CHAPTER 2

Navigating Between Dimensions

> Yet I exist in the hope that these memoirs, in some manner, I know not how, may find their way to the minds of humanity in Some Dimensions, and may stir up a race of rebels who shall refuse to be confined to limited Dimensionality.
>
> **A Square from Flatland** (; -)

1. Flatland – A Romance of Many Dimensions

Written by **Edwin Abbott Abbott** (English Teacher and Theologian; 1838 - 1926) the novella Flatland: A Romance of Many Dimension[1] is the most well-known piece of mathematical fiction of all time.[2] Flatland is both a satire of the prejudices of Victorian England and wonderful tour through the dimensions.

The setting for Flatland is a land which is a two-dimensional plane which is populated by geometric figures. Our hero, whose name is A Square, learns of the "prejudices of his own dimension" when visited by a sphere. Sphere's arrival in Flatland mystifies our hero who sees only variable circles - the *cross sections* of Sphere as he moves through Flatland. When descriptions of the third dimension do not move A Square, Sphere literally lifts our hero out of Flatland and into the third dimension, providing enlightenment.

Abbott was a dedicated teacher and a devout, albeit outspoken, believer, spending his life trying to enlighten others. The prejudices and enlightenment that motivated Abbott to write Flatland went beyond the mathematical and metaphysical. As one of his important biographers **Thomas F. Banchoff** (American Mathematician; -) tells us:

> Abbott was a social reformer who criticized a great many aspects of the limitations of Victorian society. He was a firm believer in equality of educational opportunity, across social classes and in particular for women.[3]

His approach in Flatland is to criticize via satire. His satire of the class structure is based on geometric "regularity" and the number of sides. As our hero A Square tells us:

> The irregular. . . is either destroyed, if he is found to exceed the fixed margin of deviation, or else immured in a Government Office as a clerk of the seventh class; prevented from marriage; forced to drudge at an uninteresting occupation for a miserable stipend; obliged to live and board at the office. . . [4]

For the number of sides, A Square continues on to tell us:

> Our Soldiers and Lowest Class of Workmen are Triangles with two equal sides. . . Our Middle Class consists of Equilateral or Equal-Sided Triangles. . . Our Professional Men and Gentlemen are Squares (to which class I myself belong) and Five-Sided Figures or Pentagons. . . Next above these come the Nobility, of whom there are serveral degrees, beginning at Six-Sided Figures, or Hexagons, and from thence rising in the number of their sides till they receive the honourable title of Polygonal,

[1] The book Flatland is available online for free at `http://www.ibiblio.org/eldritch/eaa/FL.HTM`.

[2] Mathematical fiction is not a small genre. See the Mathematical Fiction site at the URL `http://kasmana.people.cofc.edu/MATHFICT/` for information on almost 1,000 other works of mathematical fiction.

[3] From the "New Introduction" in the 1991 Princeton University Press edition of Flatland.

[4] Flatland, p. 30

> or many-sided. Finally, when the number of the sides becomes so numerous, and the sides themselves so small, that the figure cannot be distinguished from a circle, he is inluded in the Circular or Priestly order; and this is the highest class of all.[5]

