + 3056 + 3063 + 3070.

117) Each of the series in 112) - 116) are called arithmetic series. In general, such a series can be written as: . Determine a closed-term, algebraic function for the sum of this series as a function of the parameters a, d, and n.

118) Use the formula in 117) to check your answers to investigations 112) - 116).

As long ago as ancient Greek mathematicians such as Pythagoras (circa 580 - 500 BC), and probably longer, people looked at patterns of numbers created by shapes. These, the so-called figurate numbers, are illustrated in the figures below.

119) Show that the numbers formed by the stages in the triangle, the Triangular Numbers, are the same as the Handshake Numbers.

120) Show that the numbers formed by the stages in the square, the Square Numbers, are 1, 4, 9, 16, 25, as we might expect.

When we generated the handshake numbers one way we did this as sums of series. In this series the first difference between consecutive terms is always 1. Namely,

1 = 1 1 + 2 = 3 1 + 2 + 3 = 6 1 + 2 + 3 + 4 = 10

121) Show how the square numbers can be written as sums of series.

123) What are the first differences between the terms that make up the series?

124) You should see a pattern forming. Illustrate this pattern using the Pentagonal Numbers.

125) Repeat 124) to create the Octagonal Numbers.

Perhaps surprisingly, the Natural Numbers 1, 2, 3, 4, 5, can also be formed in this way:

126) Show how the natural numbers can be written as sums of series where the terms are constant with value 1.

127) Using 126) and the geometric patterns investigated above to explain visually why it makes sense to call the natural numbers of the Linear Numbers.

Morgan's Theorem That is perfect for an aside. It seems nicely related to the stuff that is going in here. Why not have it in this section?

7.9 Concluding Essays

35. Need to have some closing essays. Maybe something about problem solving? It would be nice to have something that forced them to reflect on the readings at the outset. Have them find a quote on problems and/or problem solving.

36. Why were there four problems at the outset?

37. We expect that you used several different strategies to solve the problems 7.8. If you look back at the strategies that were described 7.7 there were eight strategies. Is there some value in having so many different strategies? Explain.

7.10 Further Investigations

The Line Problem can be refined and extended in many ways. Beautiful patterns continue to emerge and the problems range from the level of the Line Problem to areas of open research questions.

F1. Repeat your analysis of the Line Problem by determining how many *unbounded regions* are formed by the lines.

F2. Repeat your analysis of the Line Problem by determining how many *bounded regions* are formed by the lines.

F3. Amoung the bounded regions, different shapes may be formed. Can you determine the types and number of each of these shapes?

Instead of allowing our lines to be placed arbitrarily, we can arrange the lines regularly so they form regular polygons in their center, as shown below.[2]. ???Insert p. 118 figure 11.1 from Pedersen.

F4. Into how many unbounded regions is the plane divided by the lines?

F5. Into how many bounded regions is the plane divided by the lines?

7.11 Teachers Manual - General

For the Line Problem students who collect data should readily find the following:

$n = L$	P	R
0	0	1
1	0	2
2	1	4
3	3	7
4	6	11
5	10	16

Inductive argument for the points is that each line must intersect each other line in one NEW point. So we have $p_n = p_{n-1} + (n-1)$.

Inductive argument for the regions is that when a new line is drawn, each time it intersects an old line (or infinity), one region has be dissected into two. It must intersect exactly n-1 lines and infinity once, giving n new regions; i.e. $r_n = r_{n-1} + n$

What should be clear from the algebraic comparison of these recursive relationships or, more likely, from students' observations of the data, is that $r_n = p_{n-1} + 1$ and so again, solving one problem essentially solves the other.

As indicated in the Further Investigations, one can extend these problems in many ways.

One possibility is to consider the different types of regions formed. The number of unbounded regions is exactly 1, 2, 4, 6, 8,..., 2n,... as each pair of lines forms an unbounded region on the

[2]Adapted from "Platonic Divisions of Space" by Jean Pedersen in Mathematical Adventures for Students and Amateurs

"left" and on the "right". While the shapes of the bounded regions can change (with $n = 3$ there is one bounded region, a triangle, and with $n = 4$ there are three bounded regions, 2 triangles and a quadrilateral, however, with $n = 5$ it is possible to have configurations both with and without a pentagon), the numbers of bounded regions is accessible.

It should be clear that patterns pervade this chapter. It is also interesting to note that each of the strategies in 7.7 is based on a pattern:

7.7.1	$\leftrightarrow$ Numerical Patterns
7.7.2	$\leftrightarrow$ Matched Pairs Yield Fixed Sums
7.7.3	$\leftrightarrow$ Risers of Fixed Height; 2 Staircases = 1 Rectangle
7.7.4	$\leftrightarrow$ Fundamental Counting Principle for Counting
7.7.5	$\leftrightarrow$ Theorem is a Meta-Pattern
7.7.6	$\leftrightarrow$ Factors form a Numerical Pattern
7.7.7	$\leftrightarrow$ All Terms Share Common Factors
7.7.8	$\leftrightarrow$ Algebraic Description of Pattern

Chapter 8

Pascal's Triangle

> When I consider the small span of my life absorbed in the eternity of all time, or the small part of space which I can touch or see engulfed by the infinite immensity of spaces that I know not and that know me not, I am frightened and astonished to see myself here instead of there... now instead of then.
>
> **Blaise Pascal** (French Mathematician, Physicist, and Philsopher; 1623 - 1662)

> Nature is an infinite sphere, whose center is everywhere and whose circumference is nowhere.
>
> **Blaise Pascal** (; -)

> When we cite authors we cite their demonstrations, not their names.
>
> **Blaise Pascal** (; -)

> If God does not exist, one will lose nothing by believing in him, while if he does exist, one will lose everything by not believing.
>
> [This line of reasoning has become known as ***Pascal's wager***]
>
> **Blaise Pascal** (; -)

Certainly need a better title than that.

Good to have lots of stuff about Pascal's remarkable diversity. Stuff about his theoretical computer. Etc. He was a Rennaisance Man - one that in some sense our return to the Arts is trying to encourage.

8.1 Investigations

8.1.1 The Circle Problem

We begin with a problem that looks similar to those in the previous chapter:
The Circle Problems On a circle n points are drawn. Lines are drawn which connect each of the points to all of the other points on the circle.

Circle Problem 1: How many lines must be drawn to connect all of the points?

Circle Problem 2: Into how many regions is the circle decomposed by the lines?

1. Solve Circle Problem 1.

2. Collect some data for Circle Problem 2 into a table. Do you see a pattern? If so, make a conjecture which suggests a solution to Circle Problem 2.

3. Use your conjecture from 2 or estimate how many regions you think there will be when there are $n = 6$ points.

4. Carefully draw a circle and locate $n = 6$ points symmetrically around the boundary. Carefully draw lines and determine how many regions will be formed. How does your answer compare to your prediction from 3?

5. Carefully draw a circle and locate $n = 6$ points around the boundary so that when lines are drawn between the points no more than two lines intersect at a single point. How many regions are formed? How does your answer compare to your prediction from 3?

6. How do your answers to 4 and 5 compare? What does this tell you about inductive reasoning?

7. More generally, what do your answers to 4 and 5 tell you about Circle Problem 2? What do they say about the (mis)perception that in mathematics there is always a "correct" answer?

If we are to continue to consider Circle Problem 2 we will need to determine which "pattern" we will focus on. Following the example of the Line Problem in the previous chapter, we will consider the case where no more than two lines intersect at a single point. When lines (or planes or other higher dimensional *linear subspaces*) are arranged in this way they are said to be in ***general position***.

From now on, we will consider only the general position case and we will call this problem **Circle Problem 2(GP)**.

8. Determine how many regions there are for Circle Problem 2(GP) when there are $n = 7$ points. For $n = 8$ points there are $R = 99$ regions. How many regions are there for $n = 9$ points?

(Hint: Neglecting the circle itself, the lines can be arranged to form the *complete graph* on 7 and 9 vertices which are easily found online. Question: Why do you think we didn't ask you to do this with $n = 8$?)

9. Can you find a way to describe the "pattern" that solves Circle Problem 2(GP)? Either i) describe your solution, or, ii) articulate the ideas you have tried as well as the difficulties you have encountered.

We will return to this problem/pattern later and see some beautiful ways to describe it.

8.1.2 Binomial Coefficient Calesthenics

8.1.3 Patterns in Pascal's Triangle

8.1.4 Polya's Block Walking

Include several of the summation formulas here for them to discover.

Some ideas for the introduction includes:

Polya's blockwalking idea - I remember doing this in a class one time. It was not that successful. It is hard to illustrate how to do this. Maybe do it with Ls and Rs for lefts and rights so the students can keep track of it? Maybe this is one of those things that it is better to let the students do the problem solving on their own. Don't tell them too much how to do it.

Pizza problems - I have always thought that this is nice. Might be good to have them do this for contrast.

Binomial coefficients - Maybe do this as a way to say, "'OMG, I wish I had know how to do this so long ago when I was in an algebra class!!"

In her Mathematics and Music book Chrissi does this counting the number of different rhythms of given count length with a given number of beats!!! Make a note of this in this book.

Also make a note of the dance connection with the number of different ???? of a given number of steps.

The **binomial coefficients** are defined by

$$\begin{matrix} n \\ k \end{matrix} = \frac{n!}{(n-k)!k!}.$$

8.1.5 Return to the Circle Problem

We've been using binomial coefficients to describe patterns having to do with combinatorics. Circle Problem 2(GP) is about counting regions, and it turns out, with some ingenuity, we can use them to help solve this problem as well. You'll investigate two different ways to do this below.

We'll see later (resp. we saw earlier) that Euler's formula is a beautiful pattern which simply describes the relationship between vertices, edges, and faces of certain polyhedra. While this was a result about solids, it has a natural analogue for planar regions:

Euler's Theorem for Networks For any *simply connected network* the number of vertices (v), edges (e), and faces (f) are related by [v-e+f=1.]

The circles and lines in Circle Problem 2(GP) form just these types of networks and the regions simply correspond to faces. So instead of counting regions, if we can count vertices and edges we can solve the problem.

10. Begin by collecting some data. Namely, use your figures from ??? to complete the table below. Notice that what we have called lines now consist of several edges and we must include the curves between boundary points as edges as well.

n	L	R	v	e
1	0	1	1	1
2	1	2	2	3
3	3	4	3	6
4				
5				
6				

11. The new data is the vertex and edge data. Are there any immediate patterns that you can decipher in this data? Explain.

Let's start with the vertices. There are n around the boundary of the circle. How many others?

12. How are the vertices which are not on the boundary formed?

13. In the Figure below two interior vertices have been named as indicated. 1425 is so named because it is at the intersection of the lines $\overrightarrow{14}$ between points 1 and 4 and $\overrightarrow{25}$ between points 2 and 5. Name all of the other interior points in this manner.

