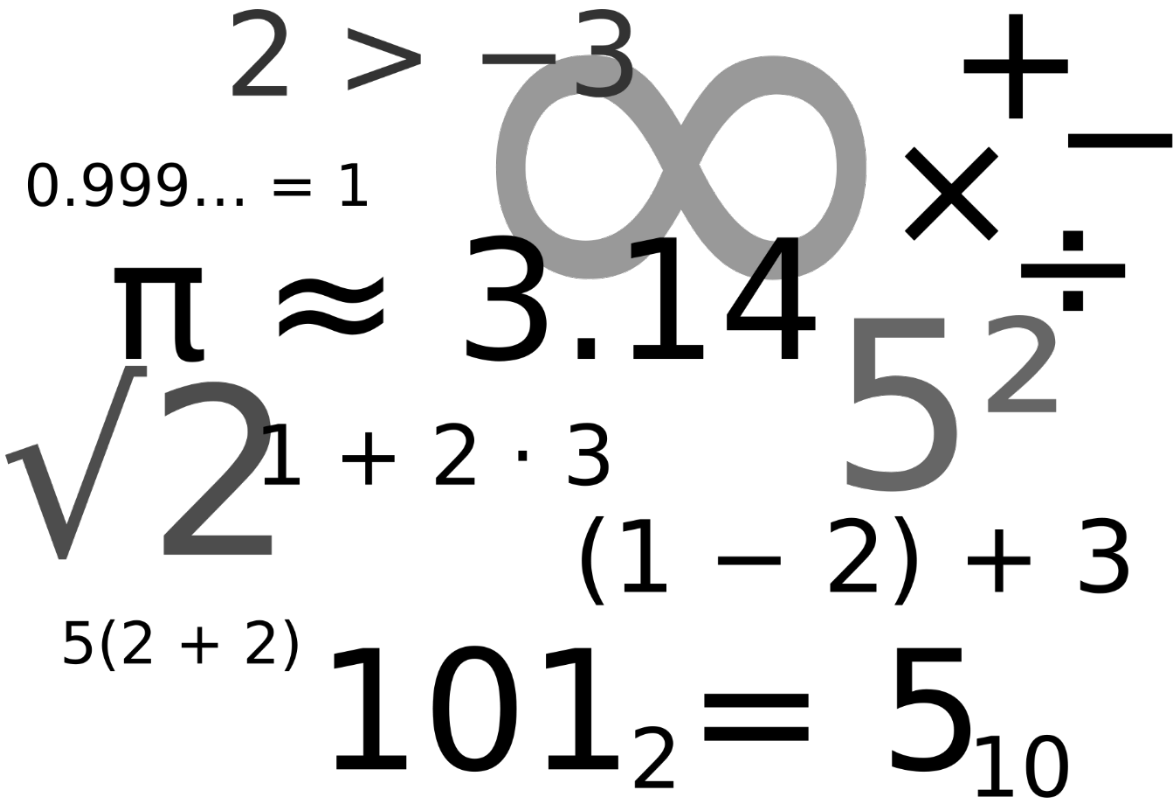
Pre-Calculus

MAT 153 Workbook



California State University, Dominguez Hills

Fall 2017

Precalculus

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June 15, 2017

Contents

Linear Functions

In this course, we will study many types of functions. Functions are mathematical objects that for each input provide exactly one output. Linear functions are used to describe patterns where the rate of change is constant, and will be the first type of function that we study.

Goals:

- L: Be able to solve a linear equation.
- L: Be able to model a situation with appropriate linear equation(s) and interpret the solution.
- L: Be able to determine the slope or equation of a linear function given its graph or a table of values.
- F: Be able to determine inputs or outputs from a function table.
- F: Be able to determine inputs or outputs from a function graph.

Example 1. *$f(x) = 2x + 3$ is a function that takes an input, x, and produces the output $2x + 3$. For instance, with the input 4, the output is $f(4) = 2(4) + 3 = 11$. Often, we will have information about a function presented in a table or a graph. Table 1 shows some input-output pairs for $f(x)$.*

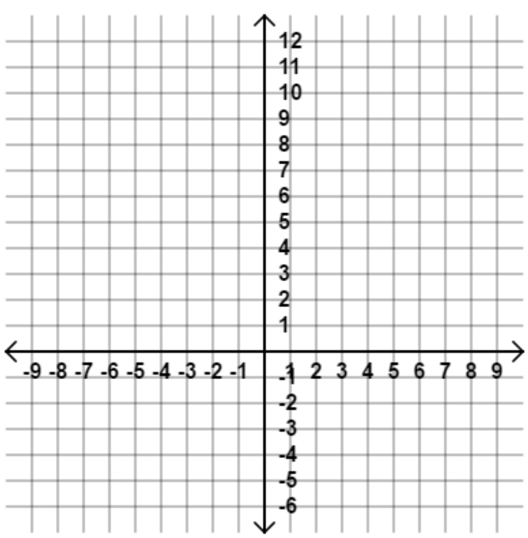
Table 1: Input-Output for $f(x)$

x	$f(x)$
0	3
1	5
2	7
3	9
4	11

Definition 2. *The **domain** of a function is the set of inputs to the function. The **range** of a function is the set of outputs from the function.*

Problem 3. *Let $f(x) = 2x + 3$.*

1. *Graph on the domain $-4 \le x \le 4$. Be sure to label the intercepts and the endpoints of the function on this domain.*



2. *Solve $f(x) = -1$ for x.*

3. *What is $f(-4)$?*

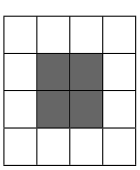
Problem 4. *Table 2 shows a number pattern. Fill in the blanks in the table, and make a graph of the values in the table. Then, write a function equation for $f(n)$.*

Table 2: Patterns

n	$f(n)$
1	1
2	3
3	5
4	7
5	
6	
20	

Problem 5. *Nancy is a landscape artist. Her specialty is a square pond that is surrounded by hand-painted tiles. Customers can order the pond in any size square, starting with sides measuring 2 feet, and available in one-foot increments thereafter (sides of 3 feet, 4 feet, 5 feet, etc.). The tiles are 1-foot square, and are placed edge-to-edge along the entire outer perimeter of the pond. An example pond is shown in Figure 1.*

Figure 1: Pond with 2-foot sides and square tile border



1. *Make a table showing the number of border tiles needed for different pond sizes, from 2-foot sides up through 6-foot sides.*

2. *If the side length of the pond is increased by 1 foot, how many more border tiles are needed? Explain why the number of tiles is increasing according to this pattern.*

3. *How many border tiles are needed for a pond with sides of length 12 feet?*

4. *If Nancy orders 64 border tiles for an upcoming job, how large is the pond the customer wants?*

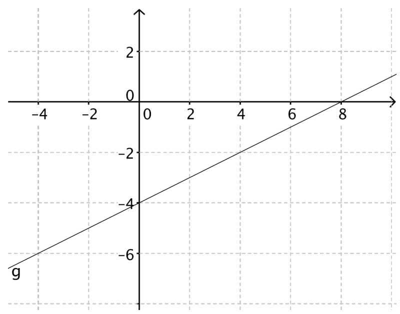
5. *Describe in words how to find the number of tiles needed for the border of a pond.*

6. *Write an equation for the number of tiles as a function of the length of one edge of the pond.*

Notes

Exercise Set 1

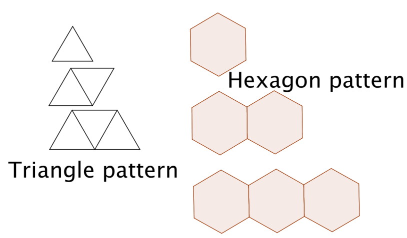
Figure 2: Graph of $g(x)$



1. Refer to the graph of $g(x)$ in Figure 2.

   (a) What is $g(3)$?
   (b) Solve $g(x) = 0$.
   (c) Write a function equation for $g(x)$.
   (d) Use your function equation for $g(x)$ to find $g(3)$. Does this match your answer to part 1a?

Figure 3: More Patterns



2. Refer to Figure 3, Triangle Pattern. Assume that each edge of the triangle measures 1 cm. Assume that one triangle is added to one figure to get the next figure. Let $P(n)$ be the function describing the perimeter as a function of the figure number.

   (a) Draw the next two figures in the pattern.
   (b) Make a table with columns for n and $P(n)$ for $1 \leq n \leq 5$.
   (c) Make a graph of the values in your table.
   (d) Write an equation for $P(n)$.
   (e) Find $P(12)$.
   (f) How many triangles are in a figure with a perimeter of 18?

3. Refer to Figure 3, Hexagon Pattern. Assume that each edge of the hexagon measures 1 cm. Assume that one hexagon is added to one figure to get the next figure. Let $H(n)$ be the function describing the perimeter as a function of the figure number.

   (a) Draw the next two figures in the pattern.

(b) Make a table with columns for n and $H(n)$ for $1 \leq n \leq 5$.
(c) Make a graph of the values in your table.
(d) Write an equation for $H(n)$.
(e) Find $H(15)$.
(f) Solve $H(n) = 38$.

4. Refer to $p(x) = -3x + 12$.

   (a) Graph $p(x)$ on the domain $-3 \leq x \leq 5$. Label the intercepts.
   (b) Solve $p(x) = 0$ for x using the function equation.
   (c) What is $p(-2)$?

5. Sally is trying to choose between two internet plans. Under the first plan, Verizon will sell her a DSL modem for \$17.99, and then she must pay \$12.99 per month. Under the second plan, ATT will give her a modem for free, but she must pay \$14.99 per month.

   (a) When does the Verizon plan become the better deal (how many months)?
   (b) Describe at least one other possible solution method for part (5a) besides the one you used.

6. Mr. and Mrs. Jones both teach mathematics. By 1999, Mrs. Jones had taught 300 students, and has an average of 150 students each year. Mr. Jones began teaching his own classes in 1999, and has an average of 100 students each year.

   (a) Write a function equation to estimate the total number of students Mrs. Jones has taught as a function of t, the number of years since 1999.
   (b) Write a function equation to estimate the total number of students Mr. Jones has taught as a function of t, the number of years since 1999.
   (c) In what year has Mrs. Jones taught 1000 *more* students than Mr. Jones?

Linear Parametric Equations

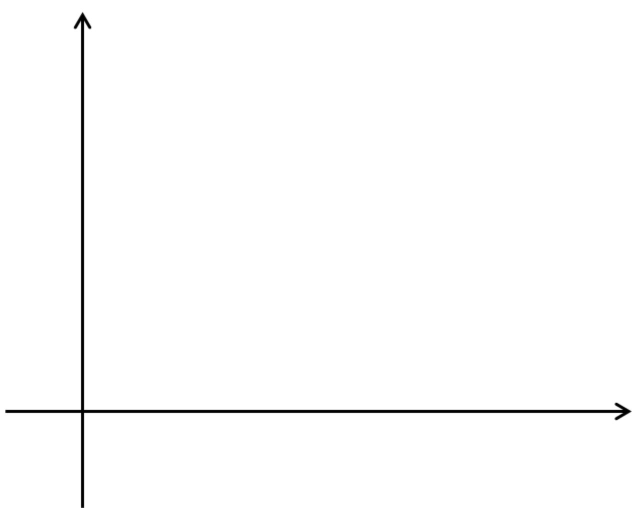
Parametric equations are often used to keep track of the position of an object in motion. In this case, rather than having y as a function of x, usually we have x as a function of t (time), and y as a function of t.

Goals:

- F: Be able to draw a diagram incorporating all of the important information in a given situation, and impose coordinates on the diagram.
- L: Be able to model a situation involving linear motion with appropriate parametric equation(s) and interpret the solution.

Example 6. *Watch the following video:* `http://tinyurl.com/freefallexample`.

1. *Sketch a graph of the person's height as a function of time.*



2. *The person's height as a function of time is given by the function* $h(t) = 100 - 16t^2$.
   (a) *What is the person's height before beginning the free fall?*

   (b) *What is the person's height at* $t = 1.5$ *seconds?*

   (c) *How long is the person free falling?*

3. *If we impose coordinates so that the person is falling directly down the y-axis, the position of the person at time t is given by $(0, 100 - 16t^2)$.*

   (a) *What are the coordinates of the person at time $t = 2$ seconds?*

   (b) *At what time will the coordinates of the person be $(0, 50)$?*

`http://tinyurl.com/desmosparex` *shows how to graph both $h(t)$ and the parametric equations in Desmos.*

Example 7. *A woman is walking in a park. She begins at a point 30 meters west of the northwest corner of the playground, and walks in a straight line to a pond 60 meters north of the same corner of the playground.*

1. *Draw a diagram showing the woman's path and impose coordinates on the diagram.*

2. *Write an equation for the line that describes the woman's path.*

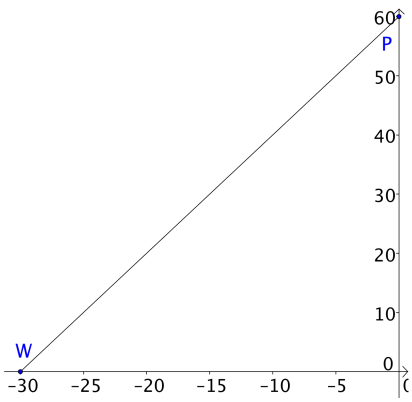
3. *It takes the woman 20 seconds to walk to the pond. Write linear parametric equations to describe her position at time t seconds.*

4. *It takes a different woman 30 seconds to walk the same path, but in the opposite direction. Write linear parametric equations to describe the second woman's position at time t.*

The steps used to solve Example 7 are outlined below:

To solve part 1, we draw Figure 4, where W is the woman's starting position, and P is the pond. (Other diagrams are possible.) To solve 2, find the equation passing through the two points $(-30, 0)$ and $(0, 60)$. The slope is $\frac{60-0}{0-(-30)} = 2$, and the y-intercept is 60, so the equation is $y = 2x + 60$.

Figure 4: A walk in the park



To solve 3, we are looking for two equations, $x(t)$ and $y(t)$.
Step 1: Get values for t, x, and y at two points in time. *In this problem, we know that at $t = 0$, we are at point W, so $x = -30$ and $y = 0$, and at $t = 20$ seconds, we are at point P, where $x = 0$ and $y = 60$.*

Step 2: Get the $x(t)$ linear equation. *We are going to find the equation of a line where we treat t as the independent value and x as the dependent value. Notice that the slope will be $\frac{\triangle x}{\triangle t}$. So we have $\frac{0-(-30)}{20-0} = \frac{3}{2}$, and since at $t = 0$, $x = -30$, then our equation is $x(t) = \frac{3}{2}t - 30$.*

Step 3: Get the $y(t)$ linear equation. *We are going to find the equation of a line where we treat t as the independent value and y as the dependent value. Notice that the slope will be $\frac{\triangle y}{\triangle t}$. We have $\frac{60-0}{20-0} = 3$. At $t = 0$, $y = 0$, so our equation is $y(t) = 3t$.*

Step 4: Verify the solution. *On Desmos, type in $(\frac{3}{2}t - 30, 3t)$, and set $0 \leq t \leq 20$. You should see the path from Figure 4.*

Problem 8. *Alberto is at the beach, and is hungry for a hot dog. Currently, he is standing at the water's edge, 100 meters west of the boardwalk. Directly in front of him on the boardwalk, the nearest food vendor is his friend Carmen, who sells pizza. The hot dog vendor is 40 meters south of Carmen's pizza stand. Alberto plans to walk directly to the hot dog vendor.*

1. *Draw a diagram showing Alberto's path and impose coordinates on the diagram.*

2. *Write an equation for the line that describes the Alberto's path.*

3. *Write parametric equations to describe Alberto's position at time t, if it takes him 90 seconds to walk from the water to the hot dog vendor.*

Notes

Exercise Set 2

1. Write parametric equations for a point traveling along the line $y = 2x - 6$, such that at $t = 0$ the point is at the x-intercept, and at $t = 1$ the point is at the y-intercept.

2. A volleyball court measures 18 meters by 9 meters. Jane takes 15 seconds to walk from one corner of the court to the diagonally opposite corner.

   (a) Draw a diagram of the court and Jane's path across the court, and impose coordinates on the diagram.

   (b) Write parametric equations to describe Jane's path.

3. Barbara is visiting another county and wants to know the sales tax rate. She just bought a pair of shorts for \$22 and paid \$1.87 in sales tax.

   (a) What tax will she pay if she buys sunblock for \$8?

   (b) What is the sales tax rate?

   (c) Write a function equation that computes the tax as a function of the item price.

4. Maria is preparing envelopes for mailing. Maria has already prepared 50 envelopes this morning. At 1 pm, she returns from lunch. She can prepare 110 envelopes per hour.

   (a) Write an equation for the total number of envelopes Maria has prepared t hours after 1 pm.

   (b) If Maria hopes to prepare 750 envelopes before going home, when can she expect to be done?

5. An object is moving along a line in the xy-plane so that at time $t = 0$, it is at the point $(8, 16)$ and at $t = 4$, it is at the point $(0, -20)$.

   (a) Draw a diagram of the xy-plane showing the object's position at time $t = 0$ and $t = 4$.

   (b) Write parametric equations $(U(t), V(t))$ to describe the position of the object at time t.

6. The owners of a movie theater have determined that the number of people who attend is a function of the price of the tickets. The theater has a capacity of 240 people. If tickets are sold for \$1, the theater will fill up completely. On the other hand, if the theater charges \$21/ticket, no one will buy tickets.

   (a) Assuming that the number of tickets sold, $S(t)$ is a linear function of the price, t, write an equation for $S(t)$.

   (b) At what price will the theater fill 150 seats?

7. The U.S. is nearly the last country to use the English system of measurement, which includes the Fahrenheit scale for temperature instead of the Celsius scale. There are two conversion formulas, one from Celsius to Fahrenheit, and one from Fahrenheit to Celsius. One of the two formulas is $F = \frac{9}{5}C + 32$, where F is the temperature in degrees Fahrenheit, and C is the temperature in degrees Celsius.

   (a) Explain the significance of the F-intercept in the given formula.

   (b) Find the conversion formula for Fahrenheit to Celsius.

   (c) When traveling abroad, it can be useful to know the formula, but it is not necessary to remember the formula. One way to generate the formula is by remembering two temperatures on both scales. Why is it enough to remember just two temperatures in both scales?

(d) In fact, this author remembers three temperatures: 0^oC is 32^oF, 20^oC is 68^oF, and 100^oC is 212^oF. Choose a pair of temperatures and use them to recover the conversion formula from Celsius to Fahrenheit.

Table 3: Temperature conversions

Temp (^oF)	59	68	77	86	95
Temp (^oC)		20			

8. Erwin is at an outdoor market. He is walking in a straight line from a fruit stand at a point 65 feet due West of a fountain to a florist at a point 420 feet due North of the fountain. He walks at a constant speed of 5 feet per second.

   (a) Draw a diagram showing the location of the fountain, the fruit stand, and the florist, and impose coordinates on the diagram.

   (b) Write an equation describing Erwin's path through the market.

   (c) Write parametric equations for Erwin's position t seconds after he begins walking.

9. Allie is at the beach, and is hungry for a hot dog. Currently, she is standing 60 meters west of the boardwalk. Directly in front of her on the boardwalk, the nearest food vendor is her friend Carmen, who sells pizza. The hot dog vendor is 40 meters south of Carmen's pizza stand. Allie plans to walk directly to the hot dog vendor.

   (a) Draw a diagram showing Allie's path and impose coordinates on the diagram.
   (b) Write an equation for the line that describes the Allie's path.
   (c) Write parametric equations to describe Allie's position at time t seconds, if it takes her 40 seconds to walk from her initial position to the hot dog vendor.

10. A basketball court is 94 feet by 50 feet. Lisa takes 20 seconds to walk from one corner of the court to the diagonally opposite corner.

    (a) Draw a diagram of the court and Lisa's path, and impose coordinates on the diagram.

    (b) Write parametric equations to describe Lisa's path.

Quadratic Functions

In this section we introduce quadratic functions. The graph of a quadratic function is a parabola, and in addition to the intercepts, we will be interested in finding the vertex of the parabola.
Goals:

- Q: Be able to solve a quadratic equation.
- Q: Be able to determine the equation or vertex of a quadratic function given its graph.
- F: Be able to give the solution to an inequality or set of inequalities using proper mathematical notation.
- F: Be able to determine the domain or range of a function given as an equation or a graph.

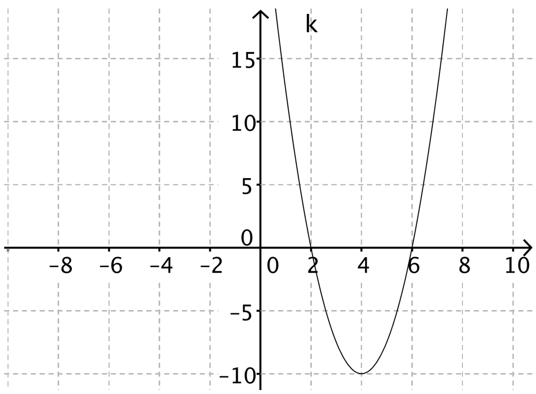
Definition 9. *A* ***quadratic function*** *is one of the form* $f(x) = a(x-h)^2 + k$*.*

In Desmos, graph $f(x)$ *and create sliders for* a*,* h *and* k*. What role do each of these constants play? Include sketches and verbal descriptions to help explain the role of each constant.*

Remark 10. *Sometimes, we will have the graph of the quadratic function. Other times, we have the equation for the quadratic function, but it is not in the form in Definition 9. Also, as we've seen with linear functions, the names of the variables don't matter. Instead of* x*, you might see* t *or another variable, and instead of* f*, the function may be named with a different letter.*

Example 11. *Refer to the graph of $k(x)$ in Figure 5.*

Figure 5: Graph of $k(x)$



1. *What are the coordinates of the vertex of $k(x)$?*

2. *Write a function equation for $k(x)$.*

3. *Solve $k(x) = 15$.*

4. *Solve the inequality* $k(x) \leq 15$.

5. *What is the range of* $k(x)$ *on the domain of all real numbers?*

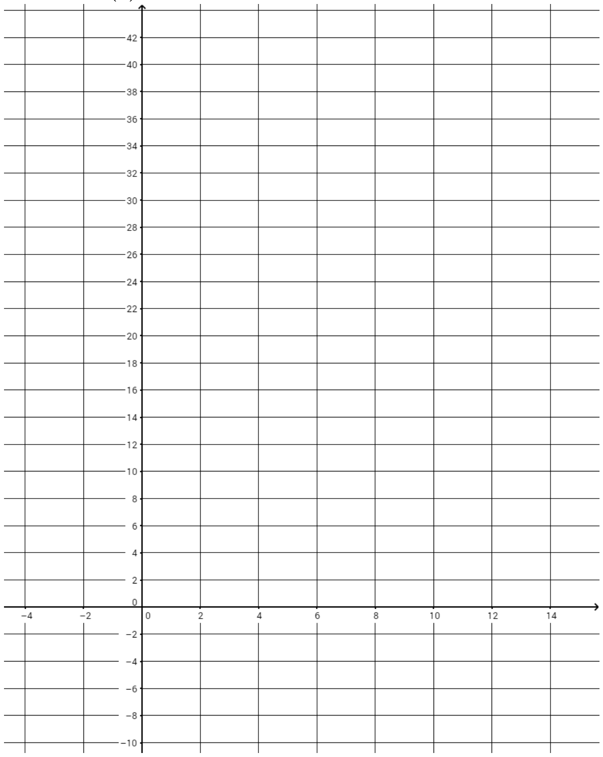
The solution to Example 11 is outlined below:
For part 1, read the vertex off the graph. The vertex is $(4, -10)$. For part 2, write an equation for the quadratic function in the form given in Definition 9, $k(x) = a(x-4)^2 - 10$. In order to find the value of a, we must use another point on the graph. We will use the point $(2, 0)$. By substituting the point $(2, 0)$ into our equation, we can solve for a.

$$\begin{aligned} k(x) &= a(x-4)^2 - 10 \\ 0 &= a(2-4)^2 - 10 \\ 10 &= 4a \\ \frac{5}{2} &= a \end{aligned}$$

So a function equation for $k(x)$ is $k(x) = \frac{5}{2}(x-4)^2 - 10$. The solutions of parts 3 and 4 are left to the reader. For part 5, the range of $k(x)$ is $[-10, \infty)$.