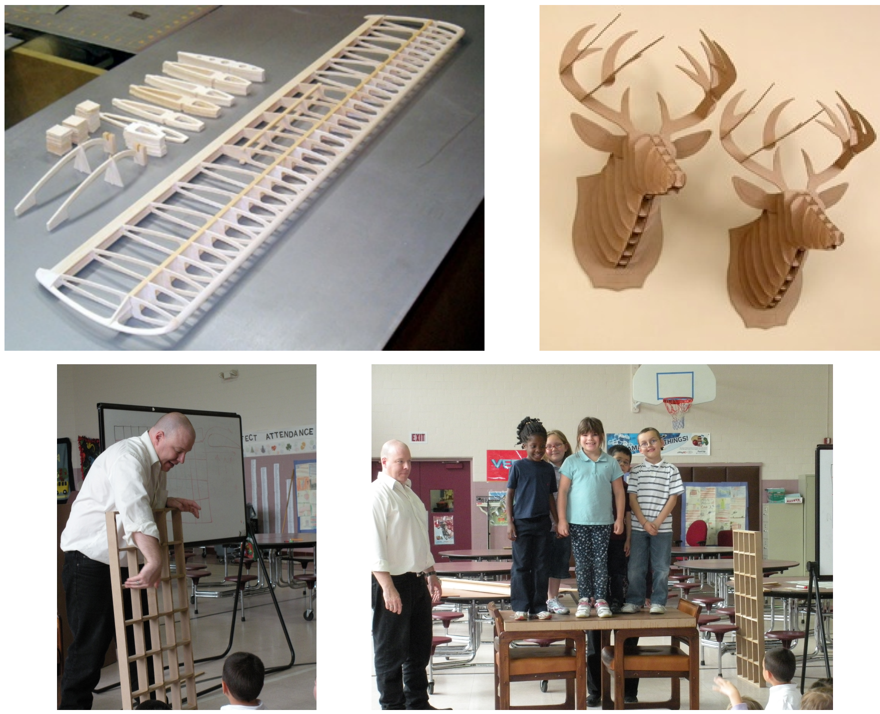

FIGURE 1. Three-dimensional objects constructed from two-dimensional components: airplane wing, Carboard Safari trophies, and the first author demonstrating the strength of *torsion boxes* with a little help from some of his friends.

And what of women? Their portrayal as the "Frail Sex" is given a geometric reworking here. Women are lines - one-dimensional rather than two-dimensional like their male counterparts. Women are "wholly devoid of brain-power, and have neither reflection, judgment nor forethought, and hardly any memory."[6] When viewed end on the women of Flatland are remarkably dangerous as they can easily pierce through the men without warning. This is just one more way that Abbott weaves together satire and geometry. His ultimate goal? To liberate us from prejudice and have our minds "opened to higher views of things."[7]

Flatland, as we shall see in Chapter **??**, is also notable in its vision of the importance of the understanding of the higher dimensions. This book predated Einstein's Theory of Relativity, CAT scans, 3D computer graphics, and many other profound advances that rest on an understanding of the higher dimensions. You'll get to explore several of these areas where interplay between the dimensions is so important in the next chapter.

1. In his description of Flatland, A Square describes the difficulty of sight in Flatland. He says,

> Place a penny on the middle of one of your tables in Space; and leaning over it, look down upon it. It will appear a circle. But now, drawing back to the edge of the table, gradually lower your eye (thus bringing yourself more and more into the

[5] Flatland, pp. 8-9.
[6] Flatland, p. 15
[7] Flatland, p. 4

condition of the inhabitants of Flatland), and you will find the penny becoming more and more oval to your view...

When you reach the viewpoint of the Flatlander, looking along the plane of the table, what does the penny look like?

2. If the object in front of you, in this Flatland view, was A Square, what would he look like?

3. Is there any way you could think of to distinguish different men in Flatland - other than moving back into three-dimensional space so you can see them from above?

4. If the object in front of you, in your Flatland view, was a woman, what would she look like? Would it depend on how she was "standing"?

5. **Classroom Discussion:**Nick Jackiw and Nathalie Sinclair have developed online scripts where Spacelanders can interactively manipulate Flatlanders to appreciate visualization in Flatland. In groups, experiment with these scripts:

`http://www.kcptech.com/dgylconf/proceedings/flatland/flat_triangle1.htm`
`http://www.kcptech.com/dgylconf/proceedings/flatland/flat_crowdscene.htm`

Do these scripts help you in visualizing some of the physical challenges as life as a Flatlander?

6. **Classroom Discussion:** Read Sections 1-3 of Abbott's Flatland to learn about other aspects of Flatland. Talk what life must be like in Flatland.

7. Watch the trailer of the Flatland movie at `http://www.flatlandthemovie.com/`. How does this help you visualize what life in Flatland would be like?

8. Because the movie was made for Spacelanders, embellishments have been made for our entertainment. Unfortunately, many are geometrically inappropriate. Find several examples of images that appear in the trailer that would not really exist in Flatland.