Sometimes one can solve a problem by translating it into a slightly different problem that can more easily be approached. Maybe we can use this naming scheme to find out how many interior vertices are created.

14. The point 1425 can also be named 2541. It has several other names, give as many of these as you can.

15. Describe what all of the names for the given point in 14 have in common.

16. Is the name 1245 an appropriate name for a vertex? If not, why not? If so, which vertex?

17. Given four numbers of vertex labels, in no particular order, how many different verticies can these labels name?

18. Use what you have learned about binomial coefficients and choosing to determine an expression for the number of interior vertices.

19. Return to 8 and check to see whether your expression gives the correct number of vertices when $n = 7$ and $n = 9$.

Now what about the number of edges?

20. Copy Figure ??? above. At each vertex, write the number of edges that is incident on this vertex. The number is called the ***degree*** of the vertex. What do you notice?

21. How does your answer to 20 change if the number of boundary points, n, is changed? Describe this change in detail.

Let's begin counting...

22. Sum the degrees of the interior vertices. Explain why this sum represents the number of terminal points of edges that the interior vertices can accomodate.

23. Sum the degrees of the boundary vertices. Explain why this sum represents the number of terminal points of edges that the boundary vertices can accomodate.

24. What is the total number of terminal points that the vertices can accomodate?

25. Since each edge has a beginning and ending point, you should now be able to determine an expression for the number of edges. Describe and explain your result.

26. Return to 8 and check to see whether your expression gives the correct number of edges when $n = 7$ and $n = 9$.

27. Use Euler's formula and your recent results to solve Circle Problem 2(GP). Leave your expression for the number of regions in terms of binomial coefficients.

28. This is harder and a bit more involved. But for those that would like to prove that the number of regions is as shown in **??**, there is an ingenious scheme which develops unique names for each the regions developed by John H. Conway and Richard K. Guy. The structure of the names allows one to count the number of regions by their names in a straightforward combinatorial way.

Read the outline on pp. 76-79 of The Book of Numbers. Describe this naming scheme in your own words and then provide examples. Explain why each region has a unique name using between zero and four of the numbers $\{0, 1, 2, ..., n-1\}$. Use basic combinatorial arguments to then complete the proof.

$$\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4}$$

and show that these answers agree with the collected data. Does the form of this equation suggest why the pattern one would inductive expect does not hold starting at stage $n = 6$?

29. This must include a discussion of why the 5-term expression shows exactly why 2^n fails.

30. This ends with a comparison of the two expressions.

31. More regions stuff goes here.

8.2 Conclusion

Linear programming can be viewed as part of a great revolutionary development which has given mankind the ability to state general goals and to lay out a path of detailed decisions to take in order to "best" achieve its goals when faced with practical situations of great complexity.

George Bernard Dantzig (American Mathematician; 1914 - 2005)

This easy-to-state example [how to best assign 70 men to 70 different jobs] illustrates why up to 1947, and for the most part even to this day, a great gulf exists between man's aspirations and his actions. Man may wish to state his wants in complex situations *in terms of a general objective to be optimized but there are so many different ways to go about it, each with its advantages and disadvantages, that it would be impossible to compare all the cases and choose which among them would be the best. Invariably, man in the past has turned to a leader whose 'experiences' and 'mature judgment' would guide the way.*

George Bernard Dantzig *(; -)*

Undertanding the way the lines in Fig. ??? intersect the *feasible region* labelled R is the key to finding an optimal solution to a *two-variable linear optimization program.* This program may be the levels of production of two different items competing for scarce resources or the assignment of two jobs to two employees. Indeed, the geometry inherent in this figure is an archetype for linear programming, an contemporary area of mathematics which distinguished science writer and National Public Radio "Math Guy" describes as:

Perhaps the single most important real-life problem.

Keith Devlin (English Mathematician; 1947 -)

.

Suppose we asked you to extend your investigations of the Line Problems and the Circle Problems to Plane Problems and Sphere Problems? Could we encourage you to think about analogous problems in four-dimensional space or even five-dimensional space? Although we can imagine such spaces mathematically and learn a great deal about them (see e.g Discovering the Art of Geometry in this series) this may seem like an dry, academic pursuit.

But in fact, understanding the interaction between *hyperplanes* and *polytopes* in higher dimensions is the fundamental geometry that underlies linear programming. The polytopes may have hundreds of thousands of sides and there may be tens of thousands of *decision variables* giving rise to a problem in tens of thousands of dimensions.

The end result of linear programming? Job statistics. Obaminoes. Airline scheduling. Waste. Give references. (Devlin? H and L).

In addition to the mathematical connections, the confluence of needs, ideas, tools, and personalities that gave rise to linear programming is worthy of attention.

Often called the "Father of Linear Programming," George Bernard Dantzig

Also, this stuff become important because of the availability of computers. So it will be nice to have this in the chapter about Pascal!!!!!!!!!!!!!!!!!!!!! Volker had an interesting idea about how this illustrates how many different things need to come together to solve a problem, make a discovery, change a paradigm. For LP to happen we needed people thinking deeply about geometry, we needed the computers, and we needed a context set by a real problem that was begging for a solution. "Confluence" of ideas, opportunities, needs, and developments. Note about his name - George Bernard for Shaw; parents wanted him to be a writer. Brother named Henri Poincare for a mathematician. Mention the homework problem thing. This is the set-up for Good Will Hunting. See Snopes.com.

Indeed, in 1975 the Nobel Prize in Economic Sciences was awarded to Professor Leonid Kantorovich and Tjalling Koopmans who:

> largely independent of one another, have renewed, generalized, and developed methods for the analysis of the classical problem of economics as regards the optimum allocation of scarce resources.
>
> (; -)

Koopmans was quite distressed that Danzig did not share in the prize. His "great intellectual honesty" in trying to make this right is described by Michel L. Balinski in "Mathematical programming: Journal, society, recollections" from History of Mathematical Programming: A Collection of Personal Reminiscences by J.K. Lenstra, A.H.G. Rinnooy Kan and A. Schrijver, Elsevier, 1991.

8.3 Further Investigations

F1. One can use the methods of **??** to determine the *degree* of the polynomial that solves Circle Problem 2GP and, with quite a bit of somewhat intense algebra, determine an expression for this polynomial. Using your data from 8 to make a table of 1st, 2nd, 3rd, 4th, and 5th differences. What does this table tell you?

F2. Use algebra to show directly that the two expressions for the number of regions in 30 are indeed equal. Compare the relative merits of your approach here to that in 30.

F3. Now that you have had some experience with some fairly sophisticated counting you might be able to count the number of regions directly. Namely, show that each line after the first creates one new region at the outset and then as many more regions as intersections with previously drawn regions. Then use your knowledge of the number of lines and the number of intersections to count the total number of regions. [1]

8.4 Teacher's Manual

A very interesting alternative approach to Pascal's triangle is through probability. This approach, in a guided discovery approach similar to ours, is used quite effectively by Harold R. Jacobs in Chapter 8 of Mathematics: A Human Endeavor.

It should be noted that this approach is not only pedagogically interesting, but has a basis in applications. The Hardy-Weinberg Equilibrium Principle is a fundamental principle of population genetics. It relies on binomial coefficients and the interpretation of these coefficients in *generating functions*. Weingerg is **Wilhelm Weinberg German physician** (???; ??? - w)ho was credited with the simultaneous discovery only many years later in 1943. Hardy is the famous mathematician **G??? H. Hardy ???** (???; w - h)o thought it somewhat absurd that this principle had not been realized prior to his description of it in 1908:

> I am reluctant to intrude in a discussion concerning matters of which I have no expert knowledge, and I should have expected the very simple point which I wish to make to have been familiar to biologists. However, some remarks of Mr. Udny Yule, to which Mr. R. C. Punnett has called my attention, suggest that it may still be worth making...[2]

Punnett is **R.C. Punnett ???** (???; ??? - w)ho invented *Punnett squares* that we all study in high school biology. Punnett and Hardy played cricket together and it is said that Hardy's discovery was made in this setting.

It is instructive to consider the forms of the answers to Circle Problem 2(GP). A quick analysis of the binomial coefficients show the number of regions will be degree 4 in n. Indeed, for those students who have considered 7.7.5 will be interested to consider the sequence of difference here:

[1] This approach, due to Marc Noy, is sufficiently original and noteworthy that it was published in the Mathematical Association of America journal *Mathematics Magazine* in 1996.

[2] *Science*, Vol. XXXVIII, July 10, 1908, pp. 49-50

n	R		1st Δ		2nd Δ		3rd Δ		4th Δ
1	1								
		>	1						
2	2			>	1				
		>	2			>	1		
3	4			>	2			>	1
		>	4			>	2		
4	8			>	4			>	1
		>	8			>	3		
5	16			>	7			>	1
		>	15			>	4		
6	31			>	11			>	1
		>	26			>	5		
7	57			>	16			>	1
		>	42			>	6		
8	99			>	22				
		>	64						
9	163								

Hence, students who investigate using this route will see from the Fundamental Theorem of Discrete Calculus that the number of regions is a quartic equation.

However, determining the coefficients is an mechanical, uninformative process that leads us to the algebraic expression for the number of regions given by:

$$R = \left(\frac{1}{24}\right) n^4 - \left(\frac{1}{4}\right) n^3 + \left(\frac{23}{24}\right) n^2 - \left(\frac{3}{4}\right) n + 1.$$

Certainly the expressions using binomial coefficients are more aesthetically pleasing, easier to work with (30), and far more instructive (29).

The data for 10 is:

n	L	R	v	e
1	0	1	1	1
2	1	2	2	3
3	3	4	3	6
4	6	8	5	12
5	10	16	10	25
6	15	31	21	51

8.4.1 Selected Answers

Answer to 17: Each set of four vertex labels uniquely defines an interior vertex.

Answer to 18: The number of interior vertices for any n is $\binom{n}{4}$.

8.4.2 Bibliography

The link between the Circle Problem and Linear Programming was suggested parenthetically by David M. Bressoud in http://www.maa.org/columns/launchings/launchings_12_07.html . He attributes much of this activity to George Polya from the video Let us Teach Guessing published by the Mathematical Association of America.

8.4.3 Random Notes

http://mathforum.org/library/drmath/view/51708.html http://en.wikipedia.org/wiki/Dividing_a_circle_into_areas p. 187 of Maurer and Ralston's Discrete Algorithmic Mathematics. Some related ones in the exercises. Seems that it is Bassarear, Chapter 8 - Basic Concepts of Geometry. Exploration 8.4 for example. Did I put any of this into a file somewhere and that is why I do not have it printed out?

Chapter 9

Library of Patterns

Throughout this volume we have emphasized the view that mathematics is the science of patterns. We have also tried to help you see more deeply how patterns play a central role in nature, in the arts, in the sciences, and throughout the human experience.