Problem 12. *Refer to the function* $r(t) = \frac{1}{2}t^2 - \frac{9}{2}t + 4$.

1. *Graph* $r(t)$ *on the domain* $-2 \leq t \leq 14$*. Be sure to label the vertex and any intercept(s).*

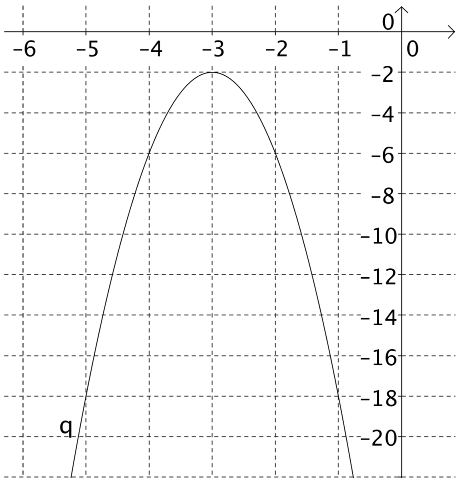


2. *What is the range of* $r(t)$ *corresponding to the domain* $-2 \leq t \leq 14$*?*

3. *What is the range of* $r(t)$ *on the domain of all real numbers?*

Problem 13. *Refer to the graph of $q(x)$ in Figure 6.*

Figure 6: Graph of $q(x)$



1. *Evaluate $q(-2)$.*

2. *What are the coordinates of the vertex of $q(x)$?*

3. *Write a function equation for $q(x)$.*

4. *Solve the inequality $q(x) \leq -6$.*

5. *Solve the inequality $q(x) > 2x - 12$.*

6. *What is the range of $q(x)$ on the domain of all real numbers?*

Supplemental Lesson: Interval Notation

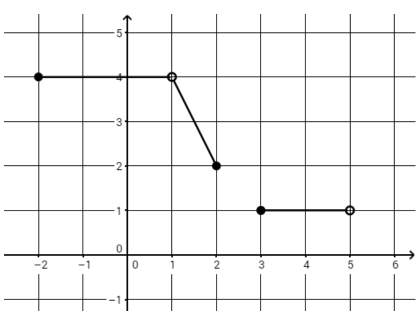
We can represent all the numbers between -2 and 7 (including -2 but not including 7) in multiple ways - using inequalities, interval notation, or a graph on a number line.

Inequalities: $-2 \leq x < 7$
Interval Notation: $[-2, 7)$

Write each of the intervals below using inequalities and interval notation.

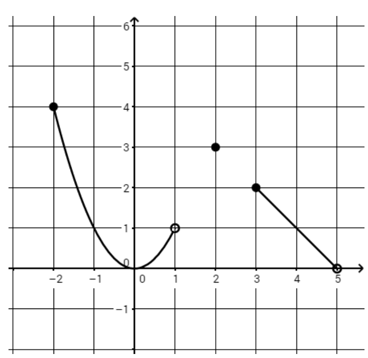
Inequalities	Interval Notation
1. $x \leq 5$	____________________
2. __________________	$(-5, 0)$
3. $-3 > x$	____________________
4. __________________	$(-\infty, 0]$
5. $2 < x \leq 8$	____________________
6. $x \leq 2$ or $x > 6$	____________________

7. Find the domain and range of each function below:



Domain:

Range:



Domain:

Range:

Notes

Quadratic Modeling

Quadratic functions are useful as mathematical models. We will use them to model objects experiencing the force of gravity, or for modeling income when demand is a linear function of the price.
Goals:

- Q: Be able to model problem situations with an appropriate quadratic function equation(s) and interpret the solution(s).
- Q: Be able to model motion of objects falling with the force of gravity with appropriate quadratic equation(s) and interpret the solution.
- Q: Be able to determine the equation of a quadratic function given its graph.
- Q: Be able to determine and interpret the vertex of a quadratic function given an equation or context.

Problem 14. *A flower pot falls from the ledge of a balcony on a high-rise building. Any object experiencing the force of gravity can be modeled by the equation $h(t) = -16t^2 + vt + c$, where t is the time in seconds, $h(t)$ is the height in feet, c is the initial height of the object, and v is the initial velocity of the object. (This model applies to any object experiencing the force of gravity, where the measurement is in English units (feet). There is another version for metric units (meters).)*

1. *If the pot falls, what is its initial velocity?*

2. *Suppose it takes 3 seconds for the pot to hit the ground. How high was the balcony?*

3. *Write the equation, $h(t)$ that models the height of the pot at time t.*

4. *What is the height of the pot at time $t = 1.5$ seconds?*

5. *When will the height of the pot be 100 feet?*

Problem 15. *Esperanza is the owner of an independent motel that has 50 rooms. She finds that if she charges \$40 per night, all the rooms will be rented. Thereafter, for every \$4 she raises the room rate, 2 fewer rooms will be rented out.*

1. *Make a table showing the price of the room, the number of rooms rented, and the total income for the hotel for prices of \$40 to \$60 per night, in \$4 increments.*

2. *Write a function equation for the number of rooms rented as a function of the price of a room.*

3. *Write an equation describing the income for the motel for one night.*

4. *Suppose Esperanza will be satisfied if the motel's income for a night is at least \$2300. What are the possible prices she can charge to earn this income?*

5. *What price will earn the maximum income for the hotel? How many rooms will be rented at this price? What is the maximum income the hotel will earn?*

Problem 16. *Adam is raising pigs on his farm. He needs to build a rectangular pen for his pigs, and he wants to give them as much area as possible. However, he only has 180 feet of fencing.*

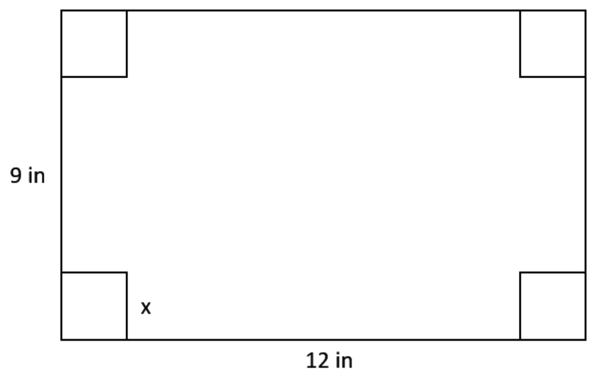
1. *Draw a diagram showing a rectangular pen, and labeling the lengh of one of the sides of the pen with a variable.*

2. *Write an equation describing the area of the pen in terms of the length variable you chose.*

3. *How should the pen be built to get the maximum possible area? What is the maximum possible area?*

Problem 17. *Ernesto is going to make an open-top cardboard box to store some items on his desk. His original piece of cardboard is 9 inches by 12 inches. He is going to cut out squares from each corner of the cardboard and fold up the resulting sides to make a box, as in Figure 7. He wants to create the largest volume possible for his box.*

Figure 7: Open Top Box



1. *Write an equation for the volume of the box in terms of x.*

2. *How large should the cutout be to maximize the volume of the box? What is the maximum volume?*

Notes

Exercise Set 3

1. A movie theater seats 240 people. For any particular show, the amount of money the theater makes is a function of the number of people, n, in attendance. Analysis of recent price and attendance data suggests that for a weekday matinee showing, if the price of a ticket is set at p dollars, then the number of people in attendance, n, is given by $n = 240 - 12p$.

   (a) At what price will no one attend the showing?

   (b) How many people will attend if prices are set at the regular price of \$12? What income will the theater earn at this price?

   (c) Write an equation that describes the income, I, that the theater will earn in terms of the ticket price, p.

   (d) At what price should tickets be sold to earn the greatest ticket income from the matinee show? What income will the theater earn at this price?

2. A toy rocket is launched from a table 2 feet above the ground. The height of the rocket above the ground (in feet) is given by the equation $h(t) = -16t^2 + vt + 2$, where v is the launch velocity.

   (a) If the rocket reached its maximum height of 38 feet above the ground at time $t = 1.5$ seconds, what was the launch velocity?

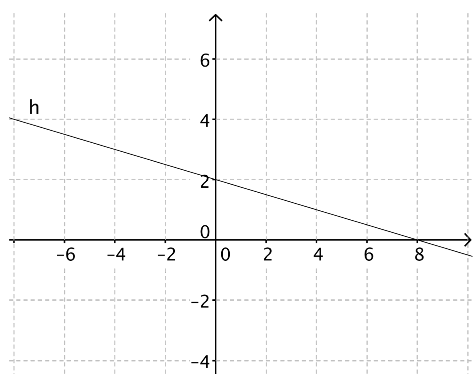
   (b) Write a function equation for $h(t)$.

   (c) How long was this rocket in the air?

3. Refer to $g(t) = t^2 - 8t + 15$.

   (a) Graph the function $g(t)$ on the domain $-6 \le t \le 6$. Be sure to label the vertex and any intercept(s).
   (b) Solve the inequality $g(t) > 3$.
   (c) Solve the inequality $g(t) < t^2 - 1$.
   (d) What is the range of $g(t)$ on the domain of all real numbers?

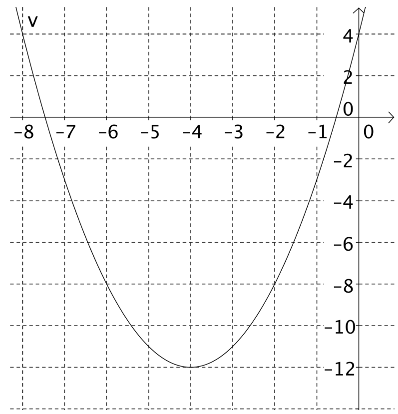
Figure 8: Graph of $h(x)$



4. Refer to the graph of $h(x)$ in Figure 8.

   (a) What is $h(4)$?
   (b) Write a function equation for $h(x)$.

Figure 9: Graph of $v(t)$



(c) Use your function equation to evaluate $h(4)$. Does this match with your answer from part 4a?

5. Refer to the graph of $v(t)$ in Figure 9.

   (a) What are the coordinates of the vertex of $v(t)$?
   (b) Write a function equation for $v(t)$.
   (c) Solve the inequality $v(t) \geq -12$.
   (d) What is the range of $v(t)$ on the domain of all real numbers?

6. A baseball is hit by a batter when it is 3 feet off the ground. Its height off the ground is given by $h(t) = -16t^2 + 80t + 3$, where h is in feet and t is in seconds. Its distance (along the ground) from home plate, where it contacted the bat, is $d(t) = 60t$.

   (a) Draw a graph of $h(t)$ on a domain so that the h- and t-intercepts are visible on the graph.
   (b) How long is the ball in the air?
   (c) How far from home plate does the ball hit the ground?
   (d) Use Desmos to graph the parametric equations $(d(t), h(t))$ on the domain $0 \leq t \leq 5.5$. Sketch the graph on your paper.

7. Adam is also raising chickens, and wants to build a fence next to the chicken coop. This time, he has 100 feet of chicken wire fencing, and the fence is going to be built so that the chicken coop wall forms one side of the enclosure, with the wire fencing along the other three sides. He wants to build a rectangular space so that his chickens as much area as possible to roam.

   (a) Draw a diagram showing the wall of the chicken coop and the three sides of wire fencing forming a rectangle, and label one of the sides of the wire fencing as x.

   (b) Write an equation for the area of the chicken enclosure in terms of x.

   (c) How should the fencing be used to get the maximum possible area? How much area will the chickens get?

8. For every 100 ticket buyers at a concert, the band expects to sell 27 T-shirts.

   (a) If they have sold 1350 tickets to their upcoming show, about how many T-shirts can they expect to sell?

(b) If each T-shirt sells for \$20, what can the band expect the total sales value of the T-shirts at the concert to be?
(c) If the band's *profit* per T-shirt is \$7, and their profit per ticket sold is \$ 2, write an equation for the band's profit in terms of the number of ticket buyers.
(d) How many people must come in order for the band to earn \$500?

9. Graph the path traveled by a point described by the parametric equations $x(t) = 3+4t$, $y(t) = 8-2t$ on the domain $-2 \leq t \leq 5$. Label the location of the point at $t = 0$.

10. Table 4 shows a linear pattern. Make a graph of the values in the table. Then, write a function equation that gives the output value corresponding to any input.

Table 4: Patterns

n	$g(n)$
1	17
2	14
3	11
4	8

11. A ball is launched vertically into the air from a height of h meters and with an initial upward velocity of v meters/second. The ball's height above ground is given by the equation $H(t) = -4.9t^2 + vt + h$, where H is in meters and t is in seconds. (This is the metric version of the gravity model.)

(a) Write an equation to model the height of a ball thrown from a height of 2 meters off the ground, with an initial upward velocity of 40 meters/second.
(b) How long is the ball in the air?
(c) What is the maximum height reached by the ball?

Exponential Functions

Definition 18. *An **exponential function** is a function of the form $f(t) = ab^t$, where a is a nonzero real number, and b is a positive real number. Whenever we have an expression like b^t, b is called the **base** and t is the **exponent**. The number e is often used as the base in exponential functions because it has a number of special properties (you will learn more about this in calculus). The value of e is about 2.718.*

Goals:

- E: Be able to solve an equation with an unknown exponent.
- E: Be able to determine the equation of an exponential function given a table of values.
- F: Be able to give the solution to an inequality or set of inequalities using proper mathematical notation.
- F: Be able to determine the domain and/or range of a function given as an equation or a graph.

Example 19. *Two companies that rent laptops have different late fee policies.*

Company 1: *For each day the laptop is late, you owe an additional \$5. On day 1, your total late penalty is \$5. On day 2, your total late penalty is \$10. On day 3, your total late penalty is \$15, and so on.*
Company 2: *For each day the laptop is late, your penalty doubles from the previous day. On day 1, your late penalty starts at \$0.25. On day 2, your late penalty doubles to \$0.50. On day 3, the late penalty doubles to \$1, and so on.*

Which company has the better late fee policy? Explain your reasoning.

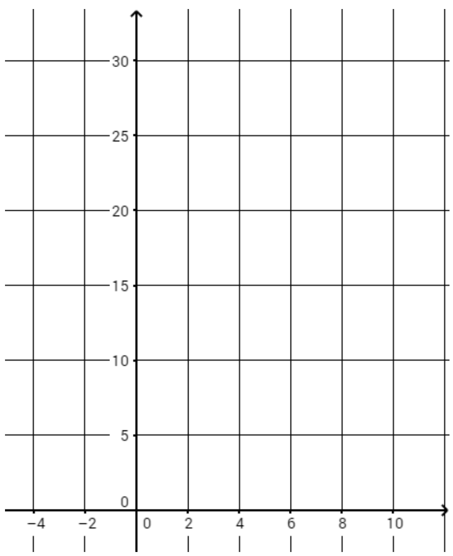
Definition 20. *The simplest form of an **exponential function** is $f(x) = a \cdot b^x$.*

In Desmos, graph $f(x)$ and create sliders for a, and b. What role do each of these constants play? Include sketches and verbal descriptions to help explain the role of each constant.

Example 21. *$f(x) = 2^x$ is an exponential function. A table of values for $f(x)$ is shown in Table 5. Recall that a negative exponent means taking the reciprocal of the positive exponent, so $f(-1) = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$. Graph $f(x)$ on the domain $-1 \le x \le 5$. Describe the shape of the graph.*

Table 5: $f(x)$

x	$f(x)$
-1	$\frac{1}{2}$
0	1
1	2
2	4



Problem 22. *Refer to $P(t)$ as given in Table 6.*

Table 6: Values of $P(t)$

t	0	2	4	6
$P(t)$	10	90	810	7290

1. *Assume that $P(t)$ is an exponential function, and write a function equation for $P(t)$.*

2. *What is $P(3)$?*

3. *Solve $P(t) \leq 10,000$.*

Problem 23. *Let $g(x) = 10 \cdot 2^x$, $h(x) = 20^x$, $k(x) = 10^x 2^x$, and $m(x) = (10 \cdot 2)^x$. By graphing these functions, decide which, if any, of these are really the same function.*

Problem 24. *Consider $2^x 3^y$, 6^{xy}, and 6^{x+y}. By plugging in pairs of values for x and y, decide whether any of these are the same.*

Problem 25. *Refer to $F(x)$ as shown in Table 7.*

Table 7: $F(x)$

x	0	1	2	3
$F(x)$	10	45	202.5	911.25

1. *Write an exponential function equation for $F(x)$.*

2. *Evaluate $F(6)$.*

3. *Solve $F(x) = 2000$.*

Notes

Exponential Modeling

Exponential functions are used to model situations in which the rate of change is proportional to the current amount. Exponential models are used in situations such as money earning interest, population growth, and radioactive decay.

Goals:

- E: Be able to solve an equation/inequality with an unknown exponent.
- E: Be able to model a situation with appropriate exponential equation(s) and interpret the solution.
- E: Be able to determine the equation of an exponential function given a table of values.
- F: Be able to compute the average rate of change of a given function on a given interval.

Example 26. *Pam opens a banking account with \$500. The account earns 1.5% compounded annually. Let t be the number of years the bank account has been open, and $B(t)$ the balance in the account.*

1. *Make a table showing the balance, $B(t)$, in the account at $t = 0$, 1, 2, 3, and 4 years.*

2. *Write an equation for the function $B(t)$.*

3. *Use your function equation to determine the balance in 20 years.*

4. *When will the balance in the account reach $800?*

Problem 27. *Tyus opens a banking account with $800. The account earns 3% compounded annually. Let t be the number of years the bank account has been open, and $A(t)$ the balance in the account.*

1. *Make a table showing the balance, $A(t)$, in the account at $t = 0$, 1, 2, 3, and 4 years.*

2. *Write an equation for the function $A(t)$.*

3. *Use your function equation to determine the balance in 20 years.*

4. *When will the balance in the account reach \$1400?*

Problem 28. *The bacteria in a dish have an initial population of 1000, and are growing such that the population doubles every 45 minutes. Let t be the number of minutes that have passed since the initial population was measured, and let $P(t)$ be the population at time t minutes.*

1. *Make a table showing the population, $P(t)$, at times $t = 0$, 45, 90, and 135 minutes.*

2. *Write an equation for the function $P(t)$.*

3. *Use your function equation to determine the population of bacteria in 6 hours (360 minutes).*

4. *When will there be 250,000 bacteria?*

5. *Determine the average number of bacteria added per hour in the first 6 hours.*

Problem 29. *Carbon dating is used to determine the age of fossils and relics. Carbon-14 has a half life of* 5728 *years, meaning that if an object is found to have* 50% *of its original carbon, it is* 5728 *years old.*

1. *Make a table for the amount of Carbon-14, $C(t)$, at time $t = 0$, 5728, and 11456 years, supposing that the initial amount of carbon is an unknown a.*

2. *Write an equation for the amount of Carbon-14, $C(t)$.*

3. *In 1990 a body was found in the Sierra Nevada mountain range. An examination of the tissue found that 27% of the carbon-14 present at the time of death had decayed. How long ago did the man die?*

Notes

Exercise Set 4

Table 8: Population, $P(t)$

t (years since 1980)	0	10	20	30
$P(t)$	1000	1344	1806	2427

1. The population of a small town has been growing. The population is shown in Table 8.
   (a) What is the average number of people added to the town per year between 1980 and 1990?
   (b) Assuming that the population continues to grow exponentially, write a function equation for $P(t)$.
   (c) If the population continues to grow according to the model, what is the population in 2015?
   (d) When will the population reach 5,000?

2. Suppose you put \$100 into a savings account paying 2.5% interest each year (compounded annually).
   (a) What percent will you earn if you leave the money in the account for 3 years?
   (b) Explain why the answer is NOT $2.5\% \times 3 = 7.5\%$.
   (c) One student solved the problem this way: $100 \times 1.025 \times 1.025 \times 1.025 = 107.69$. $107.69 - 100 = 7.69$. So the answer is 7.69%. Explain this student's work. What does 7.69% represent in this problem?

3. The pesticide DDT was used in the US and later banned. The half-life of DDT is about 15 years.
   (a) Write an exponential model for the amount of DDT, $A(t)$, remaining after t years, if the initial sample is 100 grams.
   (b) According to your model, how much of the initial sample will remain after 60 years?
   (c) How many years will it take for the sample to decay to 1 gram?

Table 9: Mosquito population, $S(t)$

t	0	1	2	3
$S(t)$	100	1600	25600	409600

4. The population of mosquitoes on a small island increases during the wet season. The population was measured once per week, as shown in Table 9.
   (a) What is the average rate of growth of the mosquito population over the three weeks shown in the table?
   (b) Assuming that the population continues to grow exponentially, write a function equation for $S(t)$.
   (c) When will the population reach 1 million (1,000,000)?

5. Plutonium 238 is a radioactive element that decays at a rate of 0.8% per year.
   (a) What percentage of an initial supply of 500 grams Plutonium 238 will remain after 40 years?
   (b) How many years will it be until an initial supply of 500 grams of Plutonium 238 has decayed to half of its initial mass?

Table 10: Values of $m(t)$

t	0	1	2	3	4
$m(t)$	12	8	4	0	-4

6. Refer to Table 10.
   (a) What is $m(1)$?
   (b) Solve $m(t) = 4$.
   (c) Write a function equation for $m(t)$, assuming that it is a linear function.
   (d) Graph $m(t)$ on the domain $0 \leq t \leq 6$.

7. Recall that any object experiencing the force of gravity can be modeled by the equation $h(t) = -16t^2 + vt + c$, where t is the time in seconds, $h(t)$ is the height in feet, c is the initial height of the object, and v is the initial velocity of the object. A football is kicked from a height of 3 feet above the ground and has an initial velocity of 60 feet per second.
   (a) What is the highest the football will go? At what time does this happen?
   (b) How long is the football in the air?

8. Adam, the farmer, is building a third enclosure, this one for his cows and bulls. He has 600 feet of fencing, and he wants to make a rectangular enclosure, but he wants to build another wall of fencing through the middle of the enclosure so that the cows and bulls each get separate halves of the space. As before, he wants to use his fencing to give the animals as much area as possible.
   (a) Draw a diagram showing the enclosure with the dividing fence, and label one of the sides of the wire fencing as x.
   (b) Write an equation for the area of the enclosure in terms of x.
   (c) How should the fencing be used to get the maximum possible area? How much area will the cows and the bulls each get?

9. Dana is standing in a mall, with rows of stores stretching along both sides of a hall 30 meters wide. From Dana's current location in front of Foot Locker, she wants to walk to Ross on the opposite side of the hall and 50 meters down the hall.
   (a) Draw a diagram showing Dana's path and impose coordinates on the diagram.
   (b) Write an equation for the line that describes the Dana's path.
   (c) Write parametric equations to describe Dana's position at time t, if it takes her 10 seconds to walk from Foot Locker to Ross.

10. The town of Allen had a population of 36,000 in 1980, and has been growing at 2.2% per year. The town of Berry had a population of 44,200 in 1980, and a population of 48,000 in 1990.
    (a) Find an exponential model, $A(t)$, for the population of Allen t years after 1980.
    (b) Find an exponential model, $B(t)$, for the population of Berry t years after 1980.
    (c) When did the population of Allen reach 45,000?
    (d) When did the population of Allen equal the population of Berry?

11. The population of India by year is given in Table 11.

Table 11: Population of India by year (Source: data.worldbank.org)

Year	2000	2003	2006	2009	2012
Pop. (millions)	1053.5	1108.4	1162.1	1214.2	1263.6

(a) Use the population in 2000 and 2012 to build an exponential model, $P(t)$, for the number of millions of people in India, with $t = 0$ corresponding to the year 2000.

(b) Plot $P(t)$ and the data in Table 11 in Desmos. Discuss how well the model fits the data.

(c) Use $P(t)$ to predict the population of India in 2014. Compare the result of the model with the actual population in 2014, which was 1295.3 million.

(d) Based on your model, predict the population of India in 2016. Do you think the actual population will be higher or lower than your prediction? Explain.

12. The average price of a home in Cincinnati, OH, in 2015 was $111,200, and growing at about 3% per year.

(a) If the current growth trend continues, write an exponential model for the average price of a home in Cincinnati t years after 2015.

(b) How long will it take for the average price to reach $130,000?

Logarithms

Logarithms are used to solve for unknown exponents in equations.

Definition 30. *If x is a positive number then $\log_b(x)$ is the exponent of b that gives x. That is, if*

$$y = \log_b(x) \qquad \textit{then} \qquad b^y = x$$

The number b is called the base of the logarithm and is always larger than 1.