2. Cross Sections

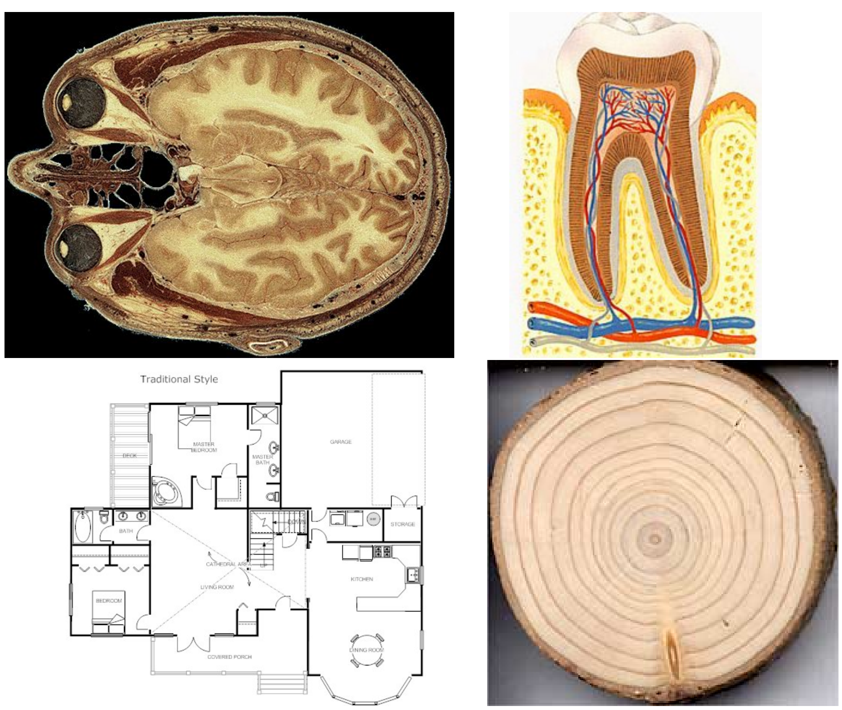

Figure 2. Cross sections of a human skull, atooth, a house, and a tree trunk.

Pictured in Figure 2 and Figure 3 are several ***cross sections***. The word cross section can have somewhat different meanings in different contexts. When mathematicians use the word they generally use it to refer to the two-dimensional faces that are revealed when a solid is sliced by a knife travelling in a plane. So when we say that Figure 4 shows cross sections of an apple, we are thinking of these slices as infinitely thin or what you would get if you dipped each in ink and stamped it to get a two-dimnensional image.

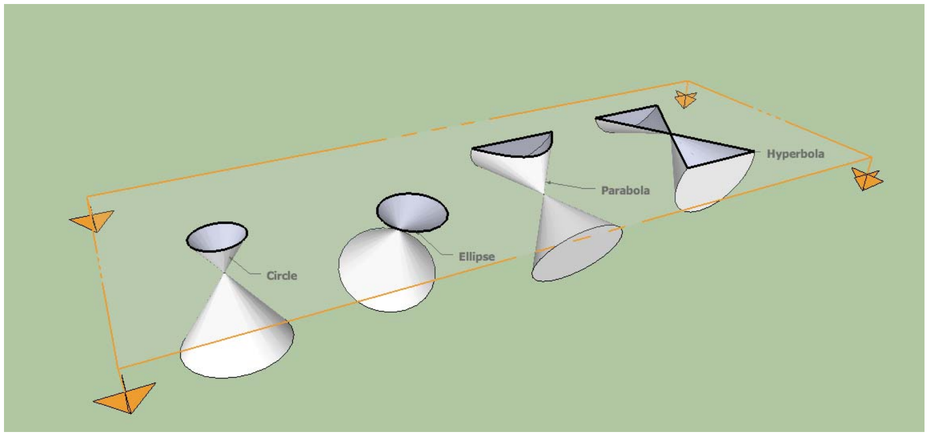

FIGURE 3. Cross of cones. Notice that these cross sections give rise to the important mathematical objects called the **conic sections**: *circle*, *ellipse*, *parabola*, and *hyperbola*.