Before continuing with detailed investigations of other patterns, this is a good juncture to pause and celebrate both the centrality and diversity of mathematical patterns in the world around us. We could do this by including a nice ??? of full-color photos. But this would be expensive. More importantly, it would put you in the role of simply an observer of mathematics while we have tried to have you be center of the action in this journey.

So your job at this point is to create a permanent collection of patterns - a **Library of Patterns** if you will. This library of patterns should be considered a central part of this book. Along with your notebook, it is a way of personalizing your mathematical journey.

There are several different ways we can suggest for you to create a library of patterns. Maybe you have your own ideas. Or maybe your guide will have some ideas or requirements.

9.1 Image Gallery

Throughout we have talked about these explorations as a journey and you as an explorer. Explorers usually come back with pictures, why not here?

Need to flush this out some more. Maybe talk to a few people about it.

Mention digital cameras, cellphones, etc. Should also be allowed to use images that they find elsewhere. It is pretty hard to take a picture of the solar system with the planets shown moving in elliptical orbits.

Again, can do this physically on hallway walls, in a display case or bulletin board, in the classroom.

As they are likely to be all digital, it is perfect for a wwwebsite.

9.2 Poster Gallery

At our college one way we create a library of patterns is for students to create a gallery of posters, each poster celebrating a different pattern. The posters are hung in Department of Mathematics hallways and classrooms for all to see - including students and faculty from other classes who are equally excited to see the many connections. Each poster hangs for a week or more until it is replaced by another student's poster. Over the course of the semester each student's poster has an opportunity to enlighten all those who pass by it. A sign-up sheet insures that all students choose different pattern. Over the course of the semester this means a general group of explorers (class) can create a gallery of 30 patterns. Additionally, each poster is accompanied by a one-page handout. Explorers collect each of these handouts and together they form a permanent record of our library of patterns.

Such a poster session can also be done electronically with posters posted online allowing the world access to your Poster Gallery of Patterns.

For those of you who have not ever seen a poster session, they are widely used to publicize, announce, and/or present the results of research investigations. They are used in professional conferences (including virtually all conferences for mathematicians and scientists), college and university courses, and meetings of all kinds. They are useful because many posters can be displayed without the time and space limitations that traditional presentations impose. Additionally, it makes it easier for participants to browse and find research of interest.

For each explorer's pattern it the poster should:

- Describe the pattern in detail, using a variety of different representations as appropriate.
- Describe why this pattern is of interest to you and may be of interest to others.
- Describe the importance of this pattern - why it appears, what is signifies, how it evolves, etc.
- Describe how this pattern can be analyzed, represented, applied, adapted, and/or related to other patterns of interest. You are free to choose topics whose mathematical analysis is much more sophisticated than you and I can currently understand - this is a survey poster. Indeed, I would encourage you to chose a topic vivaciously - the more interesting the better.
- Provide several references where the interested reader can find more information about this pattern. This should include not only print and Internet references, but also interactive online programs, methods and/or tools for constructing this pattern physically, museums where the patterns can be observed, etc.
- Include a partial history of the development and/or genesis of this pattern.

To keep everybody involved, each poster is self-reviewed, peer reviewed, and reviewed by the teacher. They are evaluated in five categories, which your group may decide to adjust however you see fit. Our categories are:

- An interesting, engaging, and/or important choice of pattern which adds vitality and breadth to our Library of Patterns.
- An informative and accessible description of this pattern, discussion where the pattern arises, description of the importance of this pattern, why this pattern is of interest to you and others, the pattern's impact, applications of this pattern, and the genesis of this pattern.
- An accessible survey of the mathematical analysis of this pattern, including as many of it's representations as appropriate. This is the more mathematical portion of your poster.

- An appropriate collection of additional information interested readers can use to pursue the topic in greater depth. These may include: book, journal, audio, video, and other media and multi-media citations; Internet resources; reviews; museum holdings; event dates; etc.
- A physical construction of a high quality poster and handout, including: appropriate design, pleasing visual layout, effectiveness, appropriate mix of media and information, effort, etc.
- A useful, interesting, appropriate handout for our library of patterns.

We have found this to be a wonderful way to celebrate patterns. We invite your group of explorers to try it as well.

Chapter 10

Circles, Stars, Gears, and Unity

The eye is the first circle; the horizon which it forms is the second; and throughout nature this primary figure is repeated without end. It is the highest emblem in the cipher of the world.
Ralph Waldo Emerson (American Philosopher and Essayist; 1803 - 1882)

Symmetry, as wide or as narrow as you define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection.
Hermann Weyl (German Mathematician; 1885 - 1955)

Figure 10.1: 19th century Tibetan mandala of the Naropa tradition; Rubin Museum of Art.

10.1 Introduction

A title for this chapter could have been "Man and His Symbols" after the last full-length work Man and His Symbols of the preeminent psychologist **Carl Gustav Jung** (Swiss psychologist;

1875 - 1961). Written to introduce general readers to his theories, mandalas played a central role.

> I had to abandon the idea of the superordinate position of the ego...I saw that everything, all paths I had been following, all steps I had taken, were leading back to a single point – namely, to the mid-point. It became increasingly plain to me that the mandala is the centre. It is the exponent of all paths. It is the path to the centre, to individuation...I knew that in finding the mandala as an expression of the self I had attained what was for me the ultimate.[1]

The word ***mandala*** derives from the Sanskrit word for "circle. Mandalas were typically thought to originate from Eastern traditions, but the term is used more generally since it was popularized by Jung who pointed out that mandala-like objects appear in central roles throughout history in most cultures.

The mandala in Figure 10.2 is "Die Philosophie mit den sieben freien Künsten" ("Lady Philosophy and the Seven Liberal Arts") from Hortus Deliciarum (Garden of Delights) by **Harrad von Landsburg** (French and German abbess, artist and author; 1130 - 1195). The book was written as an encyclopedic compendium of all that was known at that time. Lady Philosophy is in the center, attended by Socrates and Plato in dialogue. They are surrounded by the seven areas that make up the liberal arts: grammar, rhetoric, logic, music, arithmetic, geometry and astronomy. As described in "Preface: Notes to the Explorer" that opens this book, this was the view of organized knowledge at this time. Von Landsburg's mandala pays homage to this view of knowledge.

The mandala in Figure 10.3 is the *yantra* for Discovering the Art of Mathematics. It borrows heavily from von Landsburg's image as well as the "Vitruvian Man" by **Leonardo da Vinci** (Italian artist, musician, inventor, mathematician and writer; 1452 - 1519). The image in the middle represents you, our mathematical explorer, at the center of the mathematical experience. Surrounding you are many of the continents of mathematics – each of the eleven books in the Discovering the Art of Mathematics series symbolically represented.

Jung focused on the human aspects of mandalas. The aesthetic and mathematical aspects of mandalas are found throughout nature as well.

Throughout this chapter are images that illustrate the diversity of mandalas and mandala-like figures that appear in cultural, artistic, symbolic and natural roles in our world.

1. Find a dozen mandalas or mandala-like images/objects that are fundamentally different than those that appear in this chapter. For each show the image and give a brief description of both what it is and why it is important.

Our main focus in this chapter is the investigation of mathematical properties of mandala-like objects. What connects these objects mathematically is the type of symmetry that they exhibit.

10.2 Point Symmetries and Rosettes

Most of the images included in this chapter exhibit what is known as *point symmetry*. Objects that exhibit point symmetry are generally called ***rosettes***. These are the types of mathematical mandalas that we will investigate.

[1] From Memories, Dreams, Reflections.

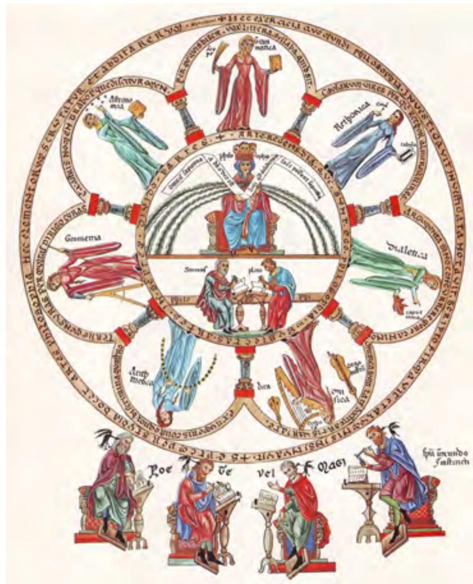


Figure 10.2: "Lady Philosophy and the Seven Liberal Arts" by Harrad von Landsburg.

It is important to note that there are other important families of symmetries that have play essential roles in the arts throughout history. These include the *line symmetries*, whose objects are known as *friezes*, and *plane symmetries* whose objects are known as *tessellations*. The study of these symmetries is wonderful and we encourage you to consider investigating them sometime. These symmetries play essential roles in many types of dance and you can learn about these connections in the chapters "Symmetry in Mathematics and Dance" and "Contra Dancing and Permutations" in Discovering the Art of Mathematics - Dance.[2]

Consider the image in Figure 10.7 as a real pinwheel with a fixed center about which the eight identical and equally spaced arms may freely rotate.

2. Determine several specific angles through which the pinwheel can be rotated leaving it looking exactly the same as in Figure 10.7.

3. Explain why there are, in fact, infinitely many angles that the pinwheel can be rotated and have it appear exactly the same as it does in Figure 10.7.

4. Find *all* angles greater than 0 and less than *or* equal to 360 degrees through which the pinwheel can be rotated and have it appear exactly the same as it does in Figure 10.7.

We call this symmetry of the pinwheel in Figure 10.7 ***rotational symmetry*** and say that the figure has 8-fold rotational symmetry because there are eight different ways it can be rotated and

[2]Other fine books that consider these topics in include Groups and Symmetry: A Guide to Discovering Mathematics by David W. Farmer and Symmetry, Shape and Space: An Introduction to Mathematics Through Geometry by L. Christine Kinsey and Teresa E. Moore.

Figure 10.3: Discovering the Art of Mathematics Yantra by Julian F. Fleron, George Ramirez, and Elaine Devine.

still appear exactly the same as when it started. It is important to remember that we count 360 degree rotational symmetry as it serves as the *group identity* as the number 1 does in multiplication. So, for example, $\Uparrow$ has one rotational symmetry - 360 degrees.

Consider the image on the left in Figure 10.9.

5. Place a mirror on one of the dotted lines. When you look toward the mirror how does the half of the figure reflected in the mirror together with the half not obscured by the mirror compare to the original image?

6. On how many other *different* lines can you put the mirror so you see the same type of ***reflection symmetry***?

We say that the figure in Figure 10.9 has 6-fold reflection symmetry.

An object has **point symmetry** if it has either rotational symmetry and/or reflection symmetry.