Definition 31. *The* ***natural logarithm****, written $\ln x$, is the logarithm with the base e.*

The natural logarithm is the inverse of the exponential function e^x, so that for positive x values, $\ln x$, $e^{\ln x} = x$, and for all real numbers, $\ln e^x = x$.

Goals:

- E: Be able to solve an equation with an unknown exponent.
- E: Be able to model a situation with appropriate exponential equation(s) and interpret the solution.
- E: Be able to determine the equation of an exponential function given a table of values.
- E: Be able to use definition and properties of logarithms to rewrite expressions involving logarithms in different forms.
- F: Be able to determine the inverse of a function given given in any form (graph, table, equation).

Problem 32. *Compute the following without using a calculator.*

1. $\ln(e^2)$
2. $\ln(e^6)$
3. $\ln(e^3)$
4. $\ln(e^{-2})$
5. $\ln(e)$
6. $\ln(1)$
7. $\ln(e^a)$

Problem 33. *Explain why $\ln(x)$ is not defined for $x \leq 0$.*

Problem 34. *What is the domain of the function* $f(x) = e^x$*?*

Problem 35. *What is the range of the function* $f(x) = e^x$*?*

Problem 36. *What is the domain of the function* $g(x) = \ln(x)$*?*

Problem 37. *What is the range of the function* $g(x) = \ln(x)$*?*

Problem 38. *Use your calculator and pick values for* A*,* B*, and* r *to test whether each of the following is true. If the statement is false, cross it out so that you do not attempt to apply it.*

1. *For every* A, B *that are positive numbers,* $\ln(AB) = \ln(A) + \ln(B)$*.*

2. *For every* A, B *that are positive numbers,* $\ln(A + B) = \ln(A) + \ln(B)$*.*

3. *For every* A *and* r *such that* A *is a positive number, and* r *is any real number,* $\ln(A^r) = r\ln(A)$*.*

4. *For every* A*,*B *that are positive numbers,* $\ln(\frac{A}{B}) = \ln(A) - \ln(B)$*.*

5. *For every* A, B *that are positive numbers,* $\frac{\ln(A)}{\ln(B)} = \ln(A) - \ln(B)$*.*

6. *For every* A, B *that are positive numbers,* $\ln(A - B) = \ln(A) - \ln(B)$*.*

Problem 39. *Rewrite the following expression using a single natural logarithm:* $2\ln u + \frac{1}{3}\ln v - 3\ln w$.

Problem 40. *Solve for* x *in the following equation:* $\ln x + \ln(x-3) = \ln 4$.

Problem 41. *Solve for* x *in the following equation:* $2^x 5^{x+2} = 250,000$.

Problem 42. *The population of bacteria in a lab culture is given by the equation* $P(t) = 1000 \cdot 2^{\frac{t}{15}}$, *where* t *is in minutes. Find out when the population of bacteria reaches 20,000. Then find an equation for the inverse function that gives the time* t *as a function of the population,* P.

Notes

Exercise Set 5

1. Rewrite the following expression using a single natural logarithm: $\ln a + 4\ln b - 3\ln c$.
2. Rewrite the following expression using a single natural logarithm: $\frac{1}{2}\ln s - 2\ln t - \frac{1}{2}\ln c$.
3. Solve for x in the following equation: $\ln(x-2) + \ln(x-5) = \ln 4$.
4. Solve for t in the following equation: $3^{t-1}7^{t-2} = 27783$.
5. Solve $e^{t^2-2t-10} = 200$.
6. Solve for t in the equation $A = 25 \cdot 3^{t/8}$.

Table 12: Population, $P(t)$, of Albuquerque t years after 1980

t	0	10	20	30
$P(t)$	331,767	386,988	450,557	545,852

7. Refer to the population of Albuquerque, NM, for given years as shown in Table 12.
   (a) Using the population of Albuquerque in 1980 and 2010, build an exponential model for $P(t)$.
   (b) Use your exponential model to predict the population of Albuquerque in 1990. How does your prediction compare with the actual population at that time?
   (c) Use your exponential model to predict in what year the population of Albuquerque will be 700,000.
   (d) Using the population of Albuquerque in 1980 and 2010, build a linear model for $P(t)$.
   (e) Do you think the linear model or the exponential model is a better fit to the data? Explain.
8. The population of Fresno, CA was 430,724 people in 2000. At that time, the population was growing at an annual rate of 1.3%.
   (a) Write an exponential equation relating the population, $P(t)$, to the number of years, t, with $t = 0$ corresponding to 2000.
   (b) If the population continued to grow at that rate, what is the population of Fresno in 2015?
   (c) According to the model, when will the population of Fresno reach 600,000?
9. Shayla opens a banking account with \$300. The account earns 2.2% compounded annually. Let t be the number of years the bank account has been open, and $A(t)$ the balance in the account.
   (a) Make a table showing the balance, $A(t)$, in the account at $t = 0$, 1, 2, 3, and 4 years.
   (b) Write an equation for the function $A(t)$.
   (c) Use your function equation to determine the balance in 20 years.
   (d) When will the balance double?
10. Recall that any object experiencing the force of gravity can be modeled by the equation $h(t) = -16t^2 + vt + c$, where t is the time in seconds, $h(t)$ is the height in feet, c is the initial height of the object, and v is the initial velocity of the object. A soccer ball is kicked from a height of 2 feet above the ground and has an initial velocity of 70 feet per second.
    (a) What is the highest the ball will go?
    (b) During what time interval is the ball at least 10 feet in the air?
    (c) How long is the ball in the air?

More Exponents and Logarithms

Both linear and exponential functions are used often to model real data. In this section, we will explore how each of these models works in the context of population data.

Goals:

- E: Be able to model a situation with appropriate exponential equation(s) and interpret the solution.
- E: Be able to determine the equation of an exponential function given a table of values.
- F: Be able to determine an appropriate function class (linear, quadratic, exponential, trigonometric) to model a particular situation.

You have been hired as a consultant for the City of Los Angeles. City planners need to understand population data and use the data to predict the population in the future. Having a good estimate of the city's population helps the planners decide how much money will be needed for city maintenance and services, and helps local officials anticipate the needs for housing, school and other construction, and much more. Your work begins with finding the population of Los Angeles in the United States Census for 1920 to 2010. That data appears in Table 13.

Table 13: U.S. Census Bureau data for Los Angeles (city), 1920-2010

x (yrs since 1920)	Year	Actual Pop.	Pop. Using Linear Model	Pop. Using Exponential Model
0	1920	576,673		
10	1930	1,238,048		
20	1940	1,504,277		
30	1950	1,970,358		
40	1960	2,479,015		
50	1970	2,816,061		
60	1980	2,966,850		
70	1990	3,485,398		
80	2000	3,694,820		
90	2010	3,792, 621		

Problem 43. *Linear Model. One way to model the population is with a linear function,* $y = mx + b$*, where* m *and* b *are real numbers. As a first attempt, find the equation of the line passing through the data for 1920 and 2010.*

Problem 44. *Use your linear model from Problem 43 to add the estimated population for values* $t = 0$ *to* $t = 90$ *to Table 13 . How do the values compare with the actual data?*

Problem 45. *At this url, you will find Desmos set up with the data:* `https://goo.gl/dynoa2`*. Using Desmos, find the values of* m *and* b *that you think make the line* $y = mx + b$ *best fit the data. Make a note of the equation. How does the equation compare to your linear model from Problem 43? How does the equation look compared to the actual data points on the graph?*

Problem 46. *Exponential Model. Find the equation of an exponential model passing through the data for 1920* $(t = 0)$ *and 2010* $(t = 90)$.

Problem 47. *Use your linear model from Problem 43 to add the estimated population for values* $t = 0$ *to* $t = 90$ *to Table 13 . How do the values compare with the actual data?*

Problem 48. *Graph the exponential model* $G(x) = 945802(1.01845)^x$ *in Desmos. How does the equation compare to your own linear model from Problem 43? How does the best-fit equation look compared to the actual data points on the graph?*

Problem 49. *Comparing the models. Your final job is to make a decision about which model will best predict the population over the next ten to twenty years. Justify your decision based on both mathematics and what you know or think will happen in Los Angeles.*

Notes

Exercise Set 6

Table 14: Population, $P(t)$, of Las Vegas t years after 1980

t	0	10	20	30
$P(t)$	164,674	259,834	484,487	583,756

1. Refer to the population of Las Vegas, NV, for given years as shown in Table 14.
   (a) Using the population of Las Vegas in 1980 and 2010, build an exponential model for $P(t)$.
   (b) Use your exponential model to predict the population of Las Vegas in 1990. How does your prediction compare with the actual population at that time?
   (c) Use your exponential model to predict in what year the population of Las Vegas will be 600,000.
   (d) Using the population of Las Vegas in 1980 and 2010, build a linear model for $P(t)$.
   (e) Do you think the linear model or the exponential model is a better fit to the data? Explain.

2. When buying a new car, one consideration is how fast the car loses value, known as the depreciation rate. For example, one version of the Jeep Liberty loses value more rapidly than some of its competitors. From the initial purchase price of \$23,395, an owner can expect the value of the car 5 years later to be \$15,239.
   (a) What is the average dollar value decline per year during the first 5 years of ownership?
   (b) Use an exponential model produce a function that gives the value of the Jeep in terms of the number of years, t, since the Jeep was new.
   (c) Use your model to predict the value of the Jeep when it was 3 years old.
   (d) When will the value of the Jeep decline to \$5,000?

3. A baseball is launched into the air from a height of 2 meters and with an initial upward velocity of 25 meters/second. The ball's height above ground is given by the equation $H(t) = -4.9t^2 + vt + h$, where H is in meters and t is in seconds.
   (a) Write an equation to model the height of a ball.
   (b) How long is the ball in the air?
   (c) What is the maximum height reached by the ball?

4. Oscar charges \$5 per linear foot to paint any standard outdoor wooden fence, plus \$20 to cover incidental items, such as brushes.
   (a) Write a function equation that gives the cost to paint a fence of length L feet.
   (b) What is the cost to paint a fence 50 feet long?
   (c) If Oscar recently painted a fence and charged \$110, how long was the fence?

5. Titanium-44 is a radioactive isotope with a half-life of 63 years.
   (a) Write an exponential equation to model the amount, $A(t)$, remaining after t years from an initial sample of 20 mg.
   (b) How much of the sample will remain after 30 years?
   (c) When will the sample decay to 2 mg?

Combining Functions

Often in mathematics, we are interested in using existing functions to create new functions. In this section and the next, we will look at some common ways of building new functions.
Goals:

- F: Be able to determine a composition of functions given in any form (graph, table, equation).
- F: Be able to perform arithmetic (sum, difference, product, quotient) on functions given in any form (graph, table, equation).

Example 50. *You have already seen that function notation uses parentheses to mean something other than multiplication. When examining functions, the function use of parentheses will be the norm. For instance, if $f(x) = 2x$ and $g(x) = 3x^2 - 2x - 5$, then we can define $h(x) = f(x) + g(x)$.*

1. *Find $h(6) = f(6) + g(6)$.*

2. *Find an algebraic equation for $h(x)$. Use this equation to find $h(6)$. How does this compare to your answer in part 1?*

Problem 51. *Refer to f and g from Example 50. Define another new function $k(x) = 2f(x) - 4g(x)$. Evaluate $k(6)$. Then find an algebraic function equation for $k(x)$. Use your function equation to evaluate $k(6)$.*

Problem 52. *Refer to f and g from Example 50. Define $m(t) = \frac{f(t)}{g(t)}$. Evaluate $m(6)$. Then find an algebraic function equation for $m(t)$.*

Another way to build a new functions is by composing them. When composing functions, the output of one function becomes the input for another function.

Definition 53. *Given functions f and g, define $p(x)$ to be the* ***composition*** $p(x) = (f \circ g)(x) = f(g(x))$.

Example 54. *Refer to f and g from Example 50. Define $p(x) = f(g(x))$.*

1. *Find $p(6) = f(g(6))$.*

2. *Find an algebraic equation for $p(x)$. Use this equation to find $p(6)$.*

3. *Define $q(x) = g(f(x))$. Find $q(6) = g(f(6))$. Find an algebraic equation for $q(x)$. Compare these to your answers in parts 1 and 2. Does it appear that $(f \circ g)(x) = (g \circ f)(x)$?*

Problem 55. *For this problem, refer to $a(t) = 4t - 7$, $b(t) = -2 \cdot 5^t$, and $r(t)$ as given in Table 15.*

Table 15: $r(t)$

t	0	1	2	3
$r(t)$	25	60	1	-10

1. *Let $c(x) = \frac{2b(x)-3}{a(x)}$. Evaluate $c(3)$. Then find an algebraic expression for $c(x)$.*

2. *Let $d(x) = x \cdot a(b(x))$. Evaluate $d(2)$. Then find an algebraic expression for $d(x)$.*

3. *Let $s(t) = a(r(t))$. Evaluate $s(3)$.*

Notes

Function Transformations

In this section, we continue to look at ways to build new functions from existing ones. In the case of transformations, we will also examine how the graph of the new function relates to the original.

Goals:

- F: Be able to determine a composition of functions given in any form (graph, table, equation).
- F: Be able to perform arithmetic (sum, difference, product, quotient) on functions given in any form (graph, table, equation).
- F: Be able to determine or describe a transformation (reflection, translation, dilation) of a function given in any form (graph, table, equation).

In groups, you are going to be assigned to one of the following graph pairs:

- $f(x) = x^2$, $g(x) = a(b(x-h))^2 + k$, at: `https://goo.gl/dEpS41`
- $f(x) = |x|$, $g(x) = a|b(x-h)| + k$, at: `https://goo.gl/GQl8t7`
- $f(x) = \sqrt{x}$, $g(x) = a\sqrt{b(x-h)} + k$, at: `https://goo.gl/4iVp3c`
- $f(x) = x^3$, $g(x) = a(b(x-h))^3 + k$, at: `https://goo.gl/6nfChx`

For your assigned function pair, answer the following questions:

1. The role of h. Begin by setting $a = 1$, $b = 1$, $h = 0$, and $k = 0$.

   (a) What happens as h increases from 0?

   (b) What happens as h decreases from 0?

   (c) What happens when $h = 0$?

   (d) Describe in words how different values of h affect the graph.

   (e) Without graphing, predict how $p(x) = (x-4)^5$ will be different from $q(x) = x^5$. Check your answer with Desmos.

2. The role of k. Begin by setting $a = 1$, $b = 1$, $h = 0$, and $k = 0$.
   (a) What happens as k increases from 0?

   (b) What happens as k decreases from 0?

   (c) What happens when $k = 0$?

   (d) Describe in words how different values of k affect the graph.

   (e) Without graphing, predict how $p(x) = x^5 - 4$ will be different from $q(x) = x^5$. Check your answer with Desmos.

3. The role of a. Begin by setting $a = 1$, $b = 1$, $h = 0$, and $k = 0$.
   (a) What happens as a increases from 1?

   (b) What happens as a decreases from 1 toward 0?

   (c) What happens when $a = 0$?

   (d) What happens when a decreases from 0 toward -1?

   (e) What happens when a decreases from -1?

   (f) Without graphing, predict how $p(x) = -4x^5$ will be different from $q(x) = x^5$. Check your answer with Desmos.

4. The role of b. Begin by setting $a = 1$, $b = 1$, $h = 0$, and $k = 0$.
   (a) What happens as b increases from 1?
   (b) What happens as b decreases from 1 toward 0?
   (c) What happens when $b = 0$?
   (d) What happens when b decreases from 0 toward -1?
   (e) What happens when b decreases from -1?
   (f) Without graphing, predict how $p(x) = (-4x)^5$ will be different from $q(x) = x^5$. Check your answer with Desmos.

5. Combining transformations. Using your assigned $f(x)$, describe how each function below will differ from $f(x)$. Check your descriptions using Desmos.
   (a) $d(x) = 3f(x - 2)$
   (b) $g(x) = f(3(x - 2))$
   (c) $m(x) = -3f(x + 2) - 5$
   (d) $n(x) = 3f(-3(x - 2)) - 7$
   (e) $r(x) = 5f(3(x + 2)) + 7$

Notes

Exercise Set 7

1. Cell phones begin to lose (resale) value immediately after purchase. After one year, an iPhone can be resold at 63% of its list price. Assume that each year after that, the resale value continues to drop to 63% of the resale price from the previous year.

   (a) Assuming the list price of an iPhone is \$649, write an exponential model for the resale price, $P(t)$, of an iPhone t *months* after purchase.
   (b) According to your model, what is the resale value of the iPhone after 18 months?
   (c) According to your model, when is the resale value of the iPhone \$150?
   (d) Assume that a particular Android phone has a value, $A(t)$, that is always \$100 less than the iPhone. Write $A(t)$ as a composition of $P(t)$ and another function, $h(t)$, so that $A(t) = h(P(t))$.
   (e) Another smartphone has a value, $V(t)$, that is always one half of the value of the iPhone. Write $V(t)$ as a transformation of $P(t)$.

2. Refer to the function $p(x) = x^2$.

   (a) Suppose $r(x) = p(x + 2) - 3$. Describe how $p(x)$ is transformed to create $r(x)$. Verify your description by graphing using Desmos.
   (b) Suppose $q(x) = -p(x - 2) + 5$. Describe how $p(x)$ is transformed to create $q(x)$. Verify your description by graphing using Desmos.
   (c) Suppose $s(x) = -3p(2(x-1)) + 5$. Describe how $p(x)$ is transformed to create $s(x)$. Verify your description by graphing using Desmos.
   (d) Suppose $u(x)$ is created by shifting $p(x)$ 3 units to the right and 4 units down. Write $u(x)$ in terms of $p(x)$, and then find an algebraic expression for $u(x)$.
   (e) Suppose $v(x)$ is created by compressing $p(x)$ horizontally by a factor of 5, and shifting 4 units up. Write $v(x)$ in terms of $p(x)$, and then find an algebraic expression for $v(x)$.

3. Refer to the functions $f(x) = 2^x$ and $g(x) = 3 \cdot 2^x - 7$.

   (a) Describe $g(x)$ as a transformation of $f(x)$.
   (b) Find a function $h(x)$ so that $g(x) = h(f(x))$.

4. Consider the functions $k(x) = 8^x$ and $m(x) = 4 \cdot 2^x$.

   (a) Are $k(x)$ and $m(x)$ the same function? Explain.
   (b) Write $m(x)$ as a composition of $f(x) = 2^x$ and another function $b(x)$, so that $m(x) = b(f(x))$.
   (c) Write $k(x)$ as a composition of $f(x) = 2^x$ and another function $d(x)$, so that $k(x) = d(f(x))$.

5. The Free Fall ride is one in which passengers are dropped and experience the acceleration of gravity, before being rapidly brought to a complete stop. One such ride, formerly at Magic Mountain, was 13 meters tall. An object falling under the force of gravity has height modeled by the equation $H(t) = -4.9t^2 + vt + h$, where H is measured in meters, v is the initial velocity in meters/second, and h is the initial height in meters.

   (a) Keeping in mind that the ride drops passengers from a stop, write a function equation to model the height of passengers as a function of time since the passengers started freefall.
   (b) If riders are allowed to free fall for 10 meters before the brakes are applied, how long (in seconds) is the free fall portion of the ride?

(c) If the speed of passengers during the freefall portion of the ride is given by the equation $S(t) = 9.8t$, where S is measured in meters per second. How fast are passengers traveling at the moment when the brakes are first applied?

(d) If you want your riders to reach speeds of 30 miles per hour, how long must riders be allowed to free fall to reach this speed? At what height would you apply the brakes? (1 meter/second is the same as 2.24 miles/hour.)

6. Consider the functions $F(x) = \ln x$, $G(x) = \ln(x-2)$, $H(x) = \ln x - 2$, and $J(x) = \ln(x+2)$. Describe $G(x)$, $H(x)$, and $J(x)$ as transformations of $F(x)$. Do any of these functions have the exact same graph?

7. The Jaguar XJ AWD loses value more rapidly than some of its competitors. From the initial purchase price of $76,700, an owner can expect the value of the car 5 years later to be $52,014.

   (a) What is the average dollar value decline per year during the first 5 years of ownership?
   (b) Use an exponential model produce a function that gives the value of the Jaguar in terms of the number of years, t, since the car was new.
   (c) Use your model to predict the value of the Jaguar at 3 years old.
   (d) When will the value of the Jaguar decline to $30,000?

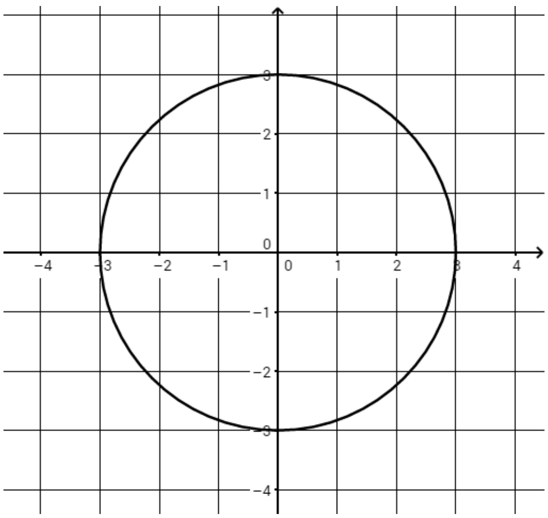
Inverse Functions

Inverse functions are functions that undo the action of a given function.
Goals:

- F: Be able to determine the inverse of a function given given in any form (graph, table, equation).
- F: Be able to determine the domain and/or range of a function given as an equation or a graph.

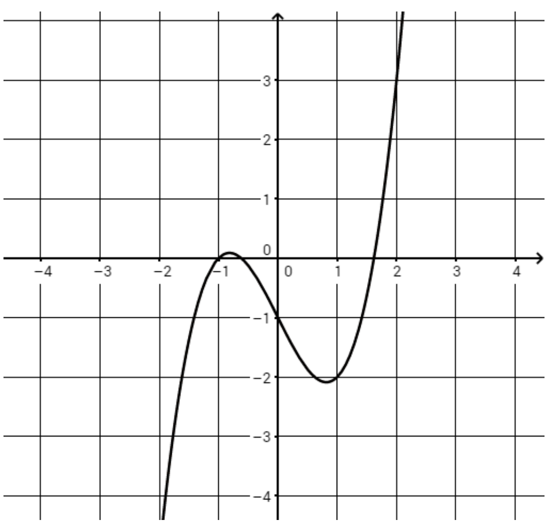
Definition 56. *A* ***relation*** *is a set of pairs of input and output values. A* ***function*** *is a relation where each input has a single output.*

Example 57. *Decide if each relation below, indicated by a graph or by a table, is a function.*



x	y
1	2
3	2
5	2
7	2
9	2

x	y
7	1
3	0
7	−1
5	−2
2	−3



x	$f(x)$
−2	−4
−1	−3
0	−2
1	−1
2	0

x	y
−3	0
−2	−1
−1	2
0	−1
1	3

Definition 58. *Given a function f defined on a domain D, a function g on a domain E is an **inverse** of f if $f(g(x)) = x$ whenever x is in the domain of g, and $g(f(x)) = x$ whenever x is in the domain of f. Often, the inverse function g is written f^{-1}.*

Example 59. *On the domain of all real numbers, $g(x) = \sqrt[3]{x}$ is the inverse of $f(x) = x^3$.*

Example 60. *The function $\ln x$ is defined on the domain of positive real numbers ($x > 0$), and e^x is defined on the domain of all real numbers, and $\ln(e^x) = x$ and $e^{\ln x} = x$, so these functions are inverses of each other.*

Example 61. *For the function $r(x)$ as defined in Table 16, the inverse function $r^{-1}(x)$ is the function given in Table 17.*

Table 16: $r(x)$

x	0	1	2	3
$r(x)$	25	60	1	-10

Table 17: $r^{-1}(x)$

x	25	60	1	-10
$r^{-1}(x)$	0	1	2	3

Problem 62. *Refer to the functions $r(x)$ and $r^{-1}(x)$ in Example 61.*

1. *What do you notice about the two rows in the tables for the functions?*

2. *Compute $r^{-1}(r(1))$, $r(r^{-1}(1))$, $r^{-1}(r(2))$ and $r(r^{-1}(25))$.*

3. *What are the domain and range of r?*

4. *What are the domain and range of r^{-1}?*

Example 63. *Consider the function $f(x) = x^2$.*

1. *Complete the following table of output values of $f(x)$:*

Table 18: $f(x)$

x	-2	-1	0	1	2
$f(x)$					

2. *Create a table for a relation, $g(x)$, that reverses the input and output of $f(x)$. Is $g(x)$ a function?*

3. *Since $g(x)$ is not a function, it cannot be the inverse of $f(x)$. In order for a function to have an inverse, it must be **one-to-one**, that is every output has exactly one input.*

4. *Sometimes we can restrict the domain of a function to make it one-to-one, so that it will have an inverse. On what domain would $f(x)$ have an inverse? What is the range of $f(x)$ on this domain?*

5. *What is the inverse of $f(x)$ on the domain $[0, \infty)$? What are the domain and range of $f^{-1}(x)$?*

Problem 64. *Refer to the function $z(t) = \sqrt{t-3}$.*

1. *What are the domain and range of z?*

2. *Find a function equation for the inverse function, z^{-1}.*

3. *What are the domain and range of* z^{-1}*?*

Problem 65. *Refer to the function* $h(x) = 3x - 7$.