Activity Preparation Collect together several dense, dry objects that can be easily by a knife or thin piece of wire. (E.g. apples, zuchinni, bagles, sponge cake, Play-Doh, cheese,...) Line a table with a large sheet of paper. Using a knife or piece of wire, you are going to slice your objects into thin, parallel, and equally spaced slices which mimic cross sections of these solids, as pictured in Figure 4.

9. Choose an object to slice. Choose a direction to slice this object - vertically, horizontally, or along some fixed angle. Before you do any slicing, think about what the resulting cross sections of your solid will look like. Draw a sketch of these cross sections in your notebook.
10. Make a number of equally spaced, parallel slices through your object. Arrange the slices linearly and sketch them in your notebook.
11. How do the actual cross sections compare to what you predicted? Is there anything that was surprising?
12. How do the cross sections change as you move through the object? Describe these changes in detail.
13. For the same object, choose a different direction to slice in. Now repeat Investigation **9** - Investigation **12**, slicing in this new direction.
14. Choose a different, second object and repeat Investigation **9** - Investigation **13**.
15. Choose a different, third object and repeat Investigation **9** - Investigation **13**.

16. Classroom Discussion: Based on your observations, what can the cross sections tell us about the three-dimensional object that is being sliced?

17. As noted in our description of Flatland above we note that the climax begins when square is visited by what he thinks is a circular object that grows and shrinks rapidly before his eyes. How is Sphere's appearance related to our discussion of cross sections?

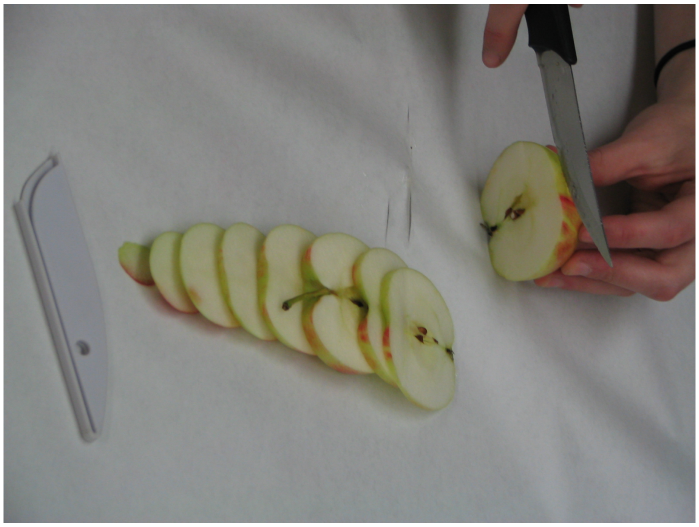

FIGURE 4. Cross sections of an apple.

18. Does it matter how Sphere was oriented as he passed through Flatland? Explain.
19. Are there any objects who would appear identical to Flatlanders no matter what orientation they passed through Flatland?
20. Are there any objects who would appear identical to Flatlanders if they passed through Flatland in several different orientations?
21. Find several objects whose cross sections in one orientation are all identical. How are these objects similar?

3. The Flatland Game

Sphere's visit to A Square in Flatland began a journey of enlightenment. We would like to put ourself in A Square's position and see how we can be enlightened. We'll do so via the **Flatland game**.

Flatland Game Goal Determine the identity of a solid object from a series of parallel cross sections.

22. Can you guess what secret solid make the cross sections shown in the first series of clues in Figure 5 as it passes through Flatland? If so, explain what the solid is and how you know its identity. If not, describe what you can ascertain about the solid from its cross sections.
23. Can you guess what secret solid make the cross sections shown in the second series of clues in Figure 5 as it passes through Flatland? If so, explain what the solid is and how you know its identity. If not, describe what you can ascertain about the solid from its cross sections.
24. Can you guess what secret solid make the cross sections shown in the third series of clues in Figure 5 as it passes through Flatland? If so, explain what the solid is and how you know its identity. If not, describe what you can ascertain about the solid from its cross sections.

Determining what a solid is from a sequence of clues is the *inverse problem* of finding the cross sections of a solid. Both are important problems. The Flatland game will give you practice with both - allowing you to move from Spaceland to Flatland and back.