7. Determine and describe all of the symmetries of each of the objects pictured in Figure 10.4, Figure 10.5, Figure 10.6, Figure 10.8, Figure 10.10, Figure 10.11, Figure 10.12, and Figure 10.14. (Please ignore minor variations and background clutter, concentrating on the larger scale makeup of the images.)

8. For each of your mandala-like objects in Investigation **1** determine and describe all of their symmetries. (You may have to make some adjustments or omissions. For example, when the

Figure 10.4: Snowflake photographs: one of the earliest recorded (circa 1900 by **Wilson A. Bentley** (American Farmer and Photographer; 1865 - 1933)) and a modern image (by **Kenneth Libbrecht** (American physicist; -)).

11 mathematical icons, inner square and Vitruvian Man are removed from Figure 10.3 the resulting figure has 11-fold rotational symmetry and 11-fold reflection symmetry.)

9. Find a dozen objects/images that are of interest to you and who exhibit point symmetry. Describe them and then draw them - at least schematically. Then determine the symmetries of each of these objects/images.

Collected together the symmetries of a given object/image is called the object's/image's ***symmetry group***. The structure of symmetry groups is essential to modern mathematics and its applications to many areas.

10. Based on your many examples from this section, make several conjectures that attempt to characterize the different structures that point symmetry groups can have. You should discover a perhaps surprising pattern which applies to *all* rosettes.

We hope you are curious about the morphogenesis of this pattern - that point symmetry groups can only have these certain structures. This is a rich and provocative area for exploration. You can explore it on your own or with the help of the wonderful book Groups and Symmetry: A Guide to Discovering Mathematics by David Farmer.

From here we will head toward particular types of artistic rosettes - star polygons and Spirograph curves - whose morphogenesis is explained by beautiful patterns.

10.3 Star Polygons

Unc, why do they call them stars when they are really round?

K.C. Fisher (American child at age 2; 1983 -)

For these investigations you will have to enlist the help of peers and will need quite a bit of string, rope, or ribbon.

In Figure **??** 5 people have formed a circle. Each person is connected to the person next to them with a length of rope. Notice that the rope forms the shape of a *regular pentagon*.

Figure 10.5: Google Earth image of Palmanova, Italy

11. Have five people form a circle. Connect every *other* person together with rope. (I.e. the first to the third to the fifth to the second, etc.) Describe the resulting shape formed by the rope and draw this shape.

The figure you created in Investigation **11** is called a *star polygon* - it is a star and, like a polygon, it is a closed, planar figure created by line segments that have been connected end to end. We will use the term ***star polygon*** to refer to all objects formed in this way, including the pentagon.

We would like to give the different star polygons names to identify specific star polygons. You should see that there are two key features in constructing them, the number of people and the count between people holding a segment of rope. Our notation to identify star polygons will be

$$\binom{\text{Number of People}}{\text{Count between people}}.$$

E.g. the pentagon is $\binom{5}{1}$ and the figure in Investigation **11** is $\binom{5}{2}$. If you had 7 people and every 4^{th} was connected by rope the star polygon would named $\binom{7}{4}$.

So there is consistency, there are a few other terms we would like to define so they can be used in a standard way. We will call star polygons like $\binom{5}{2}$ =☆***regular*** because they can be made from a single length of string. Star polygons like $\binom{6}{4}$ = ✡will be called ***compound*** because they require more than than a single length of string to form. We will say ✡has two ***components*** which are the two $\binom{3}{1}$ star polygons △ and ▽. Star polygons like $\binom{6}{3}$ = ✳will be called ***degenerate*** as they are not polygons.

Now it is your turn to explore and see what patterns you can find in the galaxy of star polygons. In the appendix are templates of the *roots of unit* that will be useful in drawing your star polygons symmetrically.

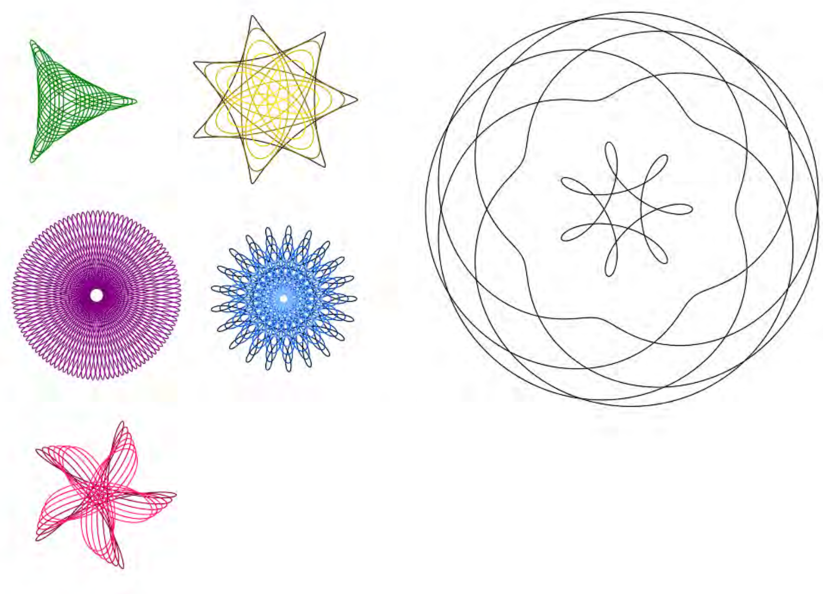


Figure 10.6: Spirograph images

> Independent Investigation 1 - A Galaxy of Stars
> Create 50 different star polygons. Record the name each, using the naming convention above, next to its image. For each, determine whether it is regular, how many components it has, whether it is degenerate, and any other interesting properties it has.

Now that you have a large collection of star polygons, you should look for patterns. As you find these patterns you may have to create other examples to test your patterns and conjectures.

> Independent Investigation 2 - Finding Star Polygon Constellations
> Using the star polygons you have just created, find 5 different patterns that help organize the star polygons into "constellations" of similar objects. Please be precise in the description of your patterns, using appropriate notation and providing examples as illustrations when they are helpful.

You should be discovering all sorts of interesting patterns, making conjectures, and discovering hidden connections. This is excellent - for this is what it means to do mathematics.

In the "Introduction" we considered morphogenesis as a way in which to view patterns - our goal is to understand the underlying process that gives rise to the pattern at hand. And in "Proof" we noted that "for every pattern that appears, the mathematician feels [s]he ought to know why

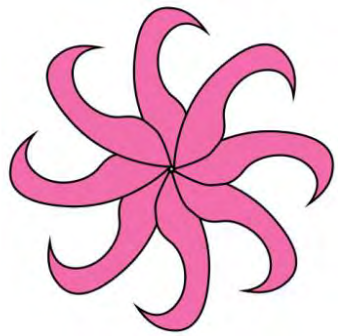


Figure 10.7: A figure with 8-fold rotational symmetry

it appears." In the star polygons there are many patterns. Mathematicians are not content until they understand the underlying processes that completely characterize the entire system. The goal is to be able to describe the class of objects in its entirety, organizing the entire system, capturing all of the possible mathematical features, and being able to determine exactly the process that controls the morphogenesis of the system.

This is what you should now turn to.

Independent Investigation 3 - A Calculus of Star Polygons Completely characterize the family of star polygons $\binom{n}{d}$.

You're back. Most of our students make quite good progress for a while, but because this is a fairly big task they often need some help. In particular, they often ask, "How do know when I am done and things are completely characterized?" Below are a few questions. If you can answer them using your characterization, you are probably done:

Question 1. Describe *all* of the star polygons $\binom{13}{d}$, all infinitely many of them. How many different $\binom{13}{d}$ star polygons are there? How many of these are regular? How many of these are degenerate?

Question 2. How many different regular star polygons $\binom{50}{d}$ are there? For those star polygons $\binom{50}{d}$ that are compound, how many components can they have? For those that are compound, how many different star polygons are there that have the same number components?

Question 3. Suppose we repeated the questions in Question 2 but asked you to answer these questions for star polygons $\binom{7917}{d}$. Without doing it, describe what you would need to do to answer these questions. What properties of the number 7917 are important?

Having completed these significant investigations you might wonder about the title of the last one - A Calculus of Star Polygons. The term *calculus* is often used to refer to the mathematical fields of *differential calculus* and *integral calculus* that were independently invented by **Isaac Newton** (English Mathematcian and Physicist; 1642 - 1727) and **Gottfried Wilhelm von Leibniz** (German Mathematician and Philosopher; 1646 - 1716) and which remain one of the most fundamental descriptive tools used by human beings to describe change over time. When you hear

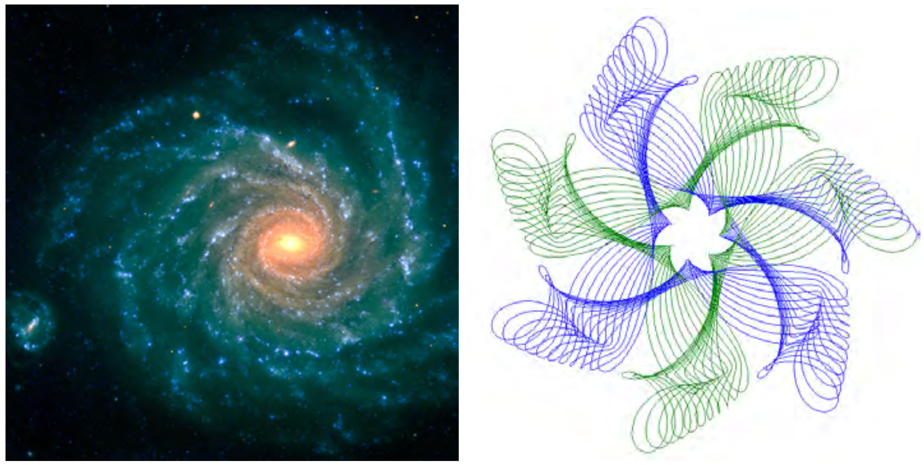


Figure 10.8: Spiral Galaxy NGC 1232 and a Spirograph image.

about AP Calculus in high school or Calculus in college, this is what is being referred to. So why use this term here?

The word calculus actually has a broader meaning - "a method of computation or calculation within a symbolic system." This meaning comes from the Latin root *calculare* which means to calculate. One uses the phrase *the* calculus to differentiate the field made up of differential and integral calculus, which is considered in Discovering the Art of Mathematics - Calculus, from other calculi such as *propositional calculus* which is a branch of mathematical logic.

You have just discovered your own calculus - one that completely describes the morphogenesis of star polygons.

10.4 Beautiful Symmetry from Toys: Spirograph and TrippingFest

As the small pebble stirs the peaceful lake;
The centre mov'd, a circle straight succeeds,
Another still, and still another spreads.