1. *What are the domain and range of* h*?*

2. *Find a function equation for the inverse function,* h^{-1}.

3. *What are the domain and range of* h^{-1}*?*

Problem 66. *Refer to the functions* g *and* f *in Example 59,* e^x *and* $\ln x$*,* z *and* z^{-1} *in Problem 64, and* h *and* h^{-1} *in Problem 65.*

1. *For each pair of functions, use Desmos to graph them on the same set of axes, along with the line* $y = x$*. Sketch each graph.*

2. *What do you notice about the relationship between the graph of a function and the graph of its inverse? Explain why this happens.*

Problem 67. *What conjecture(s) do you have about the relationship between the domain and range of a function and the domain and range of its inverse?*

Notes

Rational Functions

Goals:

- F: Be able to produce a graph of a given rational function, indicating the vertical asymptotes, and x- and y-intercepts, if any.
- F: Be able to determine the domain and/or range of a function given as an equation or a graph.

Example 68. *For this investigation, refer to the graph on Desmos at:* `http://tinyurl.com/153ratexp` *There, you will find the rational function* $y = \frac{a(x-b)}{x-d}$*, with sliders for* a*,* b*, and* d*.*

1. *Begin by setting* $a = 1$ *and* $b = 1$*. Slide the value of* d*.*

   (a) *What happens as* d *gets larger?*

   (b) *What happens when* $d = 0$*?*

   (c) *What happens when* d *is negative?*

   (d) *If you are only looking at the graph and not the function equation, what about the graph would let you know the value of* d*?*

2. *Set* $a = 1$ *and* $d = 0$*. Slide the value of* b*.*

   (a) *What happens as* b *gets larger?*

   (b) *What happens when* $b = 0$*?*

   (c) *What happens when* b *is negative?*

   (d) *If you are only looking at the graph and not the function equation, what about the graph would let you know the value of* b*?*

3. *Begin by setting $b = 1$ and $d = 0$. Slide the value of a.*

   (a) *What happens as a gets larger?*

   (b) *What happens when $a = 0$?*

   (c) *What happens when a is negative?*

   (d) *If you are only looking at the graph and not the function equation, what about the graph would let you know the value of a?*

Example 69. *Graph $r(x) = \frac{x}{x+2}$ on the domain $-4 \leq x \leq 4$. Be sure to label any asymptotes and intercepts.*

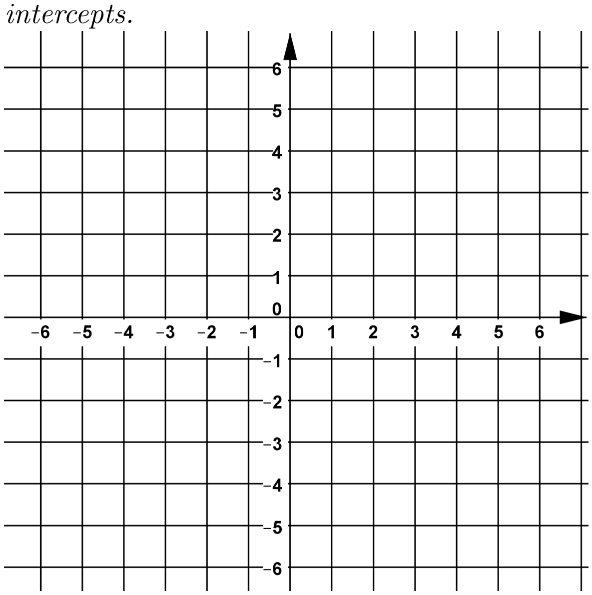
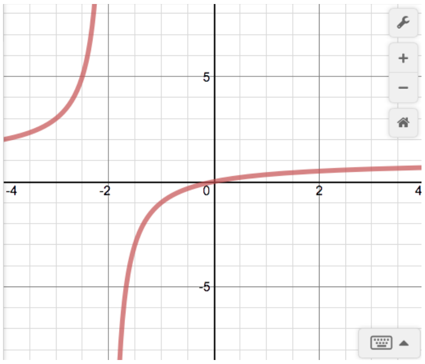
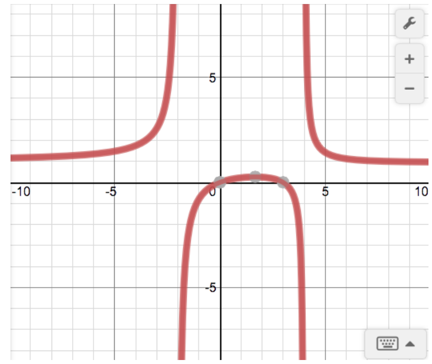


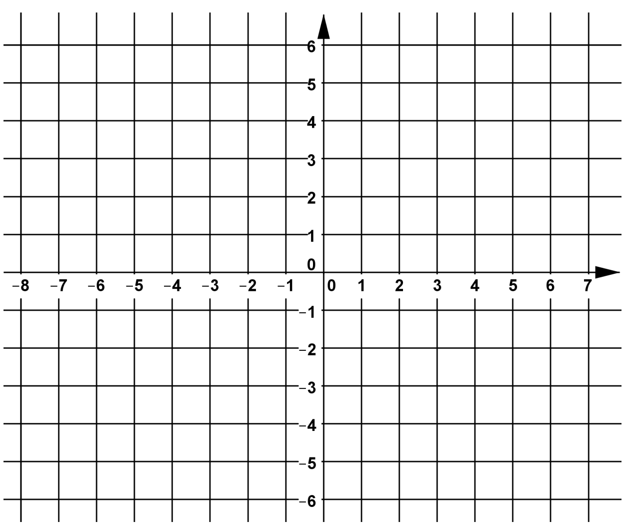
Figure 10: Graphs of $r(x) = \frac{x}{x+2}$ and $P(x) = \frac{x(x-3)}{(x+2)(x-4)}$



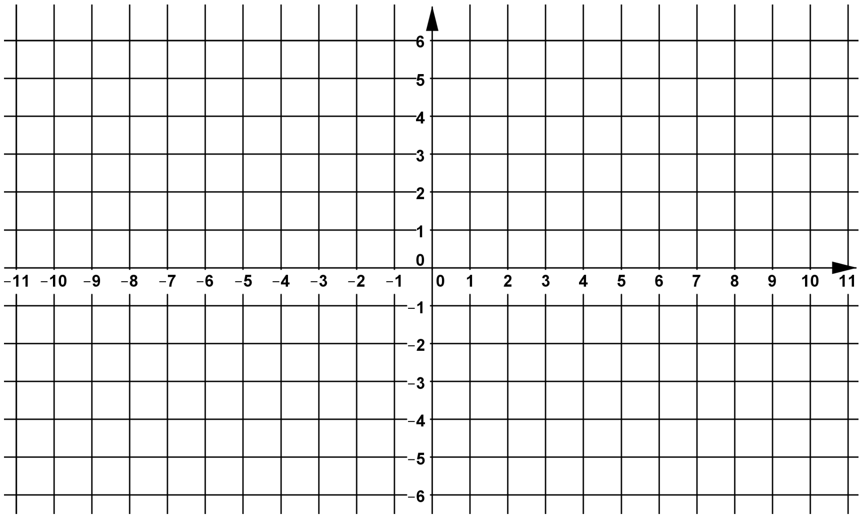


The solution to Example 69 is outlined below:
Solution: Step 1: Determine where the function is undefined. *In our example, the function is undefined when the denominator is 0. Therefore, by setting $x + 2 = 0$, we get $x = -2$. Therefore, $x = -2$ will be a **vertical asymptote**. Vertical asymptotes are values that the function approaches.*
Step 2: Graph the function in an appropriate window on Desmos. *Set up the window for x to be $-4 \leq x \leq 4$, to reflect what is given in the problem.*
Step 3: Draw your own graph. *Refer to the left-hand side graph in Figure 10, the graph from Desmos. On your own graph, draw a dotted vertical line at $x = -2$, and label the intercept at $(0, 0)$.*
NOTE: In calculus, you will learn that a function can also have horizontal asymptotes. Our example has a horizontal asymptote $y = 1$. However, for our purposes, indicating only the vertical asymptotes is enough.

Problem 70. *Graph $g(x) = \frac{x-1}{x+4}$ on the domain $-6 \leq x \leq 6$. Be sure to label any asymptotes and intercepts.*



Example 71. *Graph* $P(x) = \frac{x(x-3)}{(x+2)(x-4)}$ *on the domain* $-10 \leq x \leq 10$. *Be sure to label any asymptotes and intercepts.*



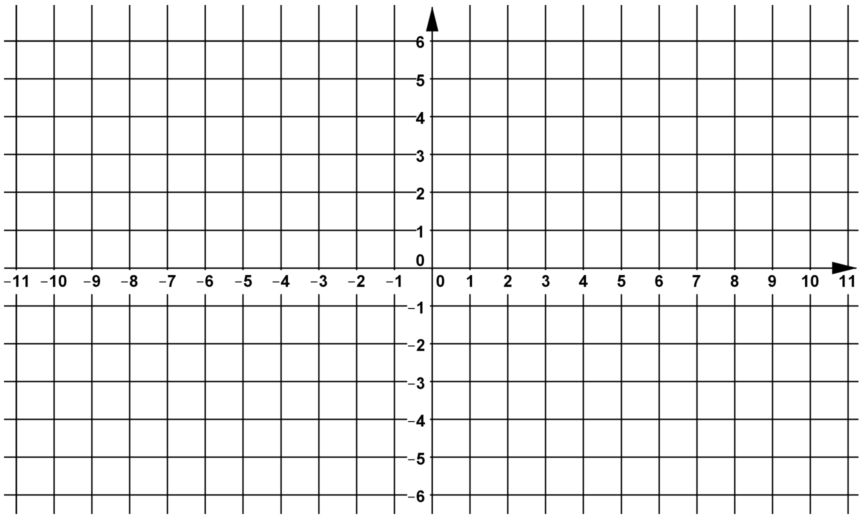
The solution to Example 71 is outlined below:

Solution: Step 1: Determine where the function is undefined. *In our example, the function is undefined when the denominator is 0. Therefore, by setting* $(x+2)(x-4) = 0$, *we get* $x = -2$ *or* $x = 4$. *Therefore,* $x = -2$ *and* $x = 4$ *will be vertical asymptotes.*

Step 2: Graph the function in an appropriate window on Desmos. *Set up the window for* x *to be* $-10 \leq x \leq 10$, *to reflect what is given in the problem.*

Step 3: Draw your own graph. *Refer to the right-hand side graph in Figure 10, the graph from Desmos. On your own graph, draw dotted vertical lines at* $x = -2$ *and* $x = 4$, *and label the intercepts at* $(0, 0)$ *and* $(3, 0)$.

Problem 72. *Graph* $Q(x) = \frac{2(x-1)(x+3)}{(x+5)(x-7)}$ *on the domain* $-10 \leq x \leq 10$*. Be sure to label any asymptotes and intercepts.*



Notes

Exercise Set 8

1. Refer to the function $g(x) = 7x + 2$.

   (a) What are the domain and range of g?
   (b) Let $h(x) = x^2 + x + 1$. Find an algebraic expression for $g(h(x))$.
   (c) Again using $h(x) = x^2 + x + 1$, find an algebraic expression for $3g(x) - h(x)$.
   (d) Find a function equation for the inverse function, g^{-1}.
   (e) What are the domain and range of g^{-1}?

2. Refer to the graph of $R(x)$ in Figure 11. The diagonal line $y = x$ is included on the graph for reference. Note that the graph of $R(x)$ includes the points $(-7.5, 0)$, $(-1, 2)$, $(0, 4.5)$, and $(6, 6)$.

   (a) What are the domain and range of R?
   (b) Draw the graph of the inverse function, R^{-1}.
   (c) What are the domain and range of R^{-1}?

Figure 11: Graph of $R(x)$

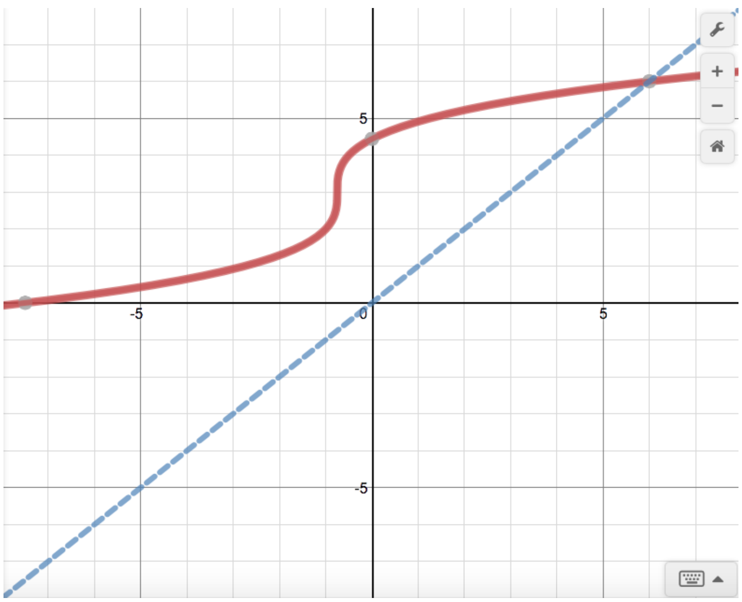


Table 19: $f(x)$

x	0	1	2	3
$f(x)$	3	0	2	-4

3. Refer to f in Table 19.

   (a) What are the domain and range of f?
   (b) Find a function table for the inverse function, f^{-1}.
   (c) What are the domain and range of f^{-1}?

4. Refer to $A(x) = \frac{4x+5}{2x-4}$.

(a) What is the domain of A?
(b) Graph $A(x)$ so that the important features are visible, and label them.
(c) Let $B(x)$ be the function whose graph is a shift of $A(x)$ down 3 units, and to the right 5 units. Write $B(x)$ as a transformation in terms of $A(x)$, and then use that to get an algebraic formula for $B(x)$.
(d) Find a function equation for A^{-1}.
(e) What is the domain of A^{-1}?
(f) Graph A^{-1} so that the important features are visible, and label them.

5. Refer to $T(x) = \frac{(x+1)^2}{x+1}$.

(a) What is the domain of T?
(b) Graph $T(x)$ so that the important features are visible, and label them.
(c) In Desmos, use the settings to view a table of values for $T(x)$. How is the domain reflected in the table?
(d) Is the graph what you expected? Explain.

6. Refer to $Q(x) = \sqrt[3]{x-3}$.

(a) What are the domain and range of Q?
(b) Let $U(x) = x^3$. Find an algebraic expression for $U(Q(x))$.
(c) Let $V(x) = 2U(x+3)$. Describe how the graph of $V(x)$ differs from $U(x)$. Verify your description by graphing both functions.
(d) Find a function equation for the inverse function, Q^{-1}.
(e) What are the domain and range of Q^{-1}?

7. Refer to the function $G(x) = \frac{2x+3}{(x+4)(x-2)}$.

(a) Graph $G(x)$ so that the important features are visible, and label them.
(b) What is the domain of G?
(c) On which interval(s) is $G(x)$ decreasing?
(d) Let $F(x) = G(x) - 3$. Graph $F(x)$ and label its features.

8. Atmospheric pressure (the force of air around you) decreases at higher altitudes. For every 1000 m gain in altitude, the air pressure decreases about 12%. The air pressure at sea level (altitude 0 meters) is 101.325 kPa (kPa are kiloPascals, metric units for pressure).

(a) Write a function equation expressing the pressure, P, in terms of the altitude a, in meters.
(b) What is the air pressure at the top of the Empire State Building (381 m)?
(c) What is the air pressure at the top of Mount Everest (8848 m)?
(d) At what altitude is air pressure half of what it is at sea level?
(e) Write an equation that gives the altitude as a function of the air pressure. State the units of the domain and the range of this new function.

Describing Change in Functions

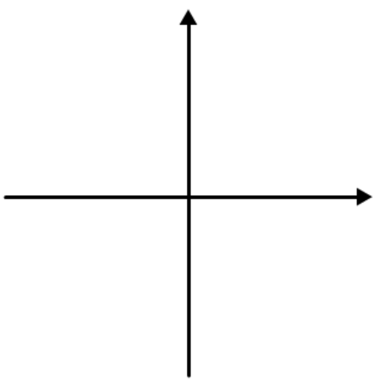
In this section, we will look at how to describe change in functions, both qualitatively and quantitatively.
Goals:

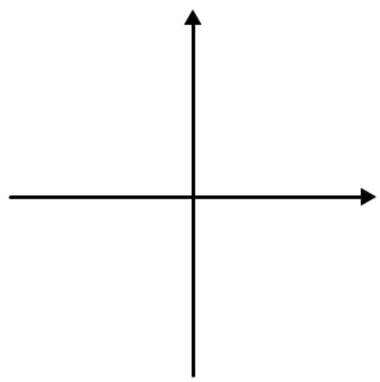
- F: Be able to identify the intervals on which a given function is increasing or decreasing.
- F: Be able to compute the average rate of change of a given function on a given interval.

Definition 73. *A function h is* ***increasing*** *on the interval $a \leq x \leq b$ if, whenever two points c and d are in the interval, with $c < d$, then $f(c) < f(d)$. A function h is* ***decreasing*** *on the interval $a \leq x \leq b$ if, whenever two points c and d are in the interval, with $c < d$, then $f(c) > f(d)$.*

Informally, the definitions for increasing and decreasing are stating that if a function goes up as we move from left to right, it is increasing, and if it goes down as we move from left to right, it is decreasing.

Below, sketch a function that is increasing on $(-\infty, \infty)$ and a function that is decreasing on $(-\infty, \infty)$.





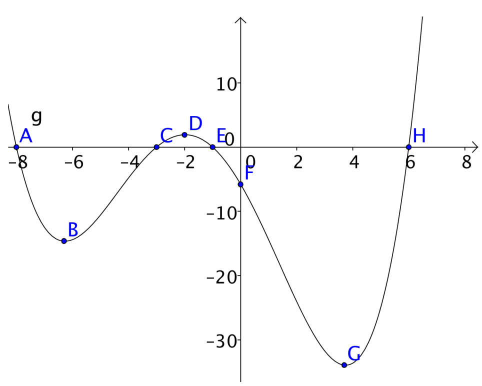
Definition 74. *The* ***average rate of change*** *of a function h defined on an interval $a \leq x \leq b$ is*

$$\frac{h(b) - h(a)}{b - a}.$$

Note that the average rate of change computes the slope of the line between the two points $(a, h(a))$ and $(b, h(b))$.

Example 75. *Refer to the graph and table for g in Figure 12.*

Figure 12: Graph and table for g



	A	B	C	D	E	F	G	H
x	-8	-6.3	-3	-2	-1	0	3.7	6
$f(x)$	0	-14.6	0	1.9	0	-5.8	-33.9	0

1. *Describe the interval(s) on which g is increasing, and the interval(s) on which g is decreasing.*

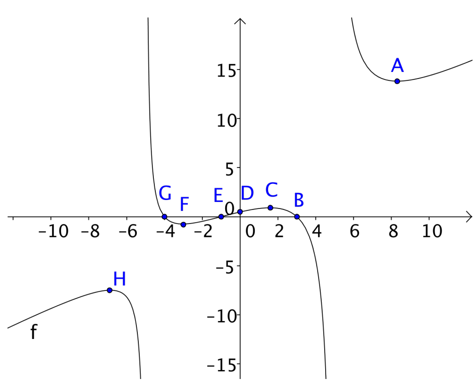
2. *Find the average rate of change of g on the interval $-6.3 \leq x \leq 3.7$.*

3. *Find the average rate of change of f on the interval $-8 \leq x \leq 6$.*

The solution to Example 75 is outlined below:
The function we are given goes down (is decreasing) from point A to point B, then increases from point B to point D, decreases again from point D to point G, and then increases from point G to point H and beyond. In this case, we only have the graph, and not the equation, so we should not make assumptions about what happens outside of the interval $-8 \leq x \leq 6$. Thus, g is increasing on the intervals $-6.3 \leq x \leq -2$, and again on $3.7 \leq x \leq 6$. The function g is decreasing on the intervals $-8 \leq x \leq -6.3$ and $-2 \leq x \leq 3.7$. The average rate of change of g on the interval $-6.3 \leq x \leq 3.7$ is $\frac{-33.9-(-14.6)}{3.7-(-6.3)} = \frac{-19.3}{10} = 1.93$. The average rate of change of g on the interval $-8 \leq x \leq 6$ is $\frac{0-0}{6-(-8)} = 0$. Can you explain why the average rate of change was 0 on $-8 \leq x \leq 6$?

Problem 76. *Refer to the graph and table for f in Figure 13. Note that f has vertical asymptotes at $x = -5$ and at $x = 5$.*

Figure 13: Graph and table for f



	H	G	F	E	D	C	B	A
x	-6.9	-4	-3	-1	0	1.6	3	8.3
$f(x)$	-7.5	0	-0.8	0	0.5	0.9	0	13.8

1. *Describe the interval(s) on which f is increasing. You may use either inequalities or interval notation.*

2. *Find the average rate of change of f on the interval $-3 \leq x \leq 1.6$.*

3. *Find the average rate of change of f on the interval $-4 \leq x \leq 3$.*

4. *Find two intervals on which the average rate of change of f is 0.*

Problem 77. *Refer to the function $h(x) = (x-1)^2 + 4$.*

1. *Describe the interval(s) on which h is decreasing.*

2. *Find the average rate of change of h on the interval $-3 \leq x \leq 1$.*

3. Find the average rate of change of h on the interval $2 \leq x \leq 3$.

4. Find two intervals on which the average rate of change of h is 0.

Notes

Exercise Set 9

1. Refer to the function $P(x) = \frac{(x-1)^2}{x(x+3)}$.

   (a) What is the domain of P?
   (b) Graph $P(x)$ so that the important features are visible, and label them.
   (c) Describe the interval(s) on which P is increasing.
   (d) Find the average rate of change of P on the interval $-8 \le x \le -4$.
   (e) Find the average rate of change of P on the interval $1 \le x \le 10$.
   (f) Find an interval on which the average rate of change of P is 0.

2. Refer to the function F given in Table 20.

Table 20: Table for F

x	-4	-3	-2	-1	0	1	2	3
$F(x)$	-7	0	8	0	5	9	13	9

   (a) Find the average rate of change of F on the interval $-3 \le x \le 3$.
   (b) Find the average rate of change of F on the interval $-2 \le x \le 1$.
   (c) Find an interval on which the average rate of change of F is 0.

3. Refer to R as given in Table 21. You may assume that R is continuous on the interval $0 \le x \le 3$.

Table 21: Table for R

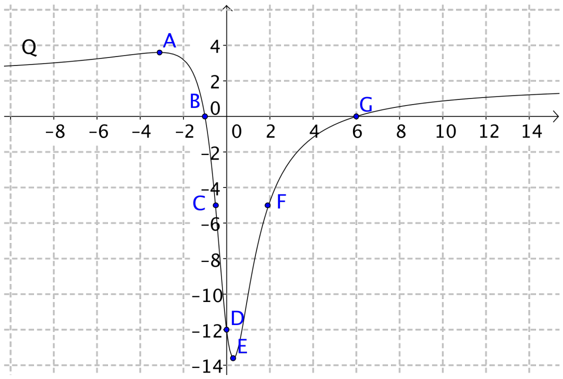
x	0	1	2	3
$R(x)$	6	24	96	384

   (a) Assuming that R is an exponential function, find a function equation for R.
   (b) Find the average rate of change of R on the interval $0 \le x \le 2$.
   (c) Using the function equation for R, find the average rate of change of R on the interval $1 \le x \le 5$.
   (d) Using the function equation for R, what is the range of R on the domain of all real numbers?
   (e) Using the function equation for R, write a function equation for the inverse, R^{-1}.
   (f) Let $U(x) = R(2x) - 3$. Describe how the graph of U will differ from the graph of R. Then verify your description using Desmos.
   (g) What is the range of U?
   (h) Let $V(x) = \ln(R(x))$. Write a function equation for V.