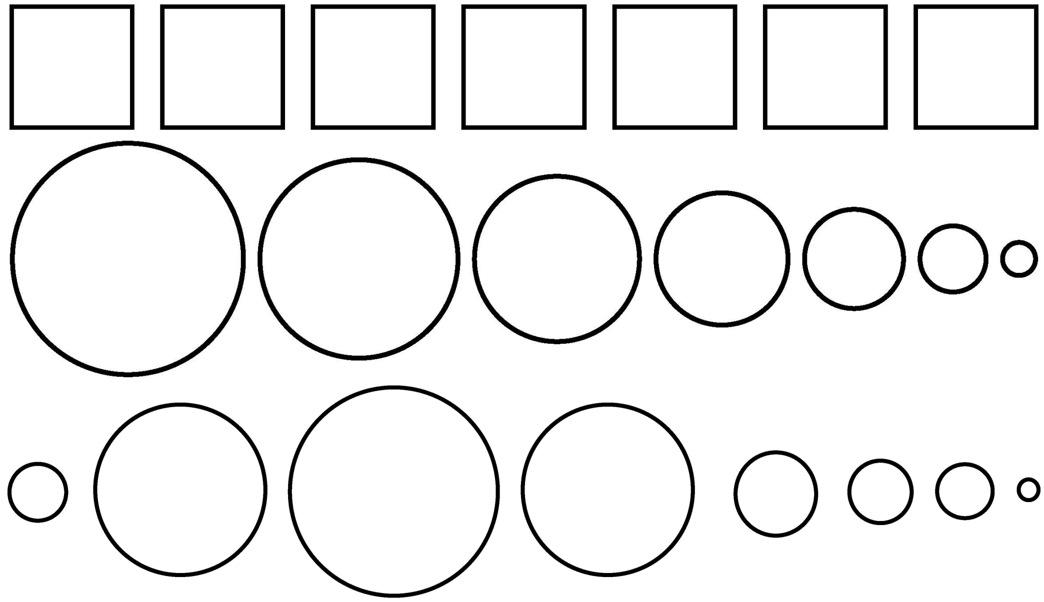

FIGURE 5. Three sets of clues for Flatland Games.

Rules and Roles for the Flatland Game

1. Choose a team of **radiographers**. This can be a single person or a small team where an illustrator has been elected.
2. Choose teams of **builders**. Teams can consist of a single person if necessary; it is best if a few small teams compete.
3. The game starts with the radiographers secretly determining a solid object from Spaceland whose identity will be the focus of the game.
4. The illustrator for the radiographers then begins play by drawing a single cross section, as viewed from above, of the secret solid.
5. The builders attempt to guess the identity of secret solid.
6. The illustrator for the radiographers then draws another cross section of the secret solid. This cross section must be parallel to earlier cross sections and the cross sections must be revealed consecutively as they would if the object actually passed through Flatland.
7. Steps 5 and 6 are repeated until:
   - **a.** a team of builders correctly guesses the identity of the secret solid, in which case they are declared the winner, or,
   - **b.** there are no more cross sections to draw and the radiographers are declared the winner.

25. Play the Flatland game. Draw the clues in your notebook and describe how the game went, noting any interesting geometrical issues that arose.

26. Play the Flatland game again with the same teams, recording the clues and any observations in your notebook.

27. Now switch roles. Let one of the teams of builders become the radiologists. Play two more games, recording the clues and any observations in your notebook.

28. Play a few more Flatland games, letting each team have a chance to be the radiographers. Each time record the clues and any observations you have in your notebook.

29. Compare your roles as radiographer and builder. Were there similar skills you needed? Were there similar challenges? In which did you learn the most about geometry?

30. Hopefully the radiographers weren't too tough. Name some objects whose identity would be really, really hard to determine in the Flatland game. Explain why they would be so hard.

4. Making Your Own Flatland Movie

As Sphere passed through Flatland A Square saw him growing and shrinking continuously, not just as a few discrete cross sections. You can mimic what it would look like as an object moves through Flatland by making a **Flatland flip book**.

31. Independent Investigation: You will need approximately two dozen small pieces of rectangular cardstock. Anything between a business card and a 3" by 5" index card will work.