Alexander Pope (English Poet; 1688 - 1744)

The rosettes pictured in Figure 10.6 and on the right of Figure 10.8 are ***Spirograph drawings*** made from the ***Spirograph toy***. Pictured in Figure 10.13, this toy is used by placing a pen in a hole in a geared, circular wheel and then moving the wheel with the pen so it travels around the edge of a geared circular ring or straight bar - as shown in Figure 10.15. Invented in 1965 by **Denys Fisher** (English Engineer; 1918 - 2002) Spirograph was a worldwide phenomena, winning Toy of the Year in 1967. Spirographs were sold in the United States by Kenner (which was later subsumed by Hasbro) through the 1980's but were discontinued for many years afterwards. While other companies sold small Spirograph-like kits, demand was high enough for the originals that by

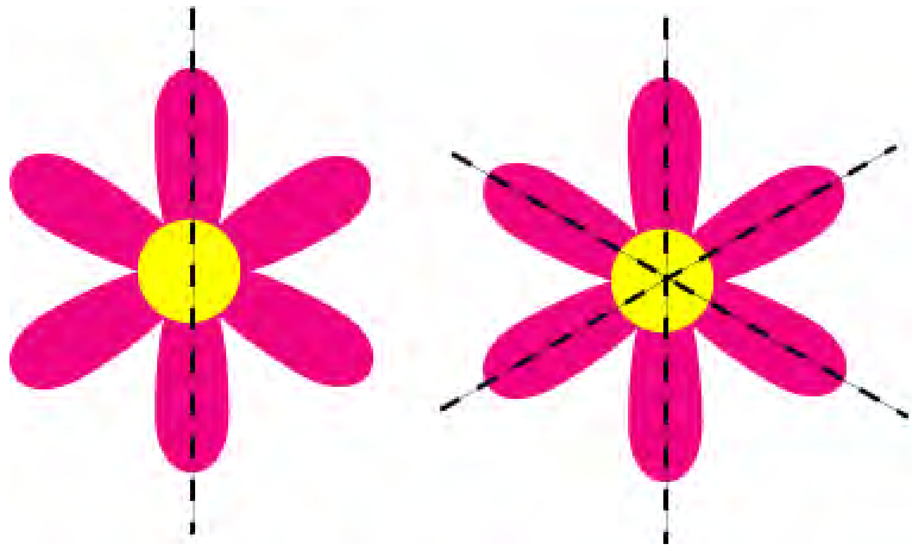


Figure 10.9: A figure with 6-fold reflection symmetry

2010 complete, good condition Spirograph sets could command as much as $100 on ebay. Kahootz Toys re-released Spirograph in what they claim is its "original product configuration" in 2013.

Spirograph brought great joy to generations of users, allowing them to make wonderful images. It is also a wonderful tool for exploring mathematics and has important historical connections. We will explore many of these connections below. Before beginning we would like to make note of a more modern rosette-generating tool.

After its release in 2009 **TrippingFest** quickly became one of the top 100 iPod/iPhone apps. Through a simple interface users can make wonderful art of all sorts, as shown in Figure 10.14.

If you have access to this app, or have a friend who does, try it out. You can use *Swirl* to make swirls - the archetypes for rosettes whose point symmetry groups consist only of rotations, and are said to have *cyclic symmetry groups*. If we use *Centered Polar* we have the archetypes for rosettes whose point symmetry groups consist of both rotations and reflections, and are said to have *dihedral symmetry groups*. If you do not have access you can see TrippingFest in action on YouTube.

The fact that this app is so popular gives added evidence of human-kind's obsession with both symmetry and art. The genesis of this app also is a wonderful story.

TrippingFest was developed by **Forrest Heller** (American College Student; -), when he was a teenager. He describes its development as follows:

> Around 2003, I wrote a drawing program with Visual Basic 5 (not 6). You could draw lots of different patterns with random colors. It had automated random drawing and around 20 different patterns. I was *in middle school at the time*, so it also made fart noises.
> After waiting for someone to make something superior and more complete (and release it for free), I could find no programs that captured the essence of TrippingFest. In

Figure 10.10: Sand Dollar and Sea Urchin Shells

Figure 10.11: Flowers on the island of St. Croix

early 2009, I embarked on efforts to make a multi-platform drawing platform based on the Visual Basic program with intent to release. I chose Java and before long I had a nice proof-of-concept. When I showed most people the demo they found it amazing. However, I swamped with irrelevant college coursework and I had to discontinue serious development. After school ended, I realized I would never finish the desktop version: I would never stop adding features.
I then realized that people who have iPhones would probably enjoy the app. I personally don't own an iPhone. I contacted some people over the Internet and found two people willing to loan me equipment for developing an iPhone application. Alex Alba lent me his iPhone for a week. (In exchange, he got to use my really old Nokia brick phone.) Kyle Coe/Sue Coe lent me their MacBook Intel with Leopard.
In one week, between work and school, I learned Objective-C, learned how to use the iPhone SDK, and finished the iPhone version of TrippingFest. Ironically, I still don't and probably never will own an iPhone.[3]

[3]From http://www.forrestheller.com/drawing/ . Emphasis added.

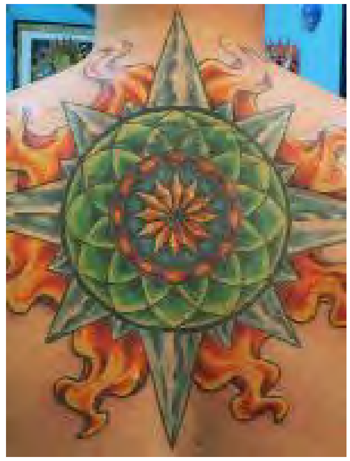


Figure 10.12: Mandala tatoo

10.5 Spirograph Morphogenesis

Spirograph is wonderful, simple tool for making art. The purpose of this section is to discover the mathematical properties that are inherent in this tool and its images. It is helpful to have an actual Spirograph which supplements the online applets, so see if you have one at home, can borrow one from a friend, or can share with some of the other explorers in your group.

Begin by playing with a Spirograph or seeing how one works in action if you don't have a physical version. There are a number of videos available on YouTube and other video sharing sites that you can check out.[4] A still picture is below in Figure 10.15.

You will also need access to programs that mimic Spirograph. There are many available that run online, on tablets, or on apps. Find your own or use one the ones below:

`http://nathanfriend.io/inspirograph/`
`http://www.mathplayground.com/Spiromath.html`
`http://cgibin.erols.com/ziring/cgi-bin/spiro.pl/spiro.html`
`http://wordsmith.org/~anu/java/spirograph.html`
`http://thinks.com/java/spiro/spiro.htm`
ÜberDoodle
RotoDoodle

[4] An original, 1970's era ad for Spirograph is available at `http://www.youtube.com/watch?v=jhWOGE95fMI&feature=rec-LGOUT-exp_fresh+div-1r-3-HM` under the name "Spirograph Commercial". Also on YouTube is a commercial for a recent knock-off, ThinkGeek's "Hypotrochoid Art Set" at `http://www.youtube.com/watch?v=Q8ZQ2INGvEw` .

Figure 10.13: The original Spirograph set

> WARNING
> The majority of online sites prescribe the sizes of the wheels and rings by their radii. Following the lead of the original Spirograph, we will refer to wheels and rings by the number of teeth. A direct translation from teeth to radii (e.g. 24 teeth to radius 24) produces the same shape as you move from actual geared Spirograph gears to online applets. You'll investigate this in Investigation **24**.

The **Spirograph figures** we will make here will all be made with *circular wheels* which roll along the inside of *circular rings*. The gears on these wheels and rings are uniformly spaced and will remain in contact at all times so there is no slipping or skipping. The figure will not be considered complete until our pen has returned to its exact starting point. The formal mathematical name of such figures are **hypotrochoids**.

We will follow the original notation for the number of teeth on these pieces: 30 denotes a wheel with 30 teeth. The circular rings have teeth on the inside as well as the outside. We'll only use those on the inside. 105 denotes a ring with 105 teeth on the inside of the ring.

As you work you should clearly label the wheel, ring and hole numbers you use to make figures as they will be critical to the analysis.

12. Choose one ring and one wheel. With your pen in any one of the holes, draw a Spirograph

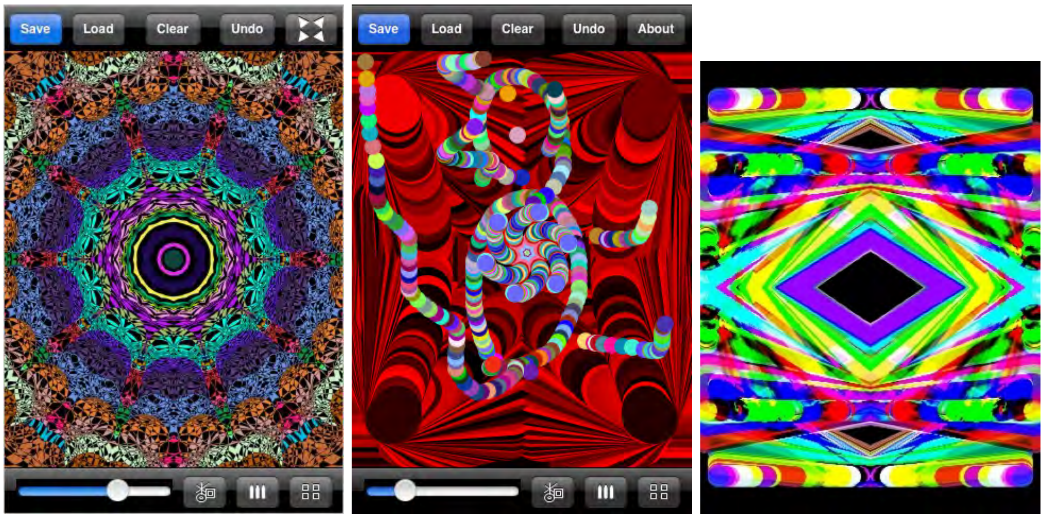


Figure 10.14: A TrippingFest images

figure. Describe its symmetries.

13. Using the same ring as in Investigation **12**, draw a Spirograph figure with your pen in a different hole. Describe its symmetries and compare it to your previous figure.

14. Using the same ring as in Investigation **12**, draw a Spirograph figure with your pen in a different hole. Describe its symmetries and compare it to your previous figures.

15. Based on Investigations Investigation **12** - Investigation **14**, what effect does the choice of hole seem to have on Spirograph figure that is drawn given a specific ring and wheel?

16. If you wanted to draw a Spirograph figure that resembled the star polygons considered above, what would the most appropriate choice of hole be?