4. Refer to $k(t) = t - \ln t$.

   (a) Describe the interval(s) on which k is increasing.
   (b) Find the average rate of change of k on the interval $1 \le t \le 4$.
   (c) What are the domain and range of k?
   (d) Find an interval on which the average rate of change of k is less than 0.
   (e) Let $W(x) = e^x$. Write a function equation for $J(x)$, where $J(x) = k(W(x))$.
   (f) Let $M(t)$ be the function obtained by shifting the function $k(t)$ to the left 3 units, and down 1 unit. Write a function equation for $M(t)$, and then use Desmos to verify that you have the correct transformation.

Figure 14: Graph and table for Q



	A	B	C	D	E	F	G
x	-3.1	-1	-0.5	0	0.3	1.9	6
$Q(x)$	3.6	0	-5	-12	-13.6	-5	0

5. Shadan opens a banking account with \$500. The account earns interest compounded annually. Let t be the number of years the bank account has been open, and $A(t)$ the balance in the account. Suppose $A(t) = 500 \cdot 1.015^t$.
   (a) What is the annual interest rate on Shadan's account?
   (b) What is the balance in Shadan's account at $t = 5$ years?
   (c) Write a function equation for A^{-1}, and give units for the input and output of A^{-1}.
   (d) How much money did the account earn per year, on average, during the first 5 years it was open?
   (e) Suppose that Shadan's twin sister, Shelly, opened a bank account at the same bank, with the same interest rate, but with twice as much money. Write a function equation for Shelly's account balance, $B(t)$, t years after the account was opened, *as a transformation of* $A(t)$.

6. Refer to the graph and table for Q in Figure 14.
   (a) Describe the interval(s) on which Q is decreasing.
   (b) Find the average rate of change of Q on the interval $-3.1 \leq x \leq -.5$.
   (c) Find the average rate of change of f on the interval $0.3 \leq x \leq 6$.
   (d) Find an interval on which the average rate of change of Q is 0.
   (e) Among points A through G, between which two points will the average rate of change be the greatest?
   (f) Among points A through G, between which two points will the average rate of change be the least?

7. Supreme Scream is a ride in which passengers are dropped and experience the acceleration of gravity, before being rapidly brought to a complete stop. The ride reaches a height of 76.8 meters. An object falling under the force of gravity has height modeled by the equation $H(t) = -4.9t^2 + vt + h$, where H is measured in meters, v is the initial velocity in meters/second, and h is the initial height in meters.
   (a) Keeping in mind that the ride drops passengers from a stop, write a function equation to model the height of passengers as a function of time since the ride started.
   (b) If riders are allowed to free fall for 10 meters before the brakes are applied, how long (in seconds) is the free fall portion of the ride?
   (c) If the speed of passengers on the ride is given by the equation $S(t) = 9.8t$ (meters/second), how fast are passengers traveling at the moment when the brakes are first applied?

(d) Suppose you are designing the next generation version of the ride, and you want riders to experience speeds of 80 miles/hour. How many meters must riders be allowed to free fall to reach this speed? (1 meter/second is the same as 2.24 miles/hour.)

8. Refer to the function $G(x) = \frac{3x^2-3}{(x-4)(x^2+1)}$.

   (a) Graph $G(x)$ so that the important features are visible, and label them.
   (b) What is the domain of $G(x)$?
   (c) On which interval(s) is $G(x)$ decreasing?
   (d) What is the average rate of change of $G(x)$ on the interval $5 \leq x \leq 8$?
   (e) Let $F(x) = G(x-2)$. Graph $F(x)$ and label its features.

Figure 15: Graph of $h(x)$

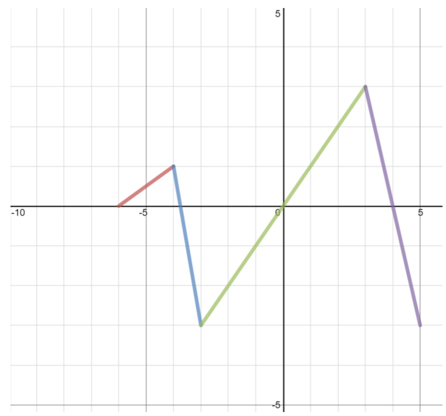


Table 22: Table of values for $a(x)$ and $b(x)$

x	$a(x)$
-1	3
0	6
1	9
2	12

x	$b(x)$
-1	3
0	6
1	12
2	24

9. Refer to $h(x)$ as shown in Figure 15 and $a(x)$ and $b(x)$ as shown in Table 22.

   (a) Evaluate $h(3)$.
   (b) What are the domain and range of $h(x)$?
   (c) Does $h(x)$ have an inverse function, h^{-1} on the domain $-5 \leq x \leq 3$? Explain.
   (d) Find two x values p and q such that the average rate of change of $h(x)$ on $p \leq x \leq q$ is less than 0.
   (e) Find two x values c and d such that the average rate of change of $h(x)$ on $c \leq x \leq d$ is 0.
   (f) Find two x values m and n such that the average rate of change of $h(x)$ on $m \leq x \leq n$ is greater than 0.
   (g) Let $f(x) = h(x-2) + 4$. Draw a graph of $f(x)$. Describe the graph of f as a transformation of h.
   (h) Solve the equation $h(x) = 1$.

(i) Solve the inequality $h(x) > 1$.

(j) Let $p(x) = a(h(x))$. Evaluate $p(-4)$.

(k) Let $q(x) = b(h(x))$. Find $q(4)$.

(l) Define $m(x) = 2b(x) - (h(x))^2$. Find $m(-1)$.

10. Refer to Table 22. You may assume that $a(x)$ and $b(x)$ are defined on the domain of all real numbers.

    (a) Decide which type of function (linear, quadratic, exponential) best fits the data for $a(x)$. Explain your reasoning.

    (b) Write a function equation for $a(x)$.

    (c) Solve the equation $a(x) = 12$.

    (d) Solve the inequality $a(x) \geq 12$.

    (e) Solve the equation $a(x) = 48$.

    (f) Find the inverse function a^{-1}.

    (g) Decide which type of function (linear, quadratic, exponential) best fits the data for $b(x)$. Explain your reasoning.

    (h) Write a function equation for $b(x)$.

    (i) Solve the equation $b(x) = 12$.

    (j) Solve the inequality $b(x) < 12$.

    (k) Solve the equation $b(x) = 48$.

    (l) Find the inverse function b^{-1}.

11. Construct an example of a rational function, $r(x)$, that has vertical asymptotes at $x = -2$ and $x = 4$, and a horizontal asymptote at $y = -1$.

Angles and Circles

In this section, we will learn about circles, how to measure angles in radians, and we will see a use for radian measure.

Goals:

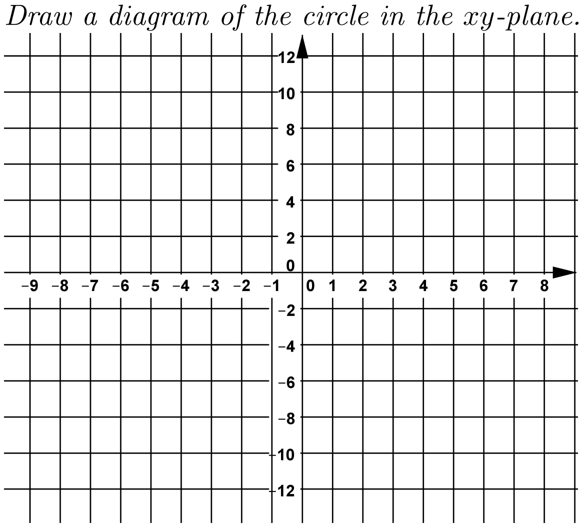
- F: Be able to draw a diagram incorporating all of the important information in a given situation, and impose coordinates on the diagram.
- T: Be able to use the distance formula or the equation of a circle in context.
- T: Be able to determine the length of an arc of a circle or the area of a sector of a circle.

A circle of radius r has a circumference of $2\pi r$ and an area of πr^2, and if the center of the circle is at (h, k), then the circle is described by the equation $(x - h)^2 + (y - k)^2 = r^2$.

Example 78. *A circle has its center at* $(3, -4)$*, and a radius of 5. Find the equation of the circle, the circumference of the circle, and its area. The equation of the circle is* $(x - 3)^2 + (y + 4)^2 = 25$*. This circle has a circumference of* $2\pi(5) = 10\pi$*, and area* $\pi(5)^2 = 25\pi$*.*

Problem 79. *A circle has its center at* $(-4, 8)$*, and just touches the* y*-axis (without crossing it).*

1. *Draw a diagram of the circle in the* xy*-plane.*

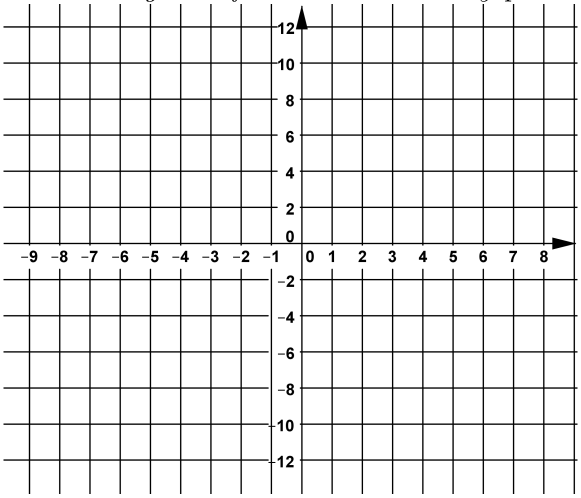


2. *Write the equation of the circle.*

3. *Find the circumference and area of the circle.*

Problem 80. *A circle has its center at* $(-6, -4)$*, and contains the point* $(-1, 8)$*.*

1. *Draw a diagram of the circle in the xy-plane.*



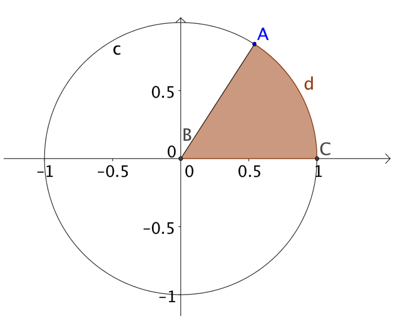
2. *Write the equation of the circle.*

3. *Find the circumference and area of the circle.*

Definition 81. *An* ***arc*** *of a circle is a piece of a circle, and has length. A* ***sector*** *is a part of the plane enclosed by two radii and an arc of a circle, and has area.* ***One radian*** *is the measure of an angle that subtends an arc of length 1 on the unit circle.*

In Figure 16, the unit circle is shown. In the figure, d is the arc and the shaded region is a sector. $m\angle ABC = 1$ radian, because the arc d, the distance along the circle from point A to point C, has length 1.

Figure 16: Unit Circle



It is sometimes handy to be able to convert degree measure to radian measure. Since the circumference of the unit circle is 2π, this means that the radian measure of an angle corresponding to a complete circle is also 2π radians. Suppose that an angle measures α (Greek letter alpha) in degrees, but x radians. We can set up a proportion $\frac{x}{2\pi} = \frac{\alpha}{360}$. If we solve for x, we have the **degrees-to-radians** conversion formula $x = \frac{\pi}{180}\alpha$.

Problem 82. *Convert the angles* 0^o, 30^o, 45^o, *and* 90^o *to radians.*

Problem 83. *Working with arc length.*

1. *What is the length of the arc along a circle of radius 7 cut out by an angle of* 90^o*?*

2. *What is the length of the arc along a circle of radius 3 cut out by an angle of* 45^o*?*

3. *What is the length of the arc along a circle of radius 5 cut out by an angle of* 30^o*?*

4. *Convert the angles in parts 1 to 3 to radians. How do the angle measures in radians relate to the arc lengths? Create a formula for the length of an arc of the circle. Explain why radians are a convenient way to measure angles.*

Problem 84. *Working with sector area.*

1. *What is the area of the sector cut out of a circle of radius 7 by an angle of* 90^o*?*

2. *What is the area of the sector cut out of a circle of radius 3 by an angle of* 45^o*?*

3. *What is the area of the sector cut out of a circle of radius 5 by an angle of* 30^o*?*

4. *Review your answers to parts 1 to 3 and compare them to the angle measures in radians. How do the angle measures in radians relate to the sector areas? Use your findings to create a formula for the area of a sector of the circle.*

Notes

Exercise Set 10

1. A circle has its center at $(5, -12)$, and contains the origin $(0, 0)$.
   (a) Draw a diagram of the circle in the xy-plane.
   (b) Write the equation of the circle.
   (c) Find the circumference and area of the circle.
   (d) Find the length of the arc of the circle cut out by an angle of $\frac{\pi}{3}$ radians.
   (e) Find the area of the sector of the circle cut out by an angle of $\frac{\pi}{3}$ radians.

2. A circle has its center at $(-2, 5)$, and contains the point $(3, 5)$.
   (a) Draw a diagram of the circle in the xy-plane.
   (b) Write the equation of the circle.
   (c) Find the circumference and area of the circle.
   (d) Find the length of the arc of the circle cut out by an angle of $\frac{\pi}{4}$ radians.
   (e) Find the area of the sector of the circle cut out by an angle of $\frac{\pi}{4}$ radians.

3. A park has a circular walking path with a diameter of 250 meters.
   (a) Draw a diagram of the path in the xy-plane.
   (b) Write the equation to describe the walking path in your diagram.
   (c) Find the distance traveled by someone walking once around the entire path.
   (d) Find the distance traveled by someone walking counterclockwise along the path from the easternmost point of the path to the northernmost point of the path.
   (e) Find the distance traveled by someone walking *straight through the park* from the easternmost point of the path to the northernmost point of the path.

4. A Ferris wheel has a diameter of 10 meters, and is mounted so that the bottom of the wheel is 1 meter off the ground. A rider boards the wheel from the 6 o'clock position and the ride turns counterclockwise as seen by the operator.
   (a) Draw a diagram of the Ferris wheel in the xy-plane.
   (b) Write an equation to describe the points on the Ferris wheel.
   (c) Find the distance traveled by someone riding through one complete turn of the wheel.
   (d) Find the distance traveled by someone ridlng from the boarding position to the 11 o'clock position.
   (e) The wheel makes 1 turn every 30 seconds. A rider boards the wheel and travels a distance of 8 meters (along the arc). How long did this take?

5. The population of Butler has been declining, as seen in Table 23.

Table 23: Population of Butler

t (years since 1990)	0	5	10	15	20
P(t) (people)	89,480	80,061	71,633	64,093	57,346

   (a) Decide what kind of function (linear, quadratic, exponential) is appropriate to model the population of Butler, and explain your reasoning.
   (b) Write a function equation for $P(t)$.
   (c) According to your function equation, what was the population of Butler in 2015?

(d) What was the average annual decline in the population of Butler from 1990 to 2010?
(e) When will the population of Butler decline to half of what it was in 1990?
(f) The population of nearby Greenville is described by the equation $G(t) = 2P(t)$. Compare the population of Greenville with the population of Butler.
(g) Write an equation for P^{-1}. State the units of the domain and range of P^{-1}.

6. A carousel (merry-go-round) has a diameter of 18 meters. The ride turns counterclockwise.

(a) Draw a diagram of the carousel in the xy-plane, with the center of the carousel at the origin.
(b) Write an equation to describe the points on the edge of the carousel.
(c) Find the distance traveled by someone riding a horse at the outer rim of the carousel through one complete turn.
(d) Find the distance traveled by someone riding a horse 2 meters from the outer rim as they make $\frac{1}{3}$ of a turn.
(e) The carousel makes 1 turn every 40 seconds. A rider on the outer rim boards the wheel and travels a distance of 15 meters. How long did this take?

7. First-class US mail flat envelopes can be shipped for \$.98 for the first ounce, and \$.21 for each additional ounce.

(a) Write a function equation that gives the cost to ship an envelope weighing w ounces.
(b) What is the cost to ship an envelope weighing 10 ounces?
(c) Customers are advised to ship via other methods if the cost of the envelope will be more than \$3.50. What is the maximum weight that should be shipped via first-class flat envelope?

8. The owners of a small concert venue (capacity 1050) have determined that the number of people who attend is a function of the price of the tickets. If tickets are sold for \$3, the venue will sell out. On the other hand, at \$38/ticket, no one will buy tickets.

(a) Assuming that the number of tickets sold, $S(t)$, is a linear function of the price, t, write an equation for $S(t)$.
(b) Write an equation for the income the theater will generate, $g(t)$, as a function of the ticket price, t.
(c) What is the maximum amount of money that the theater can earn? What price should they charge, and how many people will come at this price?

Trigonometric Functions

Trigonometric functions are used in many contexts. At the most basic level, trigonometric functions relate angle measurements to distances. We will see that the trigonometric functions are useful for modeling circular motion as well.

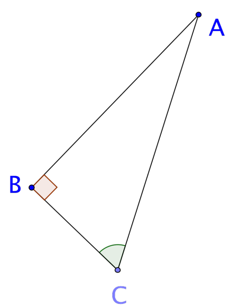
Goal:

- T: Be able to solve for an unknown angle and interpret the result in the appropriate quadrant.
- T: Be able to determine a missing angle or side in a right triangle.

Definition 85. *In a right triangle, we define for an angle α (alpha)* $\sin\alpha = \frac{\textit{opposite}}{\textit{hypotenuse}}$, $\cos\alpha = \frac{\textit{adjacent}}{\textit{hypotenuse}}$, *and* $\tan\alpha = \frac{\textit{opposite}}{\textit{adjacent}}$ *(the acronym to remember these is SOHCAHTOA).*

Problem 86. *Refer to Figure 17. Let $m\angle C = \phi$ (pronounced "fee"). Write* $\sin(\phi)$, $\cos(\phi)$, *and* $\tan(\phi)$ *in terms of the side lengths AB, AC, and BC.*

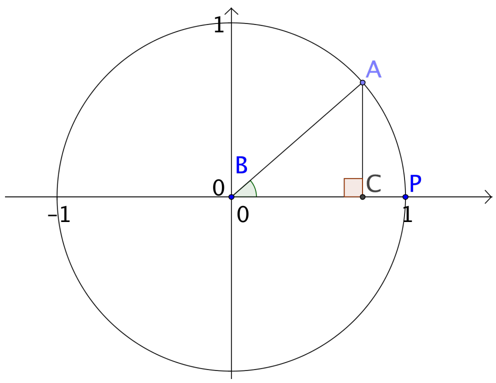
Figure 17: A Right Triangle



Problem 87. *Refer to Figure 18. Let $\theta = m\angle BAC$ (theta). Write* $\sin\theta$, $\cos\theta$, *and* $\tan\theta$ *in terms of the side lengths AB, AC, and BC.*

Definition 88. ***Alternate definition for trigonometric functions.*** *In Figure 18, notice that since point A is on the unit circle, $AB = 1$. Also, if A has coordinates (x, y), then $BC = x$, and $AC = y$. Use this information to rewrite* $\sin\theta$, $\cos\theta$, *and* $\tan\theta$ *in terms of x and y. We can now define the three trigonometric functions for any angle θ by measuring θ as the angle (in radians) traversed counterclockwise from the positive x-axis.*

Figure 18: Triangle on the Unit Circle



Problem 89. *Use what you know about circles and your work from Problem 87 to explain the basic trig identity* $\sin^2\theta + \cos^2\theta = 1$.

Problem 90. *Use the unit circle to find the sine and cosine of the following angles:*

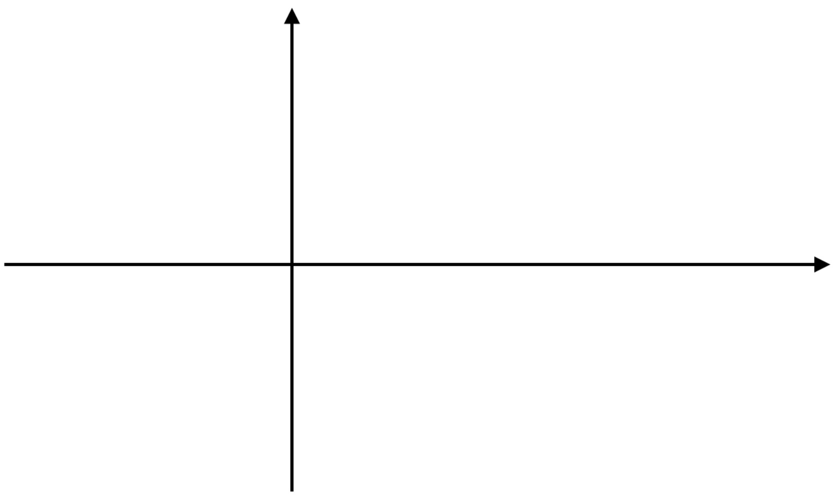
Table 24: Finding Sine and Cosine from the Unit Circle

θ	$\sin\theta$	$\cos\theta$
-2π		
$-3\pi/2$		
$-\pi$		
$-\pi/2$		
$0/2$		
$\pi/2$		
π		
$3\pi/2$		
2π		

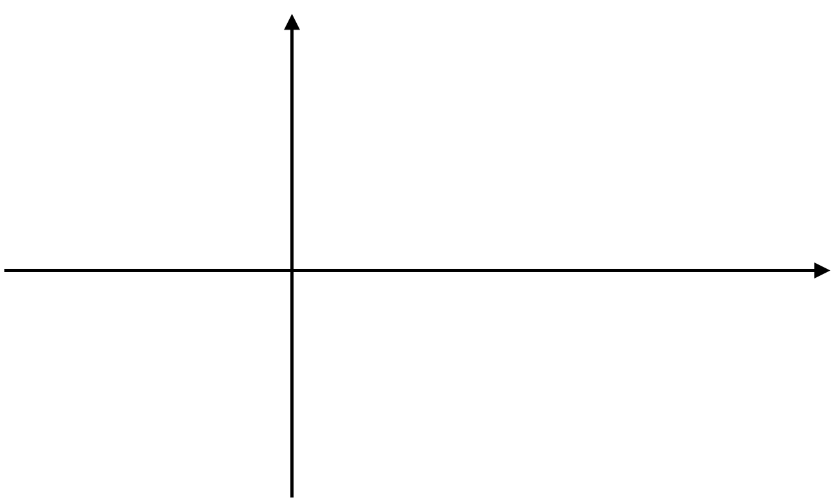
Remark 91. *While it is common to see the right triangle definitions for the trigonometric functions first, in practice, mathematicians often think of the equations in Problem 87 as the definition of the trig functions. The equations in Problem 87 lead to being able to graph the trigonometric functions for values beyond those in a right triangle, and later will be useful for modeling the motion of an object moving around a circle.*

Problem 92. *At the url* `http://tinyurl.com/153sincos` *you will find an animation in Desmos that shows how the graphs of the sine and cosine function can be constructed by examining the coordinates of points on the unit circle as the angle with the x-axis is changing.*

1. *Turn on the sine animation by clicking the circle next to 'Sine Animation'. Note how the graph of the sine function is traced out, with its values equal to the y-coordinate of points on the unit circle as the value of α changes. Sketch the graph of $f(x) = sin(x)$.*



2. *Turn off the sine animation by clicking the circle next to 'Sine Animation'. Turn on the cosine animation by clicking the circle next to 'Cosine Animation'. Note how the graph of the cosine function is traced out, with its values equal to the x-coordinate of points on the unit circle as the value of α changes. Sketch the graph of $f(x) = cos(x)$.*



Problem 93. *Refer to Figure 18. Let $\theta = m\angle ABC$ (theta). Suppose that $\theta = \frac{\pi}{3}$. What are the coordinates of point A?*

Problem 94. *Suppose that $\triangle DEF$ is a right triangle, with right angle at E. Let $m\angle D = \beta$ ("beta"), and let $DF = 4$ and $EF = 2$. Determine DE, and then use the side lengths to compute $\sin(\beta)$, $\cos(\beta)$, and $\tan(\beta)$.*

Problem 95. *A 10-foot ladder is going to be leaned against a wall. The ladder will be sturdy if the angle it makes with the wall is between $\frac{\pi}{12}$ and $\frac{\pi}{6}$. How high up the wall can the ladder make contact with the wall?*

Problem 96. *A wheelchair ramp is allowed to have a slope of no more than* $\frac{1}{12}$.