- Choose a solid object that you would like to see pass through Flatland as a movie.
- Draw successive parallel cross sections (like the clues in the Flatland game) of the solid, one on each piece of cardstock. Leave some blank space on the left (this is where you'll hold the flip book) and be sure to draw the images in consistent locations on each page.
- Assemble the pictures into a flip book, binding it with a binder clip or heavy duty staple.
- Holding it on the left, thumb through the individual pictures and you will have a movie of your object passing through Flatland!

5. Sliceforms

Box dividers are found in many packaging situations, protecting bottles, glasses, ornaments or other breakables, as shown in Figure 6. If you have not seen box dividers, see if you can locate one and figure out how it works. Mechanically they are very interesting. Simply by making regular slots in lengths of cardboard and then lacing them together to form a grid, one arrives at a very inexpensive and sturdy way to safely package a variety of objects.

In the 1870's **Olaus Henrici** (Danish Mathematician; -) discovered these same mechanical principles could be used to make beautiful, dynamic mathematical models. We say dynamic because while box dividers fold nicely for storage, Olaus' *sliceform* models deform gracefully to show varied mathematical properties of the oject - in addition to the natural interest in having three-dimensional mathematical "solids" that can be folded flat to be carried in a pocket.

Figure 6. Cardboard box dividers.

Sliceforms were resqued from relative obscurity[8] by **John Sharp** (English Teacher; -) who published a wonderful book of templates called Sliceforms: Mathematical Models from Paper Sections in 1995. Subsequently he published full-lenght book on the topic: Surfaces: Explorations with Sliceforms in 2004.

Subsequently, sliceforms have seen quite a resurgence as many of the figures below help illustrate.

FIGURE 7. Sliceforms as art; the sculpture of Richard Sweeney and the award-winning greeting card by Up With Paper.

Sliceforms are related in fundamental ways to the geometric ideas that you investigated via the Flatland game. In this section you will explore sliceforms. The ultimate goal is for you to design and build your own original sliceform.

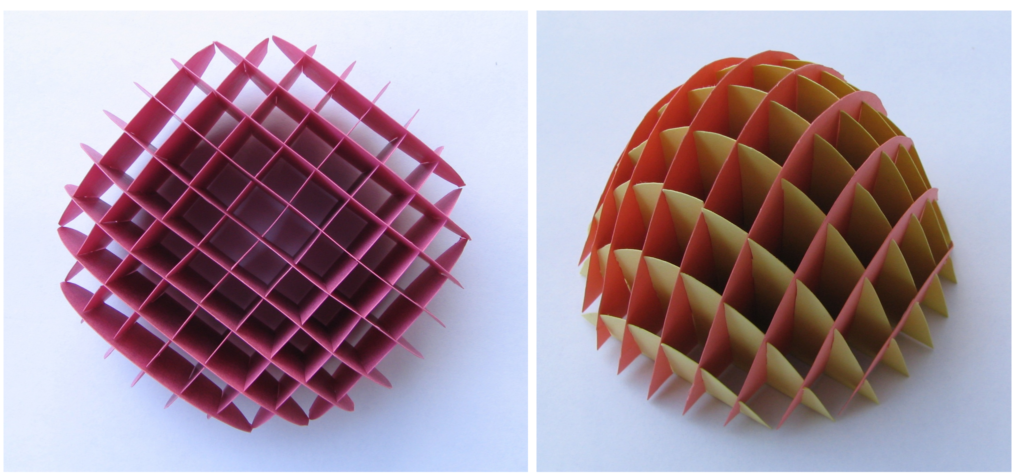

FIGURE 8. Three sets of clues for Flatland Games.

How do we go about designing and building a sliceform of a non-trivial three-dimensional object? Well, like most things you will need to explore, experiment, think, and plan. Your notebook is a perfect place to do this. The investigations below should provide some starting points.

[8]They do make a brief appearance on p. 19 of Geometry and the Imagination by **David Hilbert** (; -) and **Cohn-Vossen** (; -).

32. Look at the sliceforms pictured in Figure 7, Figure 8, and Figure 10. Imagine viewing them from the top. How are the slices positioned? What shape is formed in the openings between any four slices that meet at intersecting pairs?

33. Sliceforms are constructed so they easily fold down to be flat. As they are folded, how will the shape you described in Investigation **32** change? Explain.