17. Draw a Spirograph figure with the ring 96 and wheel 48. What is the result?

18. Now draw a Spirograph figure with the ring 96 and wheel 32. What is the result?

19. Now draw a Spirograph figure with the ring 96 and wheel 24. What is the result?

20. You should see a pattern forming. Using this pattern, what do you think would happen if you used wheel 16? Wheel 12?

21. What other size wheels could be used to illustrate the pattern you described in Investigation **20**? Explain.

22. Describe the physical characteristics of the Spirograph that explain why the pattern you have described in Investigations Investigation **20** and Investigation **21** occurs.

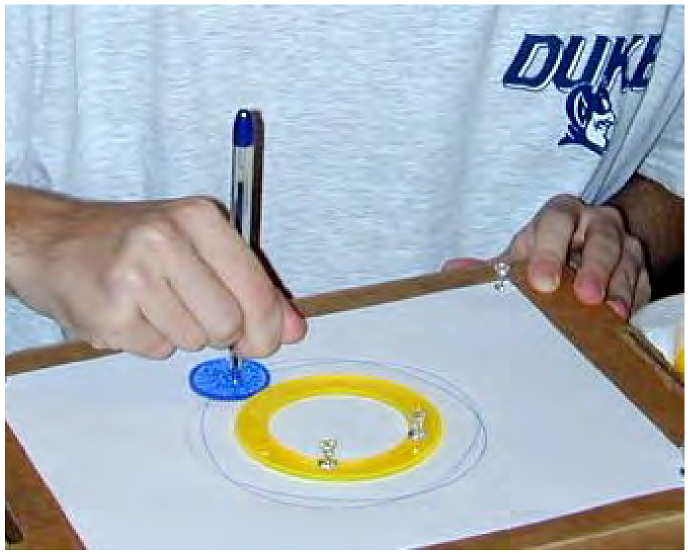


Figure 10.15: A Spirograph in action.

23. Notice that the "petals" that make up the Spirograph figures considered in Investigations Investigation **17** - Investigation **22** are drawn in order, one after another, in a circular order and the figure is complete after one full rotation of petals have been generated. Using the inside of the 105 ring, describe the Spirograph figures whose petals are generated in this same way and identify explicitly the wheels that would be necessary to make each of them.

24. Explain physically why the curve resulting from using Spirograph ring 96 and wheel 16 is, except perhaps its overall size, the exact same curve that is generated by an online script when the ring has inner radius 96 and the moving wheel has radius 16.

10.5.1 Epicycles and Astronomy

> The universe stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth.
>
> **Galileo Galilei** (Italian Astronomer, Physicist, and Philosopher; 1564 - 1642)

An often heard question in mathematics is "When will I ever use this?" Many view mathematics as an art, and think that it should be appreciated on its own for its beauty, magic and mysteries. Spirograph figures and star polygons are perfect examples that highlight this. But we would be remiss if we did not note a direct connection between Spirograph figures and the way in which people understood the universe in which we live for *thousands of years*.

The ***Pythagoreans***, the cult-like followers of **Pythagoras** (Greek Mathematician and Philosopher; ca. 570 B.C. - ca. 495 B.C.), believed that the moon, planets, and stars were carried around on a giant crystal sphere with the Earth at its center. In their motions they emitted different tones based on their locations. All told, they created a *music of the spheres*. Such a romantic vision of the heavens was compatible with the central role that both music and number played in their philosophy. This view extended nearly two centuries, all the way through **Johannes Kepler** (German Astronomer and Mathematician; 1571 - 1630) who used the five *Platonic solids* as carriers of the planets:

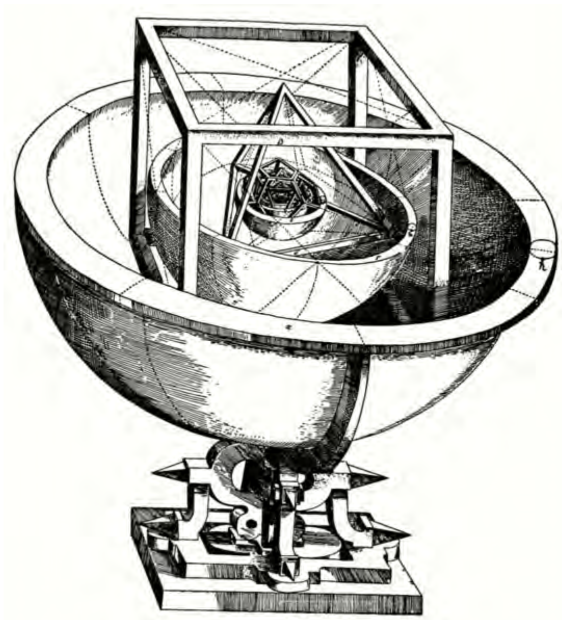


Figure 10.16: Kepler's model of the solar system using the five Platonic solids.

Yet careful observers of the night sky realized that important objects exhibited ***retrograde motion*** where they would occasionally move in the opposite direction in the sky than the normally did. This could not be explained by any fixed sphere model of the heavens nor of routine circular orbits of the heavenly bodies about a central Earth. Trying to keep the Earth at the center, new models were needed.

The Almagest by **Cladius Ptolemy** (Greek Astronomer and Mathematician; 90 - 168) is one of the most important books on astronomy ever written. It laid out a system of *epicycles* which described the retrograde motion that had been observed. These epicycles are formed by one wheel rolling along another wheel with the planet located on the smaller wheel. In this model, the paths of the heavenly bodies were *exactly* predicted to be Spirograph figures. Viewed from our perspective from Earth, at the center of the stationary wheel, such epicycles approximated retrograde motion as well as the limited observations allowed. Many interesting applets which illustrate both epicycles and retrogade motion are available online.[5]

[5] E.g. `http://www.niu.edu/geology/stoddard/JAVA/ptolemy.html` and `http://csep10.phys.utk.edu/astr161/lect/retrograde/aristotle.html`.

25. Pick a Spirograph ring and Spirograph wheel. Place a point the center of the ring to represent our position on Earth. Pick a whole on the wheel, representing the location of the planet, and draw the orbit of the planet. From our vantage point on Earth, does this orbit exhibit retrograde motion?

26. Now find a new hole in the wheel or a new wheel so that the new image you generate exhibits the opposite of the previous problem - no retrograde motion if there was before, retrograde motion if there was none before.

If you felt that there was something very "orbital" feeling about Spirograph, you are in good company. This was the prevailing model to describe our universe from the time near the birth of Christ until **Nicolas Copernicus** (Polish Astronomer, Mathematician and Scholar; 1473 - 1543) revolutionized our view of the solar system in his publication of De revolutionibus orbium coelestium in 1543.

10.5.2 More Spirograph Patterns

> Before I was two years old I had developed an intense involvement with automobiles. The names of car parts made up a very substantial portion of my vocabulary... Years later... playing with gears became a favorite pastime I became adept at turning wheels in my head and at making chains of cause and effect... Working with differentials did more for my mathematical development than anything I was taught in elementary school.[6]
>
> **Seymour Papert** (South African Mathematician and Educator; 1928 -)

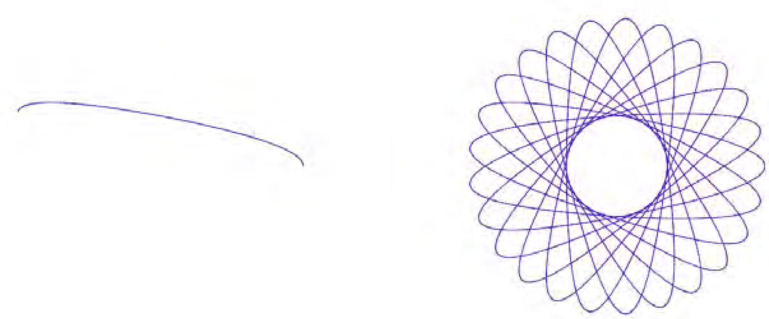


Figure 10.17: An arc of a Spirograph figure from apex to apex, left, and the entire figure, right

By the **apex** of a Spirograph figure we will mean a point where the curve is closest to the ring used to create it. Figure Figure 10.17 shows an arc of a Spirograph curve from one apex to the next.

[6] From "The Gears of My Childhood," preface to Mindstorms: Children, Computers and Powerful Ideas.

27. Create the Spirograph figure generated by wheel 40 using the 105 ring. Describe its symmetries. Label the "top" apex 1 and then continue around clockwise numbering the apexes $2, 3, 4, \ldots$ until you return again to the top.

28. Starting at apex 1, follow the Spirograph curve until you reach the next apex that the curve hits. What number apex is it?

29. Continue along the Spirograph curve to the next apex that the curve hits. What number apex is it?

30. Continue along the Spirograph curve to the next apex that the curve hits. What number apex is it?

31. You should see a pattern in Investigation **27** - Investigation **30**. Describe it precisely. Does this pattern continue? Explain.

32. In Investigation **28** - Investigation **30** you followed along the Spirograph curve from one apex to the next apex that this curve hit. Suppose you replaced each of these Spirograph arcs with a line. What figure would be formed? Do you have a precise name for this figure?

33. Create the Spirograph figure generated by wheel 42 using the 96 ring. Describe its symmetries. Label the "top" apex 1 and then continue around clockwise numbering the apexes $2, 3, 4, \ldots$ until you return again to the top.

34. Starting at apex 1, follow the Spirograph curve until you reach the next apex that the curve hits. What number apex is it?

35. Continue along the Spirograph curve to the next apex that the curve hits. What number apex is it?

36. Continue along the Spirograph curve to the next apex that the curve hits. What number apex is it?

37. You should see a pattern in Investigation **33** - Investigation **36**. Describe it precisely. Does this pattern continue? Explain.

38. In Investigation **34** - Investigation **36** you followed along the Spirograph curve from one apex to the next apex that this curve hit. Suppose you replaced each of these Spirograph arcs with a line. What figure would be formed? Do you have a precise name for this figure?

You should be seeing an important connection between Spirograph figures and star polygons. Use this pattern to:

39. Describe the symmetries and the identity of the star polygon which corresponds to the Spirograph figure generated by the wheel 60 and the ring 105 without actually drawing it first. (Feel free to draw it to check.)

40. Determine the wheel and ring used to create the Spirograph curve in Figure 10.18. (Note: Here one can only determine the relative sizes of the wheel and ring as there are infinitely many different combinations that can make this figure.)

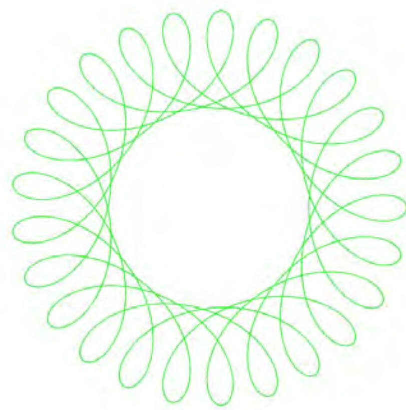


Figure 10.18: Spirograph figure for Investigation **40**

41. Describe the symmetries and the identity of the star polygon which corresponds to the Spirograph figure generated by the wheel 63 and the ring 96 without actually drawing it first. (Feel free to draw it to check.)