1. *The front entrance to a building is 3 feet above the sidewalk. Draw a diagram showing a wheelchair ramp.*

2. *How long will the base of the wheelchair ramp need to be? How long is the actual ramp itself?*

3. *What is the angle (in radians) that the ramp makes with the ground?*

Problem 97. *Suppose that* $\sin\alpha = \frac{1}{2}$. *Use Desmos to help you find possible value(s) of* α, *where* $0 \leq \alpha \leq \frac{\pi}{2}$.

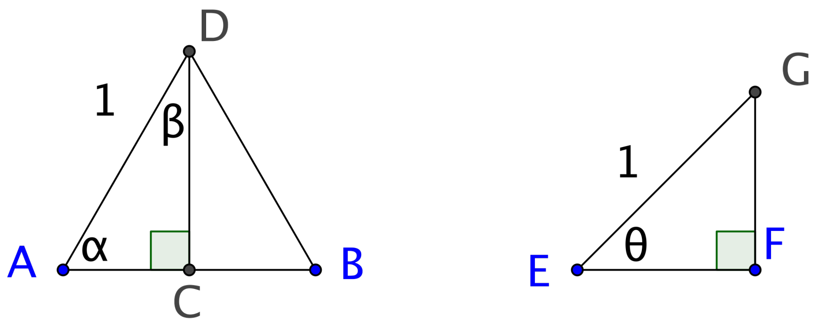
Notes

Important Angles in Trigonometry

Goal:

- T: Be able to solve for an unknown angle and interpret the result in the appropriate quadrant.

Figure 19: Triangles



Remark 98. *Perhaps because the ancient Greeks did a lot of geometry, it is common to use Greek letters in geometry, especially to label angles. In this case, the letters used are α (alpha), β (beta), and θ (theta). Also, recall that in a right triangle, we have that for any angle ϕ (pronounced "fee") the definitions $\sin\phi = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos\phi = \frac{\text{adjacent}}{\text{hypotenuse}}$, and $\tan\phi = \frac{\text{opposite}}{\text{adjacent}}$ (the acronym to remember these is SOHCAHTOA).*

Problem 99. *In Figure 19, $\triangle ABD$ is an equilateral triangle with sides of length 1. Point C is the midpoint of $\overline{AB}$. Recall that the sum of the angles in a triangle is 180^o, and that 360^o is the same as 2π radians. $\triangle EFG$ is an isosceles right triangle, meaning that the two legs of the triangle are equal in length.*

1. *What is the measure of α in radians?*

2. *What is the measure of β in radians?*

3. *What is the length of AC?*

4. *What is the length of CD? (Notice that $\triangle ACD$ is a right triangle.)*

5. *Compute $\sin\alpha$, $\cos\alpha$, $\tan\alpha$.*

6. *Compute* $\sin\beta$, $\cos\beta$, $\tan\beta$.

7. *Why is* $\sin\alpha = \cos\beta$?

8. *Why is* $\cos\alpha = \sin\beta$?

9. *Compute* $\sin^2\alpha + \cos^2\alpha$.

10. *Compute* $\sin^2\beta + \cos^2\beta$.

11. *What is the measure of* θ *in radians? (Notice that there are two equal angles in* $\triangle EFG$.*)*

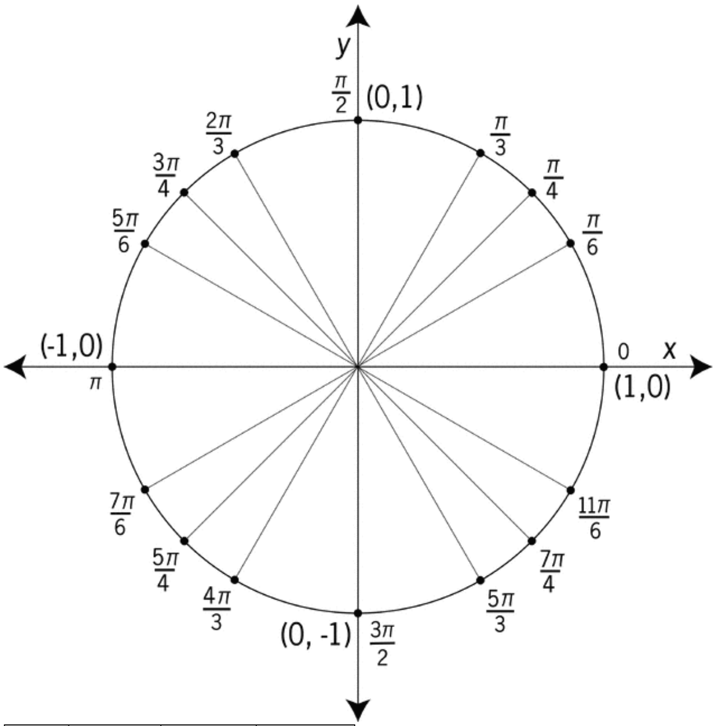
12. *What is the length of* EF?

13. *Compute* $\sin\theta$, $\cos\theta$, $\tan\theta$.

14. *Compute* $\sin^2\theta + \cos^2\theta$.

Problem 100. *Use the circle in Figure 20 and your earlier work to fill in the table.*

Figure 20: Unit circle angles



α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$
0			
$\frac{\pi}{6}$			
$\frac{\pi}{4}$			
$\frac{\pi}{3}$			
$\frac{\pi}{2}$			
$\frac{2\pi}{3}$			
$\frac{3\pi}{4}$			
$\frac{5\pi}{6}$			
π			

α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$
$\frac{7\pi}{6}$			
$\frac{5\pi}{4}$			
$\frac{4\pi}{3}$			
$\frac{3\pi}{2}$			
$\frac{5\pi}{3}$			
$\frac{7\pi}{4}$			
$\frac{11\pi}{6}$			
2π			

Problem 101. *There is a pattern to when the values are positive or negative. Recall that the quadrants are numbered I, II, III, IV starting in the upper right quadrant and proceeding counterclockwise.*

1. *In Quadrant I, which of the functions* $\sin\alpha$, $\cos\alpha$, *and* $\tan\alpha$ *are positive?*
2. *Explain your answer to 1 based on the signs of the coordinates of* $(\cos t, \sin t)$.
3. *In Quadrant II, which of the functions* $\sin\alpha$, $\cos\alpha$, *and* $\tan\alpha$ *are positive?*
4. *Explain your answer to 3 based on the signs of the coordinates of* $(\cos t, \sin t)$.
5. *In Quadrant III, which of the functions* $\sin\alpha$, $\cos\alpha$, *and* $\tan\alpha$ *are positive?*
6. *Explain your answer to 5 based on the signs of the coordinates of* $(\cos t, \sin t)$.
7. *In Quadrant IV, which of the functions* $\sin\alpha$, $\cos\alpha$, *and* $\tan\alpha$ *are positive?*
8. *Explain your answer to 7 based on the signs of the coordinates of* $(\cos t, \sin t)$.

Notes

Exercise Set 11

1. Suppose that $\sin\alpha = \frac{1}{2}$. Use your work in the "Important Angles" section to determine possible value(s) of α, where $0 \le \alpha < 2\pi$.

2. Suppose that $\cos\beta = \frac{\sqrt{2}}{2}$. Use your work in the "Important Angles" section to determine possible value(s) of β, where $0 \le \beta < 2\pi$.

3. A carousel has a diameter of 44 feet. The ride turns counterclockwise.

   (a) Draw a diagram of the carousel in the xy-plane with the center at the origin.
   (b) What is the distance traveled by a rider along the outer rim of the carousel as she travels through an angle of $\frac{\pi}{3}$? If the rider began at the point $(22, 0)$, what are the rider's coordinates after traveling through an angle of $\frac{\pi}{3}$?

4. A Ferris wheel has a diameter of 30 feet, and is mounted so that the bottom of the wheel is 4 feet off the ground. The ride turns counterclockwise as seen by the operator.

   (a) Draw a diagram of the Ferris wheel in the xy-plane.
   (b) Write an equation to describe the points on the Ferris wheel.
   (c) Find the distance traveled by someone riding through one complete turn of the wheel.
   (d) Find the distance traveled by someone ridlng from the 7 o'clock position to the 11 o'clock position.
   (e) The wheel makes 1 turn every 25 seconds. A rider boards the wheel and travels a distance of 50 feet. How long did this take?
   (f) What are the coordinates of a rider when she is in the 11 o'clock position on the wheel?
   (g) What is the height above ground of a rider when her position is at an angle of $\frac{\pi}{4}$ with the horizontal?

5. Let $P(t) = 4\cos(2t)$.

   (a) What is the average rate of change of $P(t)$ over the interval from $t = \frac{\pi}{4}$ to $t = \frac{2\pi}{3}$?
   (b) Find all possible solutions to $P(t) = 2$ on the interval $-2\pi \le t < 2\pi$.

6. Refer to the function $Q(x) = \frac{(x-4)(x+1)}{(x+2)(x+6)}$.

   (a) What is the domain of Q?
   (b) Graph $Q(x)$ so that the important features are visible, and label them.
   (c) Describe the interval(s) on which Q is increasing.
   (d) Find the average rate of change of Q on the interval $-1 \le x \le 1$.
   (e) Find the average rate of change of Q on the interval $1 \le x \le 10$.
   (f) Find an interval on which the average rate of change of P is 0.

7. Uranium-232 is a radioactive isotope with a half-life of 68.9 years.

   (a) Write an exponential equation to model the amount, $A(t)$, remaining after t years from an initial sample of 180 g.
   (b) How much of the sample will remain after 20 years?
   (c) When will the sample decay to 20 g?
   (d) Write an equation for the inverse function, A^{-1}?
   (e) What are the units (grams, years?) of the input to A^{-1}? What are the units of the output?

8. A 12-foot ladder is going to be leaned against a wall. The ladder makes an angle with the ground of 1.1 radians. How high up the wall will the ladder make contact with the wall?

Coordinating Period, Amplitude, and Midline

In this section, we revisit how to transform graphs of functions, now looking specifically at trigonometric functions.

- T: Be able to determine the equation of a trigonometric function given its graph.

Definition 102. *For a function f, the* ***period*** *is the smallest positive value t such that for all x in the domain, $f(x+t) = f(x)$.*

Informally, the period of a trigonometric function is the distance it takes to complete one cycle, before it begins to repeat.
In Desmos, at: `https://www.desmos.com/calculator/lujykscfnv`, you will find the following graphs: orange) $y = a\sin(b(x-c)) + d$, and purple) $y = a\cos(b(x-c)) + d$. You should have sliders for a, b, c, and d. To begin, set $a = 1$, $b = 1$, $c = 0$, and $d = 0$.

Problem 103. *Understanding the cosine graph.*

1. *Turn on the cosine graph (purple).*
2. *Let $a = 5$. How has the cosine changed compared to when $a = 1$? What specific part of the graph measures 5 units? This is called the* ***amplitude****. Include a sketch that shows the amplitude.*

3. *As you slide the value of b, does the cosine wave (purple) change?*

4. *Set $b = 1$. On the cosine function, what is the interval starting from $x = 0$ in order for the graph to make one complete cycle and return to its original starting position? (This is the period.)*

5. *Repeat 4 with $b = 4$, with $b = 2$, with $b = 2\pi$, $b = \frac{\pi}{2}$, and with $b = \frac{\pi}{4}$.*

b	1	2	4	2π	$\frac{\pi}{2}$	$\frac{\pi}{4}$
period						

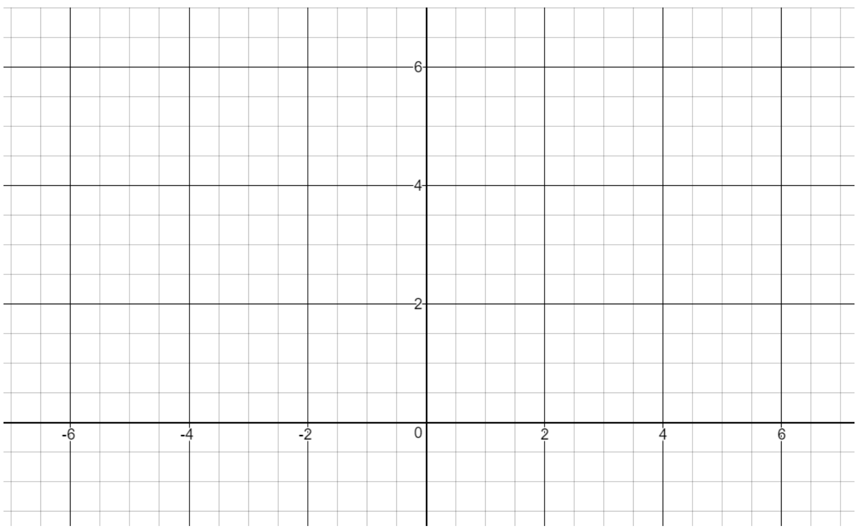
6. *You cannot directly see b on the cosine wave. Instead, b affects the period. If you know b, how can you determine the period of the cosine wave? In reverse, if you know the cosine wave period, how can you find b?*

7. *Slide the value of c. How does the cosine wave change?*

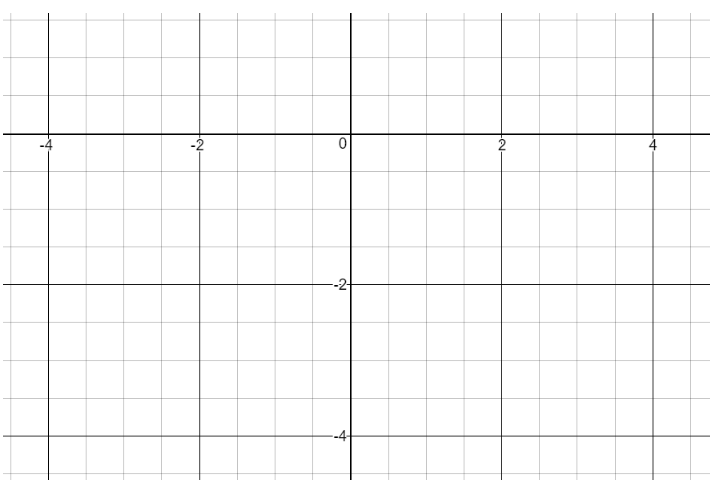
8. *Slide the value of d. How does the cosine wave change? The line $y = d$ is called the* ***midline*** *of the cosine wave. Explain why this name is used.*

Problem 104. *Predict what each of the following graphs will look like and sketch. Use Desmos to check your answer. Be sure to label the amplitude, period, midline and endpoints.*

1. $y = 2\cos(x - 2\pi) + 3$ *on* $-2\pi \leq x \leq 2\pi$



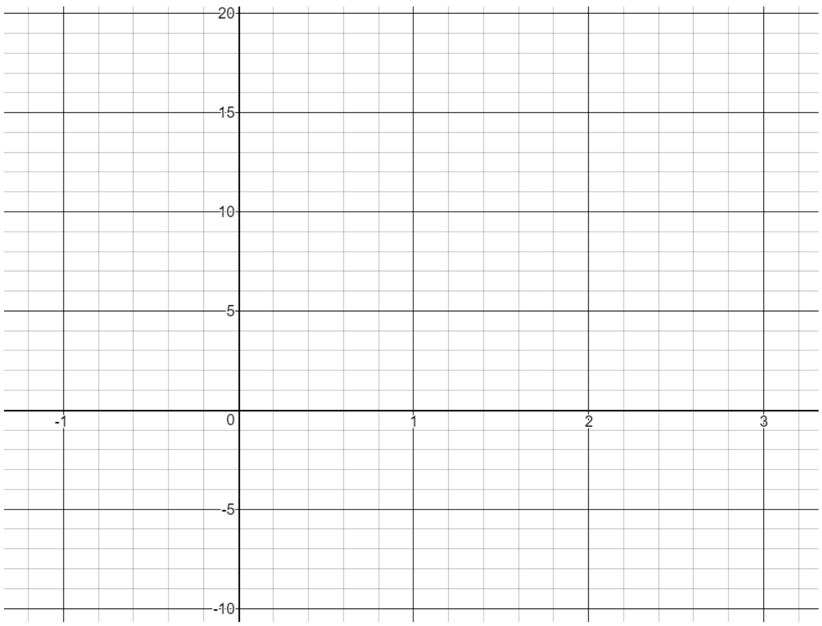
2. $y = \cos(\pi x) - 2$ *on* $-4 \leq x \leq 4$



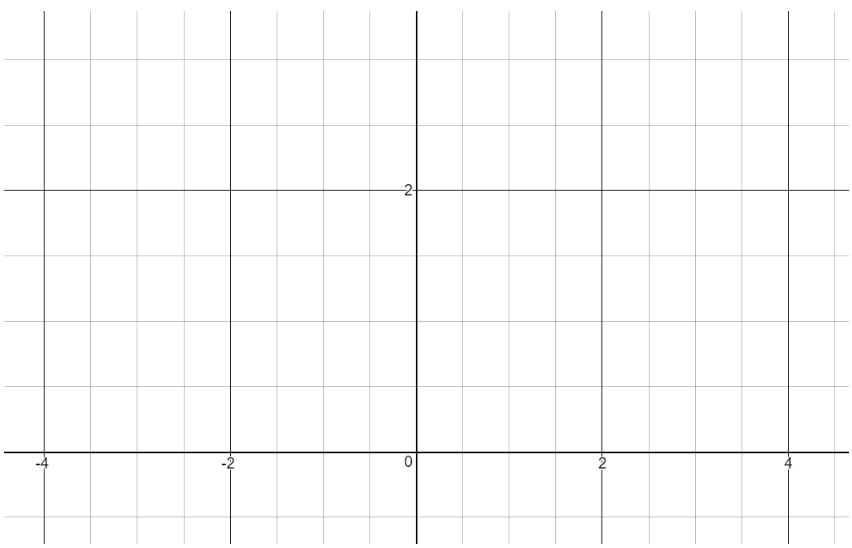
The constants a, b, c, and d operate in the same way in the sine function, $f(x) = a\sin(b(x-c)) + d$, but you begin with the base function $f(x) = \sin(x)$.

Problem 105. *Predict what each of the following graphs will look like and sketch. Use Desmos to check your answer. Be sure to label the amplitude, period, midline and endpoints.*

1. $y = 14\sin(2\pi x) + 4$ *on* $-1 \leq x \leq 2$

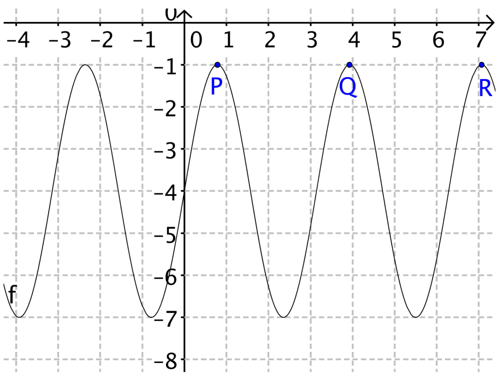


2. $y = -\frac{1}{2}\sin(\frac{\pi}{2}(x-1)) + 2$ *on* $-4 \leq x \leq 4$



Problem 106. *Refer to Figure 21. Note that* $P = (\frac{\pi}{4}, -1)$, $Q = (\frac{5\pi}{4}, -1)$, *and* $R = (\frac{9\pi}{4}, -1)$.

Figure 21: Graph of f

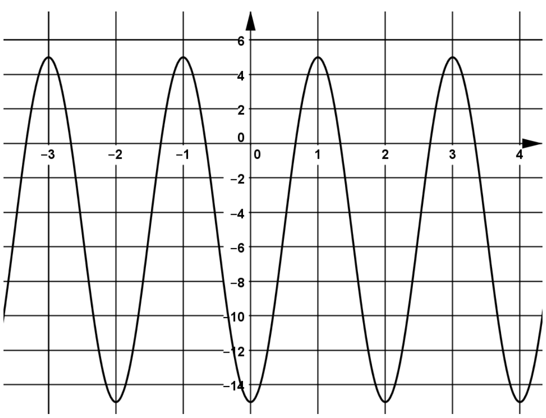


1. *Write a function equation for* f *of the form* $f(x) = a\sin(b(x-c)) + d$.

2. *Write a function equation for* f *of the form* $f(x) = a\cos(b(x-c)) + d$.

Problem 107. *Refer to Figure 22. Note that* $P = (\frac{\pi}{4}, -1)$, $Q = (\frac{5\pi}{4}, -1)$, *and* $R = (\frac{9\pi}{4}, -1)$.

Figure 22: Graph of f



1. *Write a function equation for* f *of the form* $f(x) = a\sin(b(x-c)) + d$.

2. *Write a function equation for* f *of the form* $f(x) = a\cos(b(x-c)) + d$.

Notes

Inverse Trigonometric Functions

Just as we saw that logarithms are useful for solving equations involving unknown exponents, inverse trigonometric functions are useful for solving equations involving unknown angles. Unlike logarithms, correctly solving for an unknown angle may involve interpreting the output in the context of the problem.

- T: Be able to solve for an unknown angle and interpret the result in the appropriate quadrant.

Problem 108. *Recall from our lesson on inverse functions that for a function to have an inverse, it must be one-to-one. Sketch the graph of* $f(x) = \sin(x)$ *below. Is it one-to-one?*

Also recall that we can sometimes restrict the domain of a function so that, in its restricted interval, it is one-to-one. List at least two intervals where $f(x) = \sin(x)$ *is one-to-one.*

Definition 109. *The* ***inverse sine*** *function is defined as* $\arcsin x = y$ *(also written* $\sin^{-1} x = y$*) if* $\sin y = x$ *and* $-\frac{\pi}{2} \leq y < \frac{\pi}{2}$*.*

Problem 110. *Is the function* $f(x) = \cos(x)$ *one-to-one? List at least two intervals where* $f(x) = \cos(x)$ *is one-to-one.*

Definition 111. *The* ***inverse cosine*** *function is defined as* $\arccos x = y$ *(also written* $\cos^{-1} x = y$*) if* $\cos y = x$ *and* $0 \leq y \leq \pi$*.*

Definition 112. *The* ***inverse tangent*** *function is defined as* $\arctan x = y$ *(also written* $\tan^{-1} x = y$*) if* $\tan y = x$ *and* $-\frac{\pi}{2} < y < \frac{\pi}{2}$*.*

Example 113. *Consider the equation* $\sin x = \frac{1}{2}$. *We now have two tools to solve this equation.*

1. *Use Desmos to graph* $y = \sin x$ *and* $y = \frac{1}{2}$. *Use the graphs to find at least four solutions to* $\sin x = \frac{1}{2}$. *Sketch the graph below and label the solutions to* $\sin x = \frac{1}{2}$.

2. *Calculate* $\arcsin(\frac{1}{2})$.

3. *How are your solutions to the equation* $\sin x = \frac{1}{2}$ *the same and how are they different?*

Problem 114. *An angle* t *is in the second quadrant, and* $\sin t = .3$. *Solve for* t.

Problem 115. *An angle* α *is in the fourth quadrant, and* $\cos \alpha = .2$. *Solve for* α.

Problem 116. *An angle w is in the fourth quadrant, and* $\tan w = -\frac{1}{3}$. *Solve for w.*

Problem 117. *Let* $f(z) = 5\cos(\frac{z}{7})$.

1. *Find a solution to $f(z) = -3$.*

2. *Find f^{-1}, and specify the domain and range of f^{-1}.*

Problem 118. *Let* $g(x) = 4\tan(2x)$.

1. *Find at least two solutions to $g(x) = 4$.*

2. *Find g^{-1}, and specify the domain and range of g^{-1}.*

Problem 119. *Find all solutions to the equation* $cos^2(x) - 1 = 0$ *on the interval* $[-2\pi, 2\pi]$.

Problem 120. *Find all solutions to the equation* $\sin^2(x) - 2\sin(x) + 3 = 0$ *on the interval* $[-2\pi, 2\pi]$.