34. Is there any other layout of slices that can be used to make a sliceform where the slices are all "straight" and the resulting sliceform can fold flat? I.e. will any shape other than those considered in Investigation **32** - Investigation **33** make a working sliceform?

35. In the materials list graph paper is listed. Based on Investigation **32** - Investigation **34**, describe how this graph paper can be used to help build an original sliceform.

Sliceform Materials Medium to heavy-weight cardstock. Sharp scissors. Ruler and possibly other measuring instruments such as a contour or profile gauge. Graph paper.

36. Cut out two small squares, say 3" on each side, from cardstock. In each cut a slit from the center of an edge directly to the center of the square. Insert one slit into the other. Is it easy to make the squares cross at right angles without holding them in place? Do the slits act as a smooth hinge so the object can be flattened down, easily opened back up, and then flattened down the other way?

37. Cut out two more small squares. Instead of cutting slits from the edge to the center, cut ***slots*** where a small amount of cardstock is actually removed by making two parallel cuts very close to each other. Now assemble the two pieces by mating their slots. Is it easier to make the squares cross at right angles without holding them in place? Do the slots act as a better hinge?

In Figure 9 are templates for a stock sliceform. Larger versions of these templates are included in the appendix.

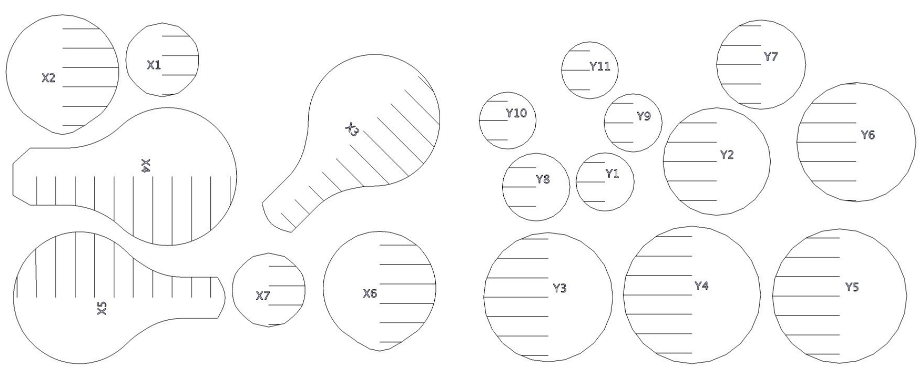

Figure 9. Templates for a sliceform model.

38. Can you guess what object this sliceform models when it is completed? Explain.

39. The individual pieces are labelled. What do you think the labels tell you? How were the names for these labels chosen?

40. Independent Investigation: Cut out the templates. Cut slots along each of the indicated lines. Then assemble the pieces into a completed sliceform. Describe the artistic, physical, and mechanical aspects of your sliceform.

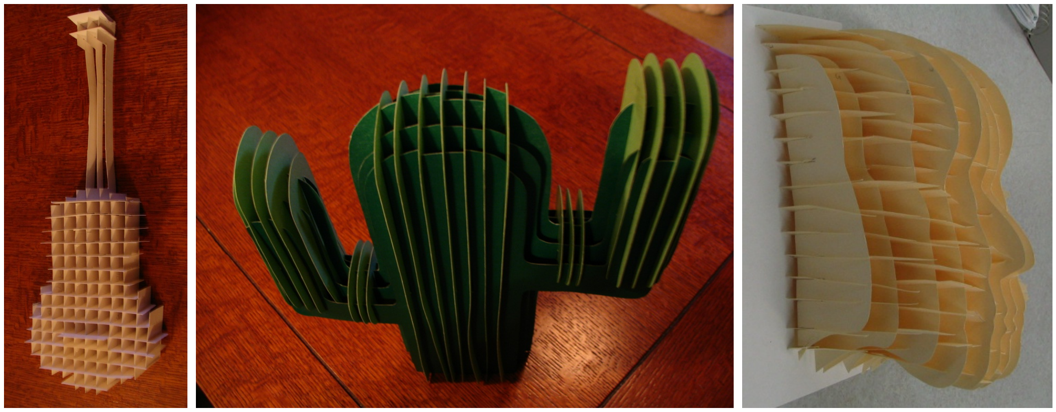

FIGURE 10. Original student sliceforms; "Guitar" by Katherine Cota, "Cactus" by Sharon Kubik-Boucher, and "The Face" the Lydia Lucia.