42. Determine the wheel and ring used to create the Spirograph curve in Figure 10.19. As it may be hard to follow the curves exactly as the reach the the inside, between successive apexes along the Spirograph curve there are fewer than 11 apexes in between. (Note: As above, you can only determine the relative sizes of the wheel and ring.)

43. Write a summary which precisely describes in a detailed, quantitative way how one navigates back and forth between the world of Spirograph figures and the world of star polygons.

44. Do you have non-school based experiences that compare with that described by Papert in the quotation that opens this section? If so, describe them. How do your experiences investigating topics from this book compare with Papert's quote that opens this section?

10.5.3 Famous Curves

Above we saw historically important connections between wheels travelling along wheels, like Spirograph curves, and astronomy. If we expand the type of objects that interact, to not only cirlces, but lines, and squares, we get many other critical links to the history, science, architecture, and art.

During the *Scientific Revolution*, roughly 1540 - 1730, mathematicians routinely challenged other mathematicians and held problem solving competitions. One important challenge is the *brachistochrone problem* which was stated eloquently by **Johann Bernoulli** (Swiss Mathematician; 1654 - 1705) in June, 1696 in the journal *Acta Eruditorum* :

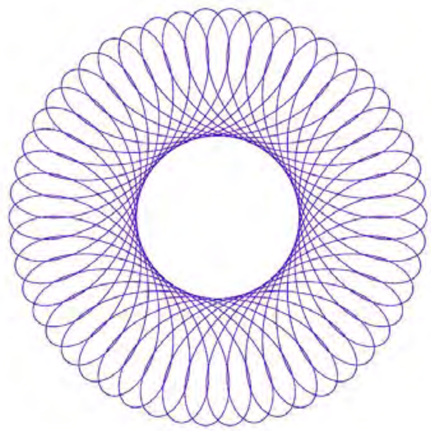


Figure 10.19: Spirograph figure for Investigation **42**.

> I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.

The problem is most easily described as a physical experiment. We have two points A and B in space and we wish to build a ramp so we can start a ball at A and have it roll to B.

Brachistichrone Problem: Of all such ramps, what is the profile of the ramp which will enable the ball to reach point B most quickly in a frictionless environment?

Skiers and skateboarders will have a sense that the latter profiles in Figure 10.21 will offer dramatically faster routes to the bottom. But what is the optimal profile?

This problem was stated not long after the independent invention of calculus by Newton and Leibniz. This development was a profound change to mathematics and science, playing a central role in the Scientific Revolution. One of calculus' main strengths was its ability to solve maxima and minima problems. Until now such solutions were to find a point in time, a speed, an optimal solution to a single problem with a fixed set-up. The Brachistichrone problem was different, the goal wasn't to find one time, one point, or one speed along a given curve, but to consider the optimal solution among all possible curves that can exist. It solution, and the subsequent blossoming of the methods used to solve similar problems, gave rise to the *Calculus of Variations* which is of fundamental importance in many areas of mathematics, physics, optics and engineering.

Said **Karl Menger** (Austrian Mathematician; 1902 - 1985) of the Calculus of Variations:

> Mathematicians study their problems on account of their intrinsic interest, and develop their theories on account of their beauty. History shows that some of these mathematical theories which were developed without any chance of immediate use later on found

Figure 10.20: A water drop.

> very important applications. Certainly this is true in the case of calculus of variations: If the cars, the locomotives, the planes, etc., produced to-day are different in form from what they used to be fifteen years ago, then a good deal of this change is due to the calculus of variations.

Returning to the problem, the optimal profile is what is now called a ***cycloid***, the curve appearing on the far right in the figure above. But what curve is this? How is it generated? It is the curve traced out by a point on a wheel moving along a straight line. In other words, it is a Spirograph curve made on a straight rod!!

The Brachistichrone problem was solved by Newton, **Jacob Bernoulli** (Swiss Mathematician; 1667 - 1748) (Johann's younger brother), Leibniz, and **Franois Antoine Marquis de L'Hopital** (French Mathematician; 1661 - 1704). Mathematical prowess was strongly valued, for both intellectual and practical reasons. When the solutions were published, Johann prefaced them by noting the solutions showed:

> The three great nations, Germany, England, France, each one of their own to unite with myself in such a beautiful search, all finding the same truth.

There is rich history and deep applications related to curves like: the *cardiod*, the *lemniscate*, and the *Witch of Agnesi*. We close with one more, the *catenary*.

A ***catenary*** is the shape taken by a weighted chain or rope under the influence of gravity. Catenaries are fundamental in the architectural design of arches, the St. Louis Gateway Arch, pictured in Figure 10.22 being a classic example.

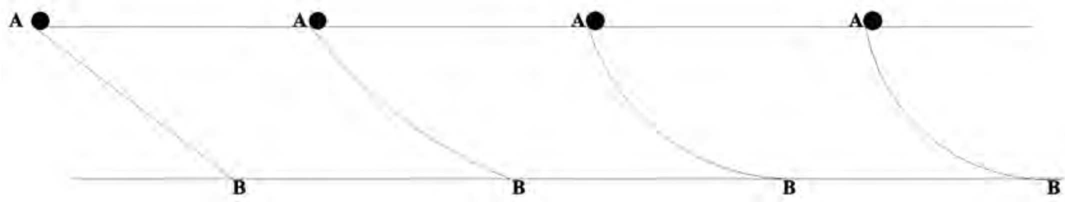


Figure 10.21: Ramps for the Brachistichrone problem

Figure 10.22: The St. Louis Gateway Arch.

While **Gerson B. Robison** (; 1909 -) was "picking up my small son's toy blocks, [he] became intriuged with the possibility of finding a cylindrical surface upon which a plank would roll in neutral equilibrium."[7] He discovered that the required curve is a catenary. And if we put a number of catenaries together end to end we get the wonderful square wheeled bicycle pictured in Figure 10.23 which was invented by **Stan Wagon** (Canadian-American Mathematician; -). The bike rides smoothly, as if it was a normal bicycle as the axles of the square wheel travel in a straight line. With Spirograph figures circles and rings made a famous curve. Here the famous curves coupled with a square bring us back to a line.

We hope you find this interesting. But the principles at work are much more than a curiousity. Many people (e.g. see Building the Great Pyramid in a Year: An Engineer's Report by **Gerard C.A. Fonte** (Engineer; -)) have argued that relics found near the pyramid suggest that the huge granite blocks used to construct the pyramids were rolled on catenaries. Fonte, a small man, regularly demonstrates this principle in action by rolling two ton concrete blocks along the

[7] From "Rockers and Rollers", *Mathematics Magazine*, vol. 33, no. 3, Jan.-Feb. 1960, p. 139.

Figure 10.23: Stan Wagon on the square wheeled bicycle he invented at Macalester College.

catenaries set up in his yard.

10.6 Further Investigations

F1. The Kakeya problem is a famous mathematical problem with a surprising solution. Find out about this problem. Then i) describe the problem in your own words and ii) describe the "answer" to this problem in your own words.

F2. If you were going to investigate the Kakeya problem on your own, are their physical, visual, and/or artistic ways you might do so? Explain.

F3. Play around with the iPod app TrippingFest. Can you recognize some other mathematical or artistic issues that are involved other than those we considered involving point symmetry?

F4. Check out Tricky Line in TrippingFest. Is there some connection to the Kakeya problem in some sense? If so, describe it.

F5. How is the Tricky Line app related to the discussion of geometric objects moving around in space above?

10.7 Connections

For those with interest in more modern physics, the wonderful paper "Black Holes through The Mirrour" by Andrew J. Simson, *Mathematics Magazine*, Vol. 82, No. 5, December 2009, pp. 372-381 offers a remarkably surprising appearance of hypotrochoids. Namely, if a sufficiently small black hole fell through the Earth, gravitational attraction from the Earth would repeatedly bring it back toward the center of the Earth. As the Earth spins, the path of the black hole traces out exactly hypotrochoids!

Chapter 11

Interlude - Mathematics and the Visual Arts

> Mighty is geometry; joined with art, resistless.
>
> **Euripides** (; -)

> Geometry is the foundation of all painting.
>
> **Albrecht Durer** (; -)

Note the profound links to linear programming when we talk about the dissection of n-space by hyperplanes. Quote about the most important real world application. Full circle back to Art with Obaminoes.

> The linear-programming was – and is – perhaps the single most important real-life problem.
>
> **Keith Devlin** (; -)

Note the word PROBLEM!!!

Give links and put a picture in. Make note about Mathematics for Liberal Sciences. This is a good topic. But not enough people know about it or other contemporary mathematical developments and how they impact our life. I.e. plug MLS. Science and Art sybiosis.

Also have links to the fourth dimension in modern art. Talk some about cubism.

Links to perspective drawing and Annalisa Crannell's book. Talk about perspective and Brunelleschi?

Fractals!! Put in the Outkast Stankonia cd cover.

BIG note that all of this involves nontrivial mathematics. We're not talking a little bit of arithmetic. We are talking about fundamental changes to art. The use of perspective, fractals, basic shapes (Susan Sheridan and Ed Emberley), CAD and Photoshop, morphing and topology, computer graphics and programming (with modeling of creatures), fractals, photomosaics, cubism,... There has always been evidence that mathematics is involved in art (see the quotes above) - but now we are talking about deep, modern mathematics that students have likely not seen much of at all. And a great amount of this can be considered as patterns. It is absurd that students have not seen any of this.

Put the stuff in here about 3D printing? Or should that be in the sculpture book?

Chapter 12

Conclusion

This is a book. All books of any merit have some sort of a conclusion, a climax, a reminder of the main theme of the work. This should as well.

However, telling/stating/professing what this is is not compatible with the pedagogical approach of the book. We want to make it clear what the main theme/conclusion/climax is - but we also want to involve the students. So how do we structure a conclusion that is compatible with both of these needs?

Essay questions?