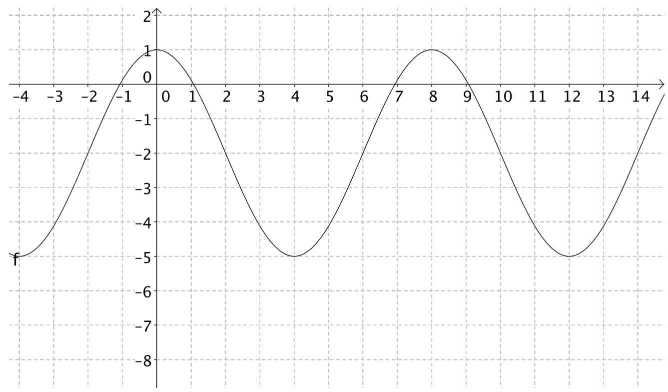
Notes

Exercise Set 12

1. Refer to Figure 23.

   (a) Write a function equation for f of the form $f(x) = a\sin(b(x-c)) + d$.
   (b) Write a function equation for f of the form $f(x) = a\cos(b(x-c)) + d$.

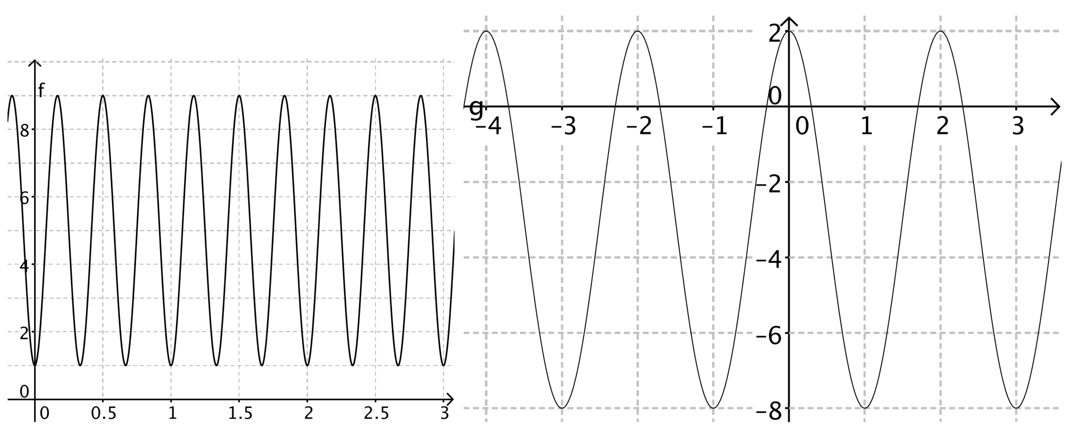
Figure 23: Graph of f



2. Predict what each of the following graphs will look like and sketch. Use Desmos to check your answer. Be sure to label the amplitude, period, midline and endpoints.

   (a) $y = 3\sin(\pi(x-1)) - 2$ on $-4 \leq x \leq 4$
   (b) $y = \frac{1}{3}\cos(\frac{\pi}{2}(x+1))$ on $-8 \leq x \leq 8$
   (c) $y = \frac{1}{4}\sin(\frac{1}{2}(x-2)) - 3$ on $0 \leq x \leq 26$
   (d) $y = -5\cos(\frac{1}{\pi}x) + 5$ on $-10 \leq x \leq 10$
   (e) $y = 4\sin(5\pi x) + 2$ on $0 \leq x \leq 6$
   (f) $y = 15\cos(\frac{2\pi}{3}(x-\pi)) + 35$ on $-6 \leq x \leq 6$

Figure 24: Graphs of f and g



3. Refer to the graph of f in Figure 24.

   (a) Write a function equation for f of the form $a\sin(b(x-c)) + d$.
   (b) Write a function equation for f of the form $a\cos(b(x-c)) + d$.

4. Refer to the graph of g in Figure 24.

   (a) Write a function equation for g of the form $a\sin(b(x-c))+d$.
   (b) Write a function equation for g of the form $a\cos(b(x-c))+d$.

5. Suppose that $\sin\alpha = \frac{\sqrt{3}}{2}$. Use Desmos to help you find possible value(s) of α, where $0 \le \alpha \le \frac{\pi}{2}$.

6. Suppose that $\triangle DEF$ is a right triangle, with right angle at E. Let $DE = 5$ and $EF = 3$. Determine DF, and then use the side lengths to compute $\sin\angle D$, $\cos\angle D$, and $\tan\angle D$.

7. Suppose that $\triangle KLM$ is a right triangle, with right angle at L. Let $KM = 8$, and let $\angle MKL = .25$ radians. Find the lengths KL and LM.

8. A circle has its center at $C = (3, 5)$, and just touches the y-axis.

   (a) Draw a diagram of the circle in the xy-plane.
   (b) Write the equation of the circle.
   (c) Find the circumference and area of the circle.
   (d) Find the length of the arc of the circle cut out by an angle of $\frac{\pi}{6}$ radians.
   (e) Find the area of the sector of the circle cut out by an angle of $\frac{\pi}{6}$ radians.
   (f) Find the coordinates of point A, if point A is in the 2 o'clock position on the circle.

9. A circle has its center at $V = (6, 0)$, and contains the point $(0, 0)$.

   (a) Draw a diagram of the circle in the xy-plane.
   (b) Write the equation of the circle.
   (c) Find the circumference and area of the circle.
   (d) Find the length of an arc of the circle cut out by an angle of $\frac{5\pi}{6}$.
   (e) Find the coordinates of the point B, if ray $\overrightarrow{VB}$ makes an angle of $\frac{\pi}{4}$ with the x-axis (there are multiple answers).
   (f) What is the measure of $\angle FVP$, where $F = (9, 3\sqrt{3})$ and $P = (12, 0)$?

Modeling with Trigonometric Functions

Early in the semester, we saw that parametric equations can be used to describe the motion of an object in the plane. More recently, we have seen that the sine and cosine functions can be used to describe the coordinates of a point on the unit circle. In this section, we bring these two ideas together to develop parametric equations that describe the position of an object as it moves in a circle.

- T: Be able to model a situation involving motion on a circle with appropriate trigonometric parametric equation(s) and interpret the solution.

Problem 121. *A Ferris wheel with a radius of 5 meters is mounted so that the center is 6 meters above the ground. At time $t = 0$, a rider is in the 3 o'clock position on the wheel. The wheel makes one complete counterclockwise turn in 30 seconds.*

1. *Draw a diagram showing the Ferris wheel, with the origin at ground level under the center of the wheel.*

2. *Write parametric equations to describe the position of the rider at time t.*

3. *How far does the rider travel in the first 5 seconds of the ride?*

4. *What are the coordinates of the rider at* $t = 70$ *seconds?*

5. *Find two times when the rider is 10 meters above ground level.*

Problem 122. *A carousel (merry-go-round) is on a pier that juts out eastward from a straight shoreline that runs north-south. The carousel has a diameter of 30 feet and takes 18 seconds to complete one counterclockwise revolution. The center of the carousel is 60 feet from the shoreline. Eden is riding one of the carousel horses, and starts at the point farthest from the shoreline. Let* $t = 0$ *denote this starting point.*

1. *Draw a diagram with the point where the pier meets the shore as the origin, and the positive* x*-axis stretching along the pier through the center of the carousel.*

2. *Graph the distance from Eden to the shoreline as a function of time for the interval from* $t = 0$ *to* $t = 36$ *seconds.*

3. *Find a function of the form $f(t) = A\cos(B(t-h)) + k$ that represents the distance in feet from Eden to the shoreline as a function of time.*

4. *Write parametric equations $(f(t), y(t))$ describing Eden's location at time t.*

Problem 123. *A circle has its center at $C = (-3, 5)$, and just touches the x-axis. An object is moving counterclockwise around the circle. At time $t = 0$, the object is at the point $(-3, 0)$, and it takes 1.5 seconds to complete a revolution.*

1. *Draw a diagram of the circle in the xy-plane.*

2. *Write an equation for the circle.*

3. *Find a function of the form $f(t) = A\cos(B(t-h)) + k$ that represents the x-coordinate of the point as a function of time.*

4. Find a function of the form $g(t) = A\sin(B(t-h)) + k$ that represents the y-coordinate of the point as a function of time.

5. Find the coordinates of the object at time $t = 4$ seconds.

6. Find all the times t, $0 \leq t \leq 1.5$, such that the y-coordinate of the point is 2.5.

7. Find all the times t, $0 \leq t \leq 1.5$, such that the x-coordinate of the point is -4.

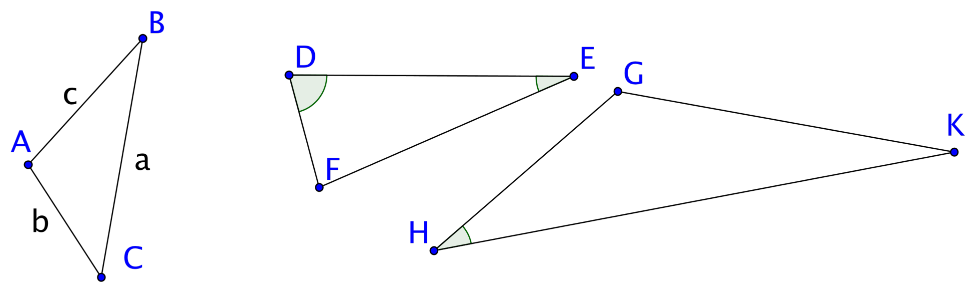
Notes

The Law of Sines and Law of Cosines

The Law of Sines and the Law of Cosines are formulas that are used to solve for unknown values in a triangle. While the SOHCAHTOA definitions apply only in right triangles, the law of sines and law of cosines apply in all triangles. Whether you need one or the other (or both) depends on what you know and what you need to know.
Goal:

- T: Be able to solve problems involving the law of sines or law of cosines.

Figure 25: Triangles



We will state the laws for $\triangle ABC$. The Law of Sines states:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

The Law of Cosines states:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Note that while the Law of Cosines looks a little like the Pythagorean Theorem, and is in fact related to it, when using the law, it does not matter which angle acts as C.

Problem 124. *Apply the Law of Sines or Law of Cosines to solve the following problems:*

1. *Suppose that in $\triangle DEF$, we know that $DF = 1.84$ cm, $m\angle DEF = .415$ radians, and $m\angle FDE = 1.303$ radians. Find the lengths of the other two sides of the triangle.*

2. *Suppose that in* $\triangle GHK$, $GH = 3.93$ *cm,* $HK = 8.51$ *cm, and* $m\angle H = \frac{\pi}{6}$*. Find the length* GK*.*

3. *Two wires are going in opposite directions from the top of a post to the ground. The angle between the two wires is* $85°$ *degrees . The ends of the wires are 15 feet apart on the ground. One of the angles forms an angle of* $35°$ *with the ground. Find the lengths of the wires.*

4. *Two friends start at the same point and begin walking away from each other. Each person is walking in a straight line, and the angle made by their two directions is* $\frac{5}{9}\pi$*. After an hour, one person has walked 5 miles and the other has walked 6.5 miles. How far apart are they?*

Notes

Exercise Set 13

1. A circle is in the third quadrant such that its center is at $(-5, -4)$, and it just touches the x-axis.
   (a) Draw a diagram with coordinates.
   (b) Write an equation for the circle.
   (c) Write parametric equations $(r(t), s(t))$ for a point traveling counterclockwise around this circle such that $t = 0$ corresponds to starting at the 3 o'clock position on the circle, and so that at $t = 2$ the point completes one turn around the circle.
   (d) At what time is the point at the position $(-5 - 2\sqrt{2}, -4 + 2\sqrt{2})$?
   (e) What is the distance traveled along the arc of the circle by the point as it moves from its initial 3 o'clock position to the 1 o'clock position (so that it traverses an angle of $\frac{\pi}{3}$)?

2. A Ferris wheel has a radius of 40 feet and is boarded in the 6 o'clock position from a platform that is 5 feet above the ground. The wheel completes a counterclockwise revolution every 2 minutes. At $t = 0$ the person is at the 3 o'clock position.
   (a) Draw a diagram and impose coordinates.
   (b) Find a function $F(t)$, using the sine function, for the height of the person above the ground after t minutes.
   (c) Find two times when a passenger is at a height of 65 feet.
   (d) Find parametric equations $(G(t), F(t))$ to describe the position of a passenger in the xy-plane.

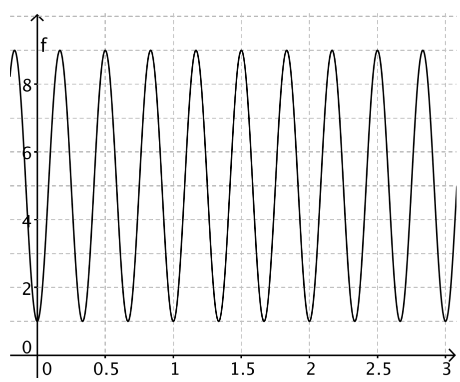


Figure 26: Graph of $f(t)$

3. The graph of $f(t)$ in Figure 26 shows the height of a rider on a Ferris wheel, with the y-axis in meters and the t-axis in minutes.
   (a) Write an equation for the function $f(t)$.
   (b) Assuming the rider boarded the Ferris wheel at $t = 0$ and finished the ride at $t = 3$, describe the rider's experience, including how high off the ground she boarded the Ferris wheel, how long it took the wheel to make one complete turn, the number of turns of the wheel during her ride, and the maximum height she reached on the ride.
   (c) Draw a diagram of the Ferris wheel, and put the origin at ground level under the center of the Ferris wheel.
   (d) Write parametric equations $(x(t), f(t))$ that describe the rider's position at time t minutes.

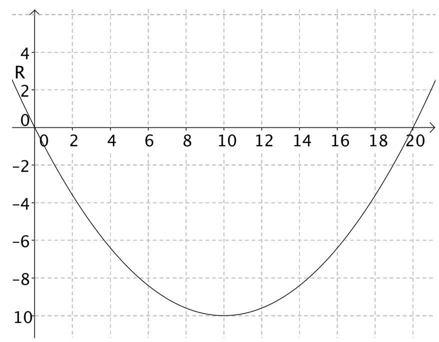
(e) How far is the rider from the boarding location at $t = 0.25$ minutes into the ride?

4. Let $g(x) = 3\sin x - 2$.

   (a) Graph the function on the domain $-2\pi \le x \le 2\pi$. Be sure to label the midline, amplitude, and period.
   (b) Graph the function $s(x) = \sin x$ on the same graph as in part 2a. Then explain how $s(x)$ is transformed to create the graph of $g(x)$.
   (c) Again referring to the graph in part 2a, write out the intervals on the domain $-2\pi \le x \le 2\pi$ on which $g(x)$ is decreasing.
   (d) List all solutions to $3\sin x - 2 = 1$ on the interval $-2\pi \le x \le 2\pi$. Give your answer in exact form.

5. A triangle DEF has measures $DE = 8$ cm and $EF = 13$ cm, and $\angle EDF = \frac{\pi}{4}$. What is $\angle DFE$ (in radians)?

Figure 27: Graph of $R(x)$



6. Refer to the graph of $R(x)$ in Figure 27.

   (a) Solve the inequality $R(x) \ge 0$.
   (b) Write a function equation for $R(x)$.
   (c) What is the range of the function $R(x)$ on the domain $2 \le x \le 18$?
   (d) What is the average rate of change of $R(x)$ on the domain $2 \le x \le 8$?
   (e) Give an example of a domain on which the average rate of change of $R(x)$ is 0.
   (f) Write a function equation for a function that has a graph that is shifted 3 units left and 4 units down from the graph of $R(x)$.

7. A carousel is in a park. The center of the carousel is 90 feet due west of the entrance to the park. The carousel has a diameter of 24 feet and takes 10 seconds to complete one clockwise revolution. Ana is riding one of the carousel horses, and starts at the point nearest to the park entrance. Let $t = 0$ denote this starting point.

   (a) Draw a diagram with the entrance to the park as the origin, and the negative x-axis stretching through the center of the carousel.
   (b) Graph Ana's y-coordinate as a function of time for the interval from $t = 0$ to $t = 30$ seconds.
   (c) Find a function of the form $f(t) = A\sin(B(t - h)) + k$ that represents Ana's y-coordinate as a function of time.
   (d) Write parametric equations $(x(t), f(t))$ describing Ana's location at time t.
   (e) What is Ana's location at $t = 6$ seconds?

(f) Find the straight-line distance between Ana and her sister Carla, if Carla is on the carousel at the 1 o'clock position when Ana is in the 3 o'clock position.

8. Suppose that in $\triangle GHK$, $GH = 3$ cm, $HK = 8$ cm, and $m\angle K = \frac{\pi}{6}$. Find the length GK.

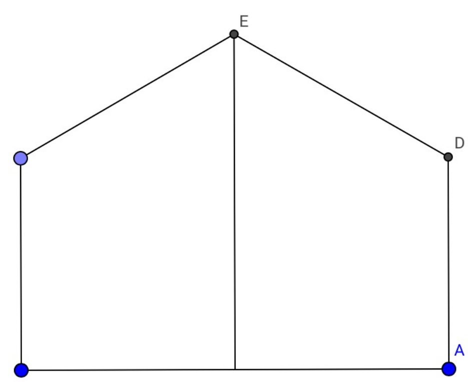
9. Let $Z(x) = \frac{(x+3)(x-4)}{x-1}$.
   (a) What is the domain of Z?
   (b) Graph Z so that the important features are visible, and label them.
   (c) Describe the interval(s) on which Z is increasing.
   (d) Find the average rate of change of Z on the interval $-5 \leq x \leq -4$.

10. Jessica is sitting 30 feet in front in a movie theater screen so that her eyes are 5 feet above the level of the bottom of the screen. The screen itself is 28 feet high. At what angle does Jessica have to look up to see the top of the screen?

11. Point $P = (8, 17)$ and $Q = (-2, 17)$ are the endpoints of a diameter of a circle.
   (a) What is the equation of the circle?
   (b) What is the length of the arc of the circle between point P and point $R = (0, 13)$?
   (c) What angle is formed by a sector whose area is $\frac{25}{12}\pi$?

12. A circle with center at $K = (5, 7)$ has a radius of 10 units.
   (a) What is the equation of the circle?
   (b) Find two points L and M on the circle with center K for which the length of the arc between them is $\frac{5}{2}\pi$.
   (c) What is the area of a sector cut by an angle of $\frac{5}{6}\pi$ radians?

Figure 28: Schematic Diagram of a House



13. The roof of a house is sloped at an angle of $\frac{\pi}{6}$ up from the horizontal. The length of the roof from the peak to the edge ($\overline{DE}$) is 28 feet, and point D is 10 feet above ground level. How high is the peak of the roof?

14. A carousel (merry-go-round) is on a pier that juts out westward from a straight shoreline that runs north-south. The carousel has a radius of 20 feet and takes 25 seconds to complete one counterclockwise revolution. The center of the carousel is 40 feet from the shoreline. Jenny is riding one of the carousel horses, and starts at the point farthest from the shoreline. Let $t = 0$ denote this starting point.

    (a) Draw a diagram with the point where the pier meets the shore as the origin, and the negative x-axis stretching along the pier through the center of the carousel.

    (b) Graph the distance from Jenny to the shoreline as a function of time for the interval from $t = 0$ to $t = 25$ seconds.

    (c) Find a function of the form $f(t) = A\cos(B(t - h)) + k$ that represents Jenny's x-coordinate as a function of time.

    (d) Write parametric equations $(f(t), y(t))$ describing Jenny's location at time t.

    (e) How far does Jenny travel in the first 10 seconds of her ride?

    (f) What are Jenny's coordinates at $t = 10$ seconds?

    (g) Find two times when Jenny is 35 feet from the shoreline.

15. A circle has its center at $C = (3, -4)$, and includes the point $(0, 0)$. An object is moving counterclockwise around the circle. At time $t = 0$, the object is at the point $(-2, -4)$, and it takes 12 seconds to complete a revolution.

    (a) Draw a diagram of the circle in the xy-plane.
    (b) Write an equation for the circle.
    (c) Find a function of the form $f(t) = A\cos(B(t - h)) + k$ that represents the x-coordinate of the point as a function of time.
    (d) Find a function of the form $g(t) = A\sin(B(t - h)) + k$ that represents the y-coordinate of the point as a function of time.
    (e) Find the coordinates of the object at time $t = 3$ seconds.
    (f) Find all the times t, $0 \le t \le 12$, such that the y-coordinate of the point is -1.
    (g) Find all the times t, $0 \le t \le 12$, such that the x-coordinate of the point is 4.
    (h) At what time does the object first reach the origin $(0, 0)$?

Trigonometric Identities

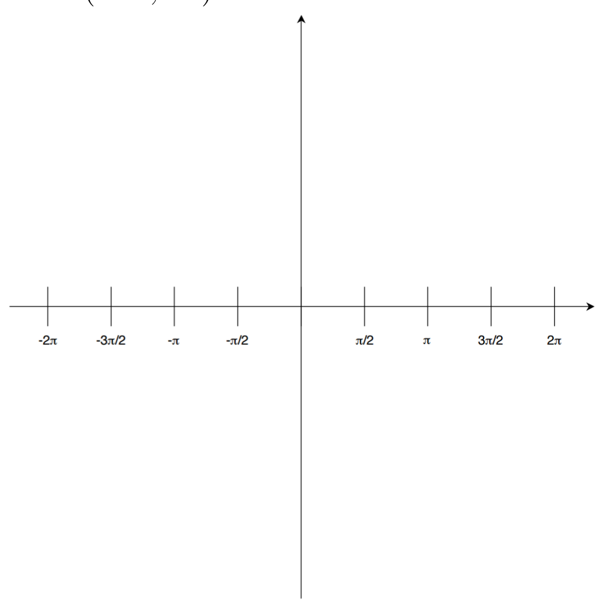
We use the word identity in everyday language to describe characteristics of people and things. In mathematics, the word 'identity' has a very specific meaning. Use your internet connected device to find out what we mean by a mathematical identity. Write a few sentences summarizing your findings which includes at least one example. (Cite your sources.)

Goal:

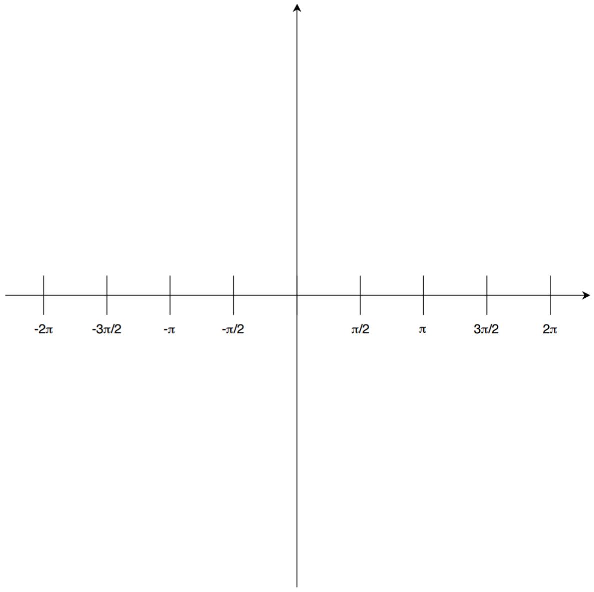
- T: Be able to prove trigonometric identities.

Definition 125. *The function* $\tan x$ *can be written in terms of* $\sin x$ *and* $\cos x$, $\tan x = \frac{\sin x}{\cos x}$. *We will define some new functions in terms of* $\sin x$ *and* $\cos x$.

1. *We will define the cosecant function,* $\csc x = \frac{1}{\sin x}$. *Use Desmos to graph the cosecant function on the interval* $(-2\pi,\, 2\pi)$.