41. Look at your sliceform and those pictured in the figures in this chapter. When vertical and horizontal slices instersect - and are in fact joined at - a slot, how are the heights of the two slices related?

42. When you make your original sliceform you will have to figure out how deep to make each slot. Does Investigation **41** provide a good rule of thumb for doing this? Explain.

43. Suppose you loose a single slice of your model. How could you use your model to recreate this missing piece?

44. Does this give you an idea how you can create horizontal slices once you have all of the veritcal slices of your sliceform created appopriately? Explain.

45. INDEPENDENT INVESTIGATION: Design and build your own original sliceform.

This process may take you several hours over a few sittings. The design phase is quite important. As you design you should consider what you learned in the Flatland game and supplement it with tools like Play-Doh to make mock models, measuring tools like profile or contour gauges (see Figure **??**), profiles cast by shadows of light, actual cross sectioning of the object, or the use of Computer Aided Design software like the free Google SketchUp.

FIGURE 11. A profile gauge; used regularly by carpenters and machinists.

6. More General Cross Sections

End with Biesty books, general cross section, body worlds, etc. Now the possibilities are limitless. How do you want to explore our 3D world and 3D objects therein?

Body Worlds intro. There have been ethical issues related to this and similar exhibits. There are also religous and personal decisions that individuals must make. But as this exhibit travels the world thousands of people see the human body in entirely different ways - learning to see this miraculous machine in entirely new ways. Each time the authors have visited the exhibit the audience has included professionals - including doctors, nurses, athletic trainers, physical therapists - seeing new things. It is regular to hear things like “Wow, now I really see how the position of the —- impacts the motion of the —-.”

FIGURE 12. Images from the Body Worlds exhibit.

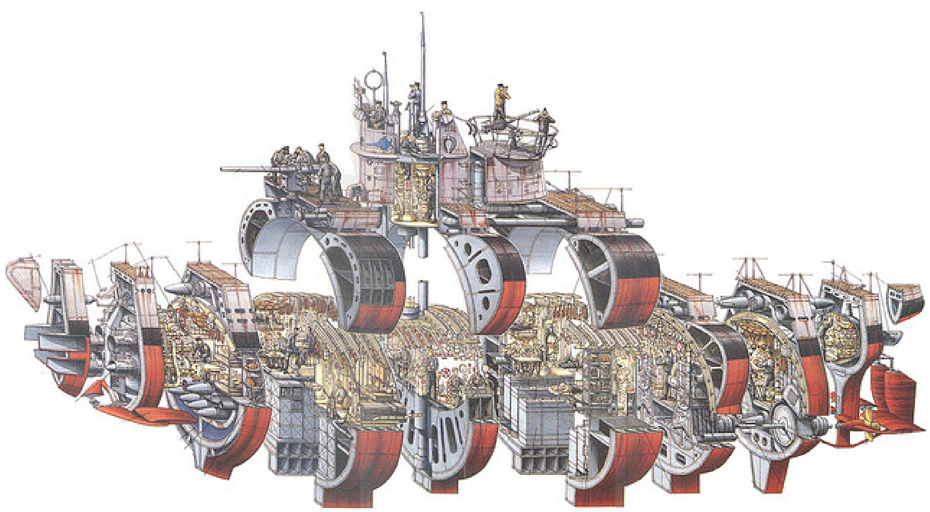

FIGURE 13. UBoat Cross Section by Stephen Biesty.

7. Alternate Approach: Projections

8. Connections

Needs to be something about Plato’s “Cave” giving a context where we might think about a two-dimensional world. The context is quite different - but both are getting at our ability to think about the limits of our perception providing notable limits to knowledge.

Needs to have something about Google SketchUp and its ability to do cross sections. Mention Google 3D Warehouse and let them know that they can take cross sections of any of the models that are available in here. This might also be good for near the conclusion.