Appendix

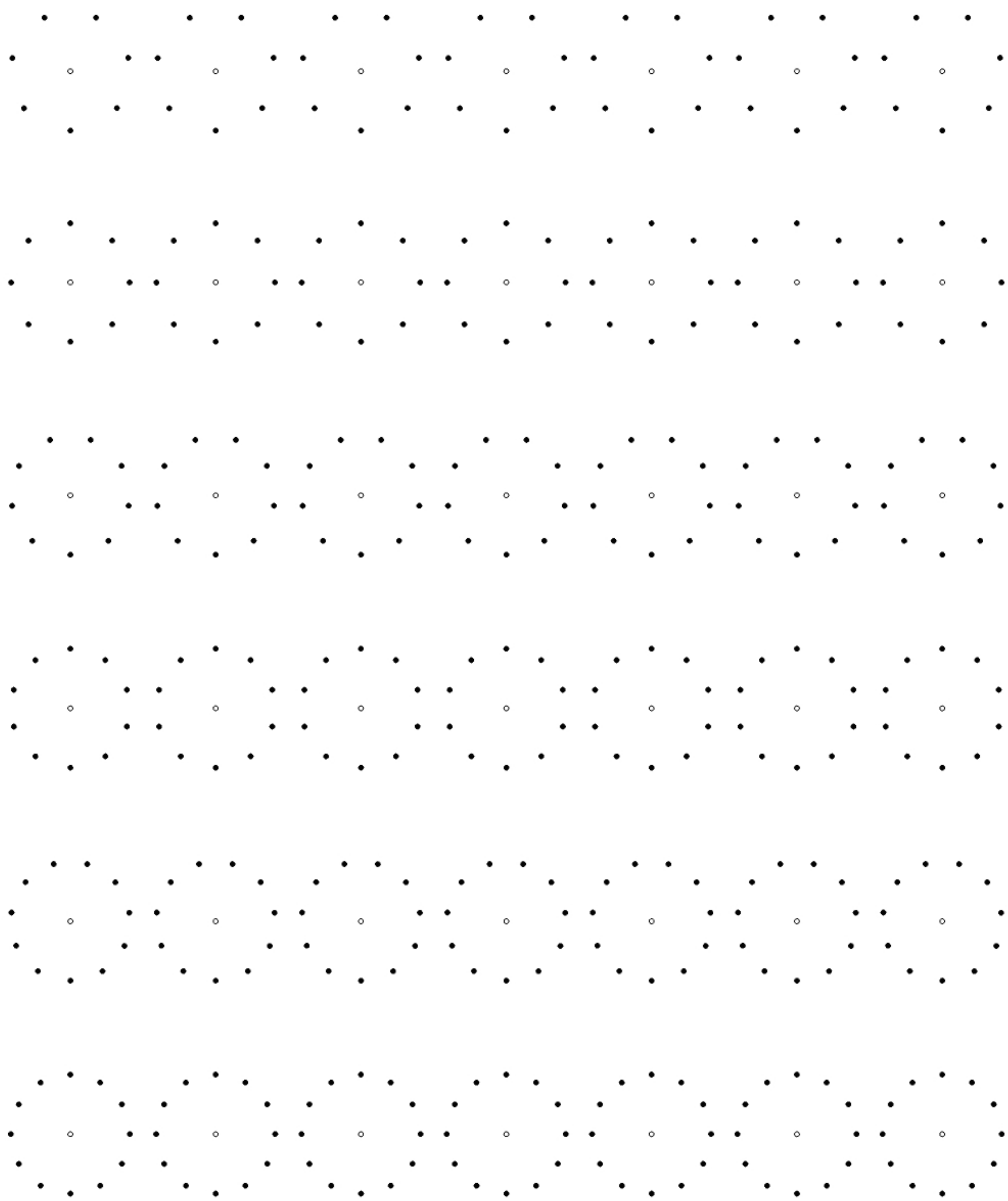


Figure 12.1: Roots of unity templates for making star polygons.

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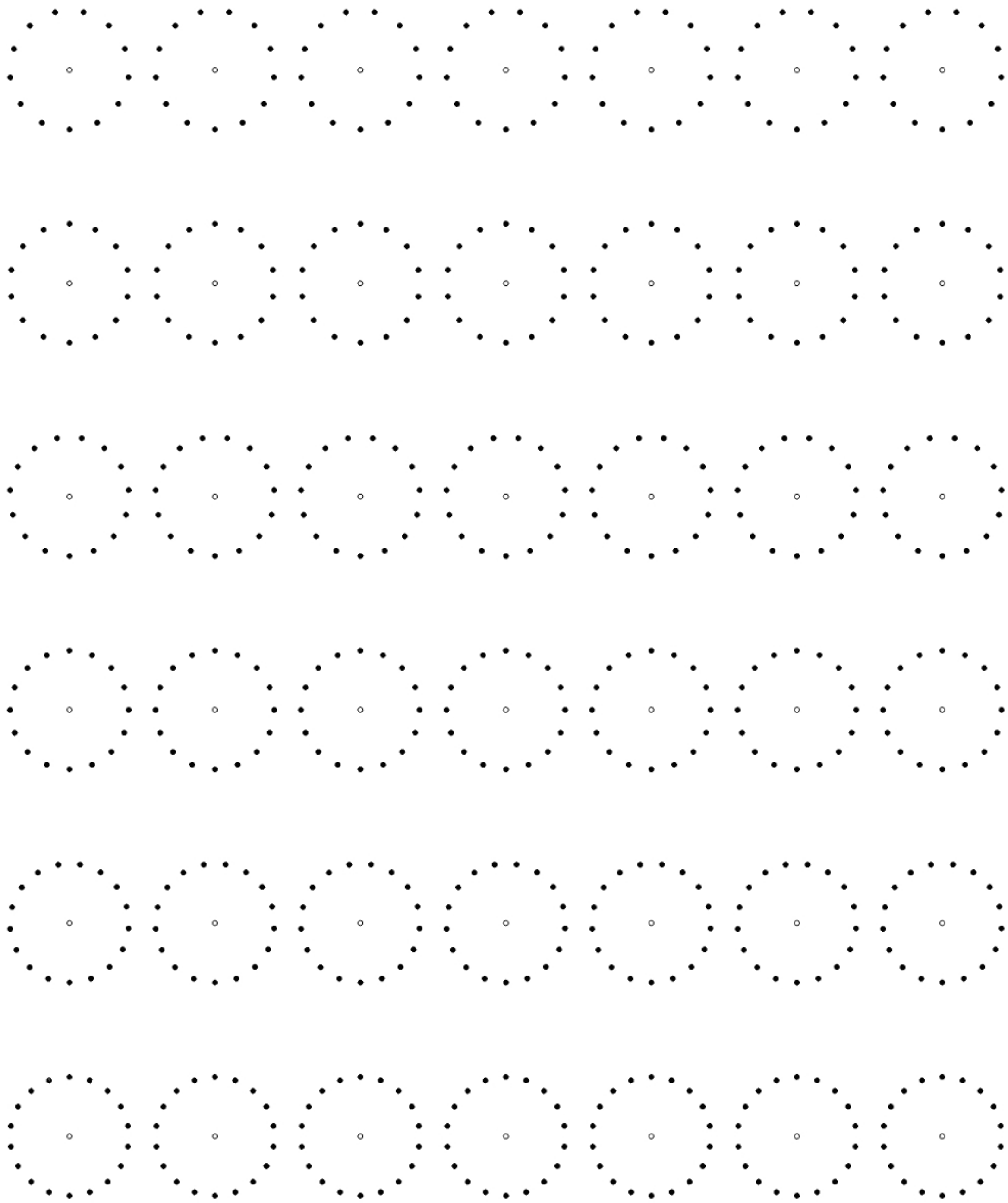

Figure 12.2: Roots of unity templates for making star polygons.

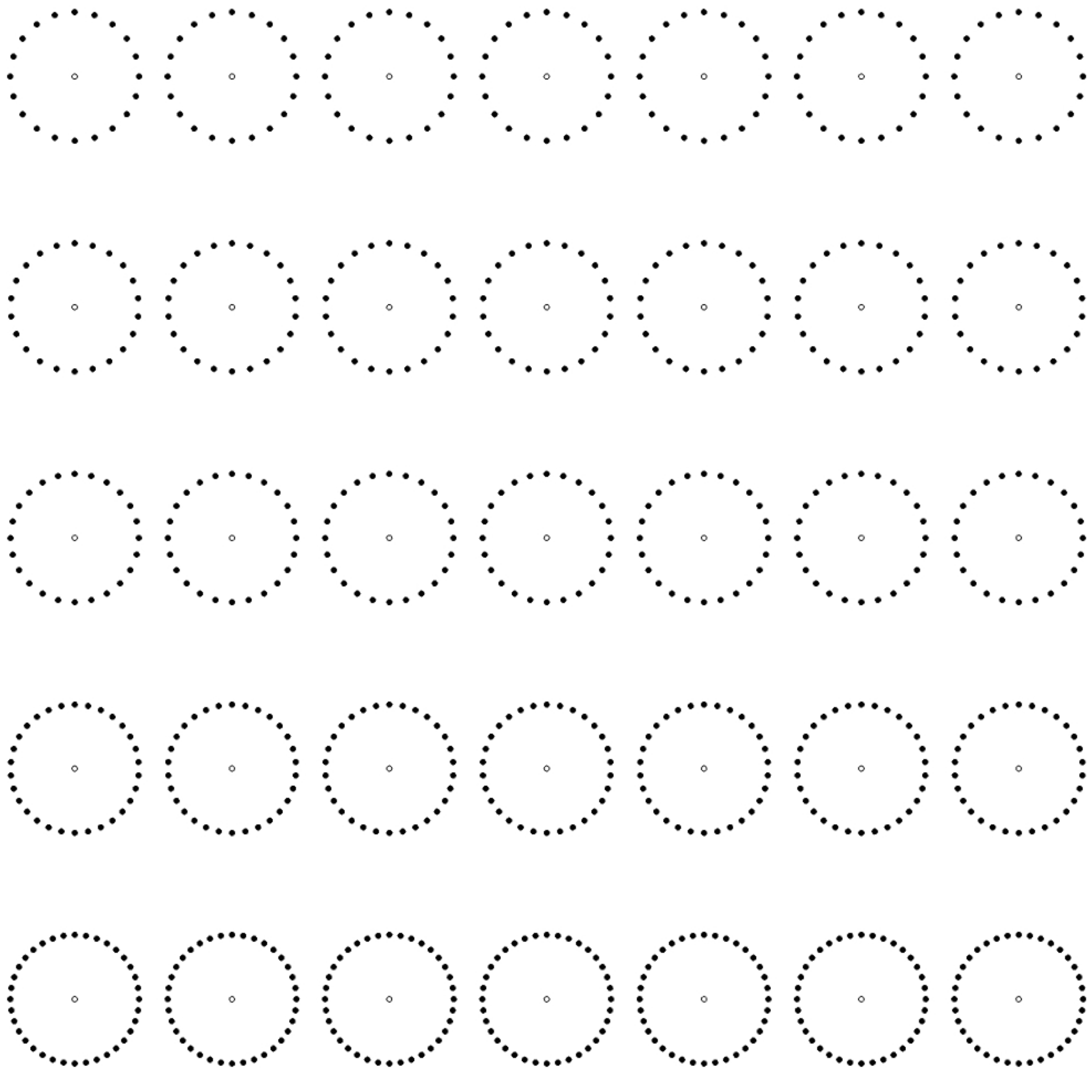

Figure 12.3: Roots of unity templates for making star polygons.

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