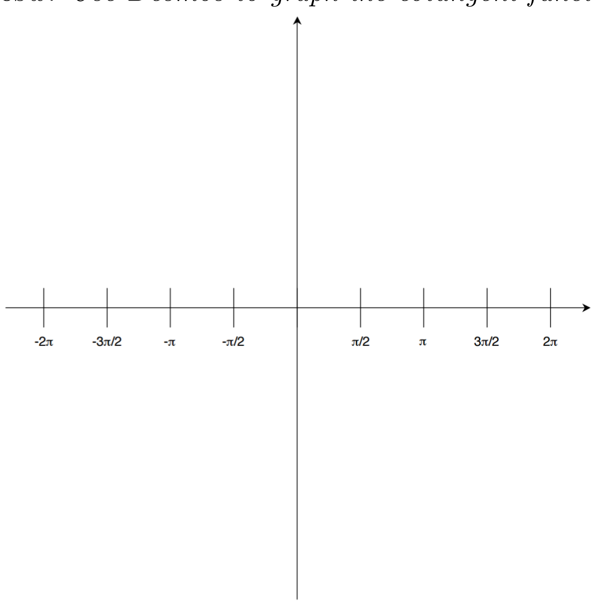
2. *We will define the secant function,* $\sec x = \frac{1}{\cos x}$. *Use Desmos to graph the cosecant function on the interval* $(-2\pi,\, 2\pi)$.



3. *Write an expression for* $\sin x$ *in terms of* $\csc x$.

4. *Write an expression for* $\cos x$ *in terms of* $\sec x$.

5. *If we define the cotangent function,* $\cot x = \frac{1}{\tan x}$, *write an expression for* $\cot x$ *in terms of* $\sin x$ *and* $\cos x$. *Use Desmos to graph the cotangent function on the interval* $(-2\pi,\, 2\pi)$.



From the unit circle, you have already seen that $\sin x^2 + \cos x^2 = 1$. This is one of the most fundamental and most used trigonometric identities, but by the definition above, there are many more identities. Some of these can be seen easily using functions.

Problem 126.

In Desmos, create $f(x) = \sin x$ and $g(x) = \cos(k - x)$, with a slider for k. Move the slider until the two graphs line up.

1. *What does it mean for two functions to 'line up'?*

2. *What is the value of k between 0 and π that makes this happen? (Hint: you can represent this number as an exact value in terms of π.)*

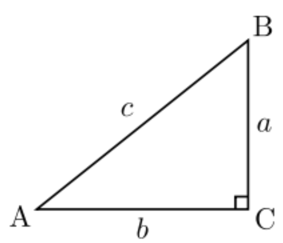


Figure 29: Right Triangle

   (a) *Refer to Figure 29.*

   (b) *Write an equation relating $m\angle A$ and $m\angle B$. Solve for $m\angle B$ in your equation.*

   (c) *Write $\sin(m\angle A)$ and $\cos(m\angle B)$ in terms of a, b, and/or c. Now substitute your result from part 2b into $\cos(m\angle B)$. Explain how this fits with part 2.*

Problem 127. *In Desmos, create* $f(x) = \cos x$ *and* $g(x) = \sin(k - x)$*, with a slider for* k *. Move the slider until the two graphs line up.*

1. *What is the value of* k *between* 0 *and* π *that makes this happen? (Hint: you can represent this number as an exact value in terms of* π*.)*

2. *Refer to Figure 29. Write* $\cos(m\angle A)$ *and* $\sin(m\angle B)$ *in terms of* a*,* b*, and/or* c*. Now substitute your result from part 2b into* $\sin(m\angle B)$*. Explain how this fits with part 1.*

Problem 128. *Use Desmos to verify the following identities:*

1. $\sin(-x) = -\sin(x)$
2. $\cos(-x) = \cos(x)$
3. $\tan(-x) = -\tan(x)$
4. $\csc(-x) = -\csc(x)$
5. $\sec(-x) = \sec(x)$
6. $\cot(-x) = -\cot(x)$
7. *In general, functions are called even if* $f(-x) = f(x)$*, and odd if* $f(-x) = -f(x)$*. Write 'even' or 'odd' next to each function above.*

There are many other useful trigonometric identities. See `http://www.pleacher.com/mp/mlessons/trig/ident2.html` for a more complete list.

In addition to using function graphs to show that trigonometric identities are true, we can also use previously proved identities and algebra.

Example 129. *Prove that* $\tan x \sin x + \cos x = \sec x$. *Choose one side of the equation and transform it into the other side using algebra and previously used identities. NOTE: You may be tempted to perform operations (addition, subtraction, multiplication or division) to both sides of the equation. If you do this, you are using properties of equality. You are using the fact that the equation is true before you have proved that it is!*

$$\begin{aligned}
\tan x \sin x + \cos x &= \\
(\frac{\sin x}{\cos x}) \sin x + \cos x &= \\
\frac{\sin^2 x}{\cos x} + \cos x &= \\
\frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} &= \\
\frac{\sin^2 x + \cos^2 x}{\cos x} &= \\
\frac{1}{\cos x} &= \\
&= \sec x
\end{aligned}$$

Prove the following trigonometric equations using the method in Example 129.

1. $\csc x \cos x = \cot x$

2. $\frac{\tan x}{\csc x} = \frac{\sin^2 x}{\cos x}$

3. $\sec x - \tan x = \frac{1 - \sin x}{\cos x}$

4. $\frac{1}{\tan x} + \tan x = \frac{\csc x}{\cos x}$

Notes

Exercise Set 14

1. Prove the trig identity $\sec x - \cos x = \tan x \sin x$.

2. Prove the trig identity $(\csc^2 x - 1)(\tan^2 x) = 1$.

3. Prove the trig identity $\csc^2 x + \sec^2 x - \cot^2 x = 2 + \tan^2 x$.

4. A triangle ABC has vertices $A = (1, 0)$, $B = (4, 4)$, and $C = (-4, 12)$.

   (a) What are the lengths of the edges of the triangle?
   (b) What are the measures of the angles of the triangle (in radians)?
   (c) What is the area of the triangle?

5. A triangle DEF has measures $DE = 12$ cm and $DF = 10$ cm, and $\angle EDF = \frac{\pi}{6}$. What is the length EF?

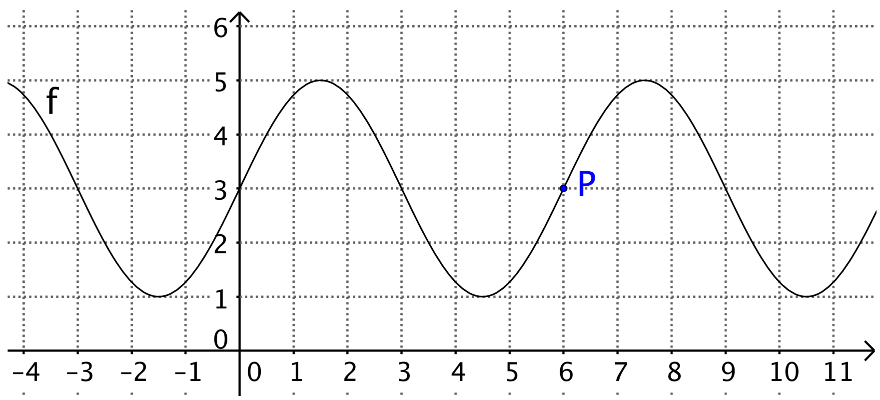


Figure 30: Graph of f

6. Refer to f as shown in Figure 30. Note that $P = (6, 3)$.

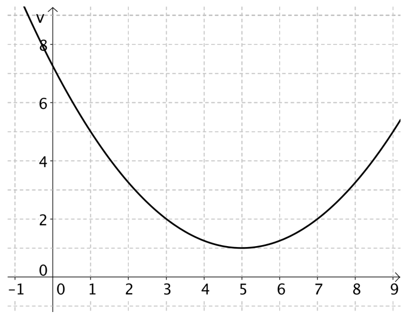
   (a) Write a function equation for f of the form $a \sin(b(x - h)) + k$.
   (b) Use your function equation to give all solutions to $f(x) = 5$ on the domain $0 \le x \le 10$. Verify your answer using the graph of f.
   (c) Evaluate $f(4)$. Verify your answer using the graph of f.
   (d) Write a function equation for f of the form $A \cos(B(x - H)) + K$.
   (e) Considering only the domain $0 \le x \le 10$, on which intervals is f increasing?

7. A carousel is in a park. The center of the carousel is 60 feet due south of the entrance to the park. The carousel has a diameter of 28 feet and takes 14 seconds to complete one counterclockwise revolution. Simon is riding one of the carousel horses, and starts at the point farthest from the park entrance. Let $t = 0$ denote this starting point.

   (a) Draw a diagram with the entrance to the park as the origin, and the y-axis stretching through the center of the carousel.
   (b) Graph Simon's y-coordinate as a function of time for the interval from $t = 0$ to $t = 28$ seconds.
   (c) Find a function of the form $f(t) = A \sin(B(t - h)) + k$ that represents the y-coordinate in feet for Simon as a function of time.
   (d) Write parametric equations $(x(t), f(t))$ describing Simon's location at time t.

(e) What is Simon's location at $t = 10$ seconds?
(f) What distance around the circle does Simon travel in 10 seconds on the ride?

Figure 31: Graph of $v(x)$



8. Refer to the graph of $v(x)$ in Figure 31.

   (a) On which interval(s) is $v(x)$ increasing?
   (b) What is the average rate of change of $v(x)$ on the interval $2 \leq x \leq 5$?
   (c) Solve the inequality $v(x) \geq 0$.
   (d) Write a function equation for $v(x)$.

Exercise Set 15

1. Use the identity $\cos(2x) = \cos^2 x - \sin^2 x$ to prove the identity $\cos(2x) = 1 - 2\sin^2 x$.
2. Suppose that in $\triangle ABC$, $m\angle B = \frac{3\pi}{5}$, $m\angle C = \frac{\pi}{18}$, and $AC = 10$. Find the length AB.
3. Let $z(t) = 3\cos(4t)$.
   (a) On the domain $0 \le t \le 2\pi$, list all intervals on which z is decreasing.
   (b) What is the average rate of change of z on the interval $\frac{\pi}{6} \le t \le \frac{\pi}{3}$?
   (c) Find the inverse function z^{-1} and state the domain and range of z^{-1}.
4. A platform is in the shape of a circle, and has 12 wedge-shaped tiles of equal size. The circle has a radius of 8 feet.
   (a) What is the length of the arc along the edge of one tile?
   (b) Draw a diagram and impose coordinates so that the center of the circle is at $(0, 0)$, and the edge of one tile lines up along the positive x-axis.
   (c) A bug is crawling clockwise along the edge of the circle, and begins at the point $(8, 0)$. Suppose he crawls at a rate of 1 foot/second. Write parametric equations for the bug's position at time t seconds after he begins his journey.

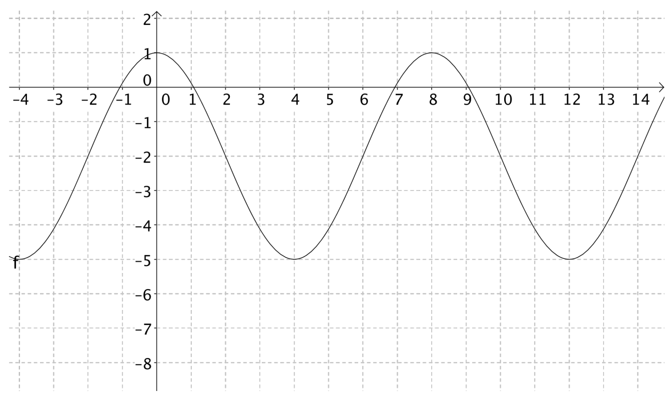


Figure 32: Graph of $f(x)$

5. Refer to $f(x)$ in Figure 32.
   (a) Write an equation for $f(x)$.
   (b) Draw a graph of $f(2x)$ on the interval $0 \le x \le 12$.
6. Refer to the function $a(x) = 10e^x$. Let $b(x) = 4x + 3$.
   (a) Graph $a(x)$ on the domain $-2 \le x \le 5$.
   (b) Solve $a(x) > 50$.
   (c) Write a function equation for the inverse function, $a^{-1}(x)$.
   (d) Let $c(x) = a(b(x))$. Write a function equation for $c(x)$.
   (e) Let $d(x) = a(x) + b(x)$. Write a function equation for $d(x)$.
7. Refer to the functions $f(x) = 2x + 3$ and $g(x) = 10 - 5x$.
   (a) Find the solution to $f(x) \le g(x)$.

(b) Write a function equation for the inverse, $f^{-1}(x)$.
(c) Let $h(x) = 2f(x)$. Write a function equation for $h(x)$.
(d) Let $k(x) = f(x) - g(x)$. Graph $k(x)$ on the domain $0 \leq x \leq 6$.
(e) Let $m(x) = \frac{f(x)}{g(x)}$. What is the set of values for which $m(x)$ is defined?
(f) Graph $m(x)$ and indicate the important features of the graph.

Exercise Set 16

1. Prove the trig identity $\sin t + (\sin t)(\cot^2 t) = \csc t$.

2. Refer to the function $h(t) = -4\cos(2(t - \frac{\pi}{4})) + 1$.

   (a) Graph h on the interval $-\pi \le t \le 2\pi$. Label the period, amplitude, and midline of the graph.
   (b) List all intervals on which h is decreasing on the domain $-\pi \le t \le 2\pi$.
   (c) List all solutions to $h(t) = -2$ on the interval $-\pi \le t \le 2\pi$.
   (d) What is the average rate of change of h on the interval $\frac{\pi}{4} \le t \le \frac{\pi}{3}$?

3. A carousel (merry-go-round) is on a pier that juts out from a straight shoreline. The carousel has a diameter of 40 feet and takes 20 seconds to complete one counterclockwise revolution. The center of the carousel is 150 feet from the shoreline. Carolina is riding one of the carousel horses, and starts at the point farthest from the shoreline. Let $t = 0$ denote this starting point.

   (a) Draw a diagram with the point where the pier meets the shore as the origin, and the positive x-axis stretching along the pier toward the carousel.
   (b) Graph the distance from Carolina to the shoreline as a function of time.
   (c) Find a function of the form $g(t) = A\cos(B(t - h)) + k$ that represents the distance in feet from Carolina to the shoreline as a function of time.
   (d) Write parametric equations $(g(t), f(t))$ describing Carolina's location at time t.
   (e) What are the coordinates of Carolina at time $t = 15$?

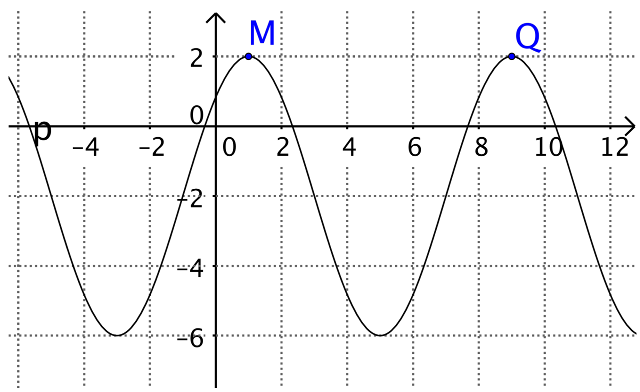


Figure 33: Graph of $p(x)$

4. Refer to $p(x)$ in Figure 33. Note that $M = (1, 2)$ and $Q = (9, 2)$.

   (a) Write an equation for $p(x)$ in the form $a\cos(b(x - h)) + k$.
   (b) Draw a graph of $p(2x)$ on the interval $0 \le x \le 12$.

5. Let $g(x) = 6\cos(\pi x) - 2$.

   (a) Graph the function on the domain $-2 \le x \le 2$. Be sure to label the midline, amplitude, and period.
   (b) Graph the function $b(x) = \cos x$ on the same graph as in part 5a. Then explain how $b(x)$ is transformed to create the graph of $g(x)$.
   (c) Again referring to the graph in part 5a, write out the intervals on the domain $-2 \le x \le 2$ on which $g(x)$ is increasing.

(d) List all solutions to $6\cos(\pi x) - 2 = 4$ on the interval $-2 \le x \le 2$. Give your answer in exact form.

6. A circle is in the fourth quadrant such that its center is at $(4, -3)$, and it just touches the x-axis.

   (a) Draw a diagram with coordinates.
   (b) Write an equation for the circle.
   (c) Write parametric equations $(r(t), s(t))$ for a point traveling counterclockwise around this circle such that $t = 0$ corresponds to starting at the 3 o'clock position on the circle, and so that at $t = 1$ the point has traveled halfway around the circle to the 9 o'clock position.
   (d) What is the distance traveled along the arc of the circle by the point as it moves from its initial 3 o'clock position to the 11 o'clock position (so that it traverses an angle of $\frac{2\pi}{3}$)?

7. A triangle DEF has measures $DE = 8$ cm and $DF = 13$ cm, and $\angle EDF = \frac{\pi}{4}$. What is the length EF?

8. Pablo skates downhill from his house to the end of his street. His distance from the end of the street is given by $D(t) = 300 - 10t^2$, where $D(t)$ is measured in feet and t is time measured in seconds.

   (a) Sketch the graph of $D(t)$, with $D(t)$ on the vertical axis and t on the horizontal axis.
   (b) How far does Pablo travel in the interval from $t = 0$ second to $t = 1$ seconds?
   (c) How far does Pablo travel in the interval from $t = 1$ second to $t = 2$ seconds?
   (d) When does Pablo reach the end of the street?
   (e) When is Pablo traveling the fastest?

9. Suppose that in $\triangle ABC$, $m\angle B = \frac{\pi}{5}$, $m\angle C = \frac{5\pi}{18}$, and $AC = 10$. Find the length AB.

10. A triangle ABC has side AC measuring 12 cm and side BC measuring 17 cm. Also, $m\angle ACB = \frac{3\pi}{4}$. What is the length of side AB?

11. To measure the CN Tower in Canada, a person stands a distance away from the base and measures the angle of elevation to be $\frac{7\pi}{18}$. The same person moves 104 meters closer to the tower and finds the angle of elevation to be $\frac{4\pi}{9}$. How tall is the CN Tower?

12. The diameter of a typical car tire is 25 inches. Suppose that a car is moving at 65 miles/hour.

   (a) How fast is the wheel rotating?
   (b) Suppose a piece of gum is stuck to the tire tread, and at $t = 0$ seconds the gum is in the 12 o'clock position. Write parametric equations to describe the position of the gum at time t seconds.

13. A circle is in the third quadrant such that its center is at $(-5, -4)$, and the point $(0, 8)$ is on the circle.

   (a) Draw a diagram with coordinates.
   (b) Write an equation for the circle.
   (c) Write parametric equations $(r(t), s(t))$ for a point traveling counterclockwise around this circle such that $t = 0$ corresponds to starting at the 3 o'clock position on the circle, and so that at $t = 2$ the point completes one turn around the circle.
   (d) At what time is the point at the position $(0, 8)$?

(e) What is the distance traveled along the arc of the circle by the point as it moves from its initial 3 o'clock position to the 1 o'clock position?

(f) Write new parametric equations $(x(t), y(t))$ for a point traveling counterclockwise around this circle such that $t = 0$ corresponds to starting at the 3 o'clock position on the circle, and so that at $t = 4\pi$ the point completes one turn around the circle.

14. A circle is positioned such that $(-5, -4)$ is a point on the circle, and the center is at $C = (0, 8)$.

(a) Draw a diagram with coordinates.

(b) Write an equation for the circle.

(c) Write parametric equations $(m(t), n(t))$ for a point traveling counterclockwise around this circle such that $t = 0$ corresponds to starting at the 3 o'clock position on the circle, and so that at $t = 2$ the point completes one turn around the circle.

(d) At what time is the point at the position $R = (-5, -4)$?

(e) At what time is the point at the position $S = (\frac{13}{2}, 8 + \frac{13\sqrt{3}}{2})$?

(f) Write new parametric equations $(u(t), v(t))$ for a point traveling counterclockwise around this circle such that $t = 0$ corresponds to starting at the 6 o'clock position on the circle, and so that at $t = 2$ the point completes one turn around the circle.

(g) Write new parametric equations $(p(t), q(t))$ for a point traveling counterclockwise around this circle such that $t = 0$ corresponds to starting at the 9 o'clock position on the circle, and so that at $t = 2$ the point completes one turn around the circle.

Give exact solutions, in radians, for the following equations on the interval $[0, 2\pi)$.

15. Solve for θ in the equation $-2\sin\theta = \sqrt{3}$.

16. Solve for y in the equation $2\cos y + 1 = \cos y$.

17. Solve for α in the equation $\sin^2\alpha - 1 = 0$.

18. Solve for β in the equation $2\tan^2\beta = 2$.

19. Solve for y in the equation $\sin^2 y + \cos y + 1 = 0$.

20. Solve for θ in the equation $\sec\theta + \tan\theta = 0$.

21. Solve for t in the equation $-\sqrt{3}\tan\frac{t}{2} = 1$.

22. Solve for x in the equation $\sec x = \sqrt{2}$.

Find the principal solution(s), in radians, to the following equations.

23. Solve for x in the equation $\tan^2 x + 2\tan x - 3 = 0$.

24. Solve for x in the equation $2\sin^2 x - \sin x - 3 = 0$.

25. Solve for x in the equation $4\cos^2 x + 5\cos x + 1 = 0$.

26. Solve for x in the equation $8\sin^2 x - 3\sin x + 1 = 0$.

Use identities (a) to (l) to verify the additional identities in questions 27 through 34.

(a) $\sin^2 u + \cos^2 u = 1$

(b) $\tan u = \frac{\sin u}{\cos u}$

(c) $\sec u = \frac{1}{\cos u}$

(d) $\csc u = \frac{1}{\sin u}$

(e) $\cot u = \frac{1}{\tan u}$

(f) $\cot u = \frac{1}{\tan u}$

(g) $\sin(-u) = -\sin u$

(h) $\cos(-u) = \cos u$

(i) $\tan(-u) = -\tan u$

(j) $\sin 2u = 2 \sin u \cos u$

(k) $\cos 2u = \cos^2 u - \sin^2 u$

(l) $\tan 2u = \frac{2 \tan u}{1-\tan^2 u}$

27. $\cot(-x) = -\cot(x)$

28. $\frac{\sin x}{1-\cos(-x)} = \frac{1+\cos(-x)}{\sin x}$

29. $\frac{\tan(-x)}{\sin(-x)} = \sec x$

30. $(2\cos^2 x - 1)^2 + (2\cos x \sin x)^2 = 1$

31. $\sin(2x) = \frac{2 \tan x}{1+\tan^2 x}$

32. $(\cos x - \sin x)^2 = 1 - \sin(2x)$

33. $(\cos x + \sin x)^2 = 1 + \sin(2x)$

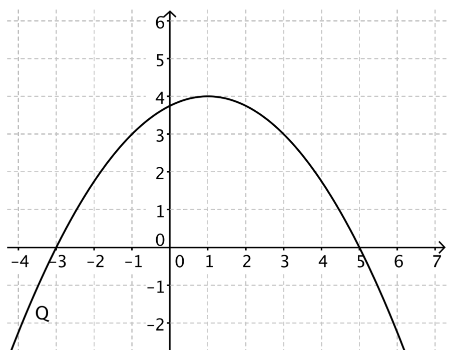
34. $\csc(2x) - \cot(2x) = \tan x$

35. Is it true that $\frac{\sin(2x)}{2} = \sin x$? Explain.

Exercise Set 17

1. Prove the trig identity $\tan x + \cot x = (\sec x)(\csc x)$.
2. Let $g(x) = 5\cos(\pi(x-1)) + 3$.
   (a) Graph the function on the domain $-2 \le x \le 3$. Be sure to label the midline, amplitude, and period.
   (b) Referring to the graph in part 2a, write out the intervals on the domain $-2 \le x \le 2$ on which $g(x)$ is increasing.
   (c) List all solutions to $5\cos(\pi(x-1)) + 3 = 8$ on the interval $-2 \le x \le 3$. Give your answer in exact form.

Figure 34: Graph of $Q(x)$



3. Refer to the graph of $Q(x)$ in Figure 34.
   (a) On which interval(s) is $Q(x)$ decreasing?
   (b) What is the average rate of change of $Q(x)$ on the interval $3 \le x \le 5$?
   (c) Write a function equation for $Q(x)$.
   (d) Solve the inequality $Q(x) \ge -5$.
   (e) Let $R(x) = Q(x-2)$. Graph $R(x)$ on the domain $-7 \le x \le 7$.
   (f) Let $S(x) = Q(x)R(x)$. Solve $S(x) = 0$.
4. Suppose that Ronda skates, and that her rate is given by a function $R(t) = 6t$, where $R(t)$ is measured in feet per second and t is time measured in seconds.
   (a) Sketch $R(t)$ on the domain $0 \le t \le 5$.
   (b) Describe Ronda's trip in this short time interval.
   (c) Find out how far Ronda travels in 5 seconds. You may only be able to get an approximation.