Beyond Whole Number Bases

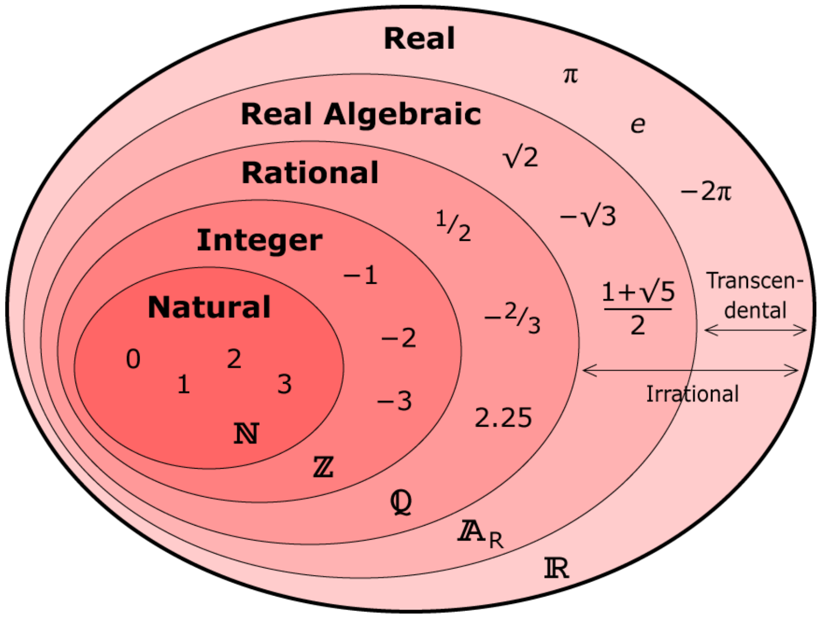

Figure 12: Here is a Venn diagram representing the various subsets of the real numbers. As you can see there are many types of real numbers, why restrict ourselves to positive whole number bases? Picture from `http://thinkzone.wlonk.com/`

The positional numeral systems (place value number systems with whole number bases) have become the standard for number systems (sorry Romans) for a variety of reasons, most especially because they greatly simplify arithmetic (if you don't believe me, try subtracting large numbers with only Roman numerals). The most widely used of these systems by modern civilizations is the very familiar decimal (base ten) system, see `https://en.wikipedia.org/wiki/Decimal`, though the binary (base two) system is foundational to digital electronics, see `https://en.wikipedia.org/wiki/Binary_number`, and the Hexadecimal (base sixteen) system is often used in applications in modern computing, see `https://en.wikipedia.org/wiki/Hexadecimal`).

Historically, the Babylonians were the first civilization to employ a positional number system now known as the sexagesimal (base sixty) system. The Babylonian numerals are displayed below:

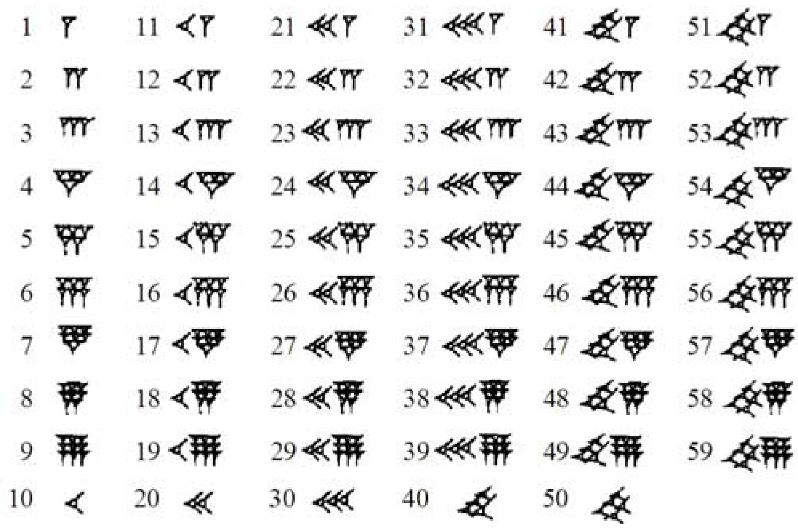

Figure 13: The Babylonian numerals paired with their base ten counterparts. Picture from `https://commons.wikimedia.org/w/index.php?curid=9862983`

Note that the Babylonian number system predated the development of the Arabic numerals (a.k.a. Hindu-Arabic numerals, and Indo-Arabic numerals which are the very familiar symbols 0,1,2,3,4,5,6,7,8,9) by over two thousand years, so they use a completely different set of symbols to construct numbers. (If a base sixty system sounds crazy, just note that the vestiges of that system still exist today, namely as part of how we keep track of time...sixty seconds in a minute, sixty minutes in an hour.) And just for interesting contrast, the Mayan's used a base twenty number system, see `https://en.wikipedia.org/wiki/Maya_numerals`, with symbols:

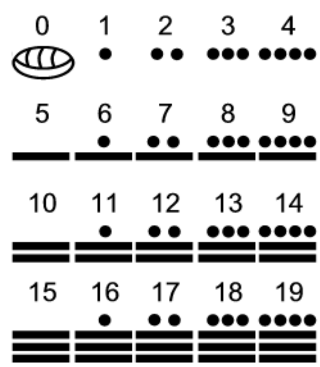

Figure 14: The Mayan numerals paired with their base ten counterparts. Picture from `https://commons.wikimedia.org/w/index.php?curid=1404491`

Notice that due to the geographic isolation of the Americas from Europe, Asia, and Africa in ancient times, we again have a number system based on a set of symbols that are very different than the symbols we commonly employ for numbers today.

The construction of numbers in a positional number system (with base that is positive integer that is greater than 2) works analogously to how we construct numbers in the familiar decimal (base ten) system.

Namely the base tells us how many distinct symbols we have (in the decimal system the symbols are 0,1,2,3,4,5,6,7,8,9) and then numbers larger than those that are denoted by the symbols are constructed by systematically combining the symbols. (The combination of symbols does not have to be in strings as it is in our modern base ten system, it wasn't in the Babylonian system.) Let's quickly review how numbers are constructed in several modern positional number systems in order to set the stage for the explorations below. The tables below contain a collection of amounts represented in words, in our usual decimal (base 10) system, and in another positional numeral system.

The Binary (Base Two) Number System:

There are two basic symbols, 0,1; where 0 represents zero, and 1 represents one. These symbols are combined to denote larger numbers by combining them together in strings where each position in the string denotes a unit that is twice as large as the unit of the position to the right of it (or said differently, each position in the string denotes a unit that is half as large as the unit to the left). For positive integers we assume that the right most position denotes a unit of one. In order to represent positive fractions we utilize place-value positions to the right of the ones place, with a '.' between the ones place and the largest fractional place, namely the one-half position. Let's look at some examples of numbers represented in the binary system:

Amount (in Words)	Amount (in base ten)	Amount (in base two)
three	3	11
eighteen	18	10010
one hundred thirty one	131	10000011
sixty eight and one half	68.5	1000100.1
twenty three and three hundred seventy five thousandths	23.375	10111.011
one tenth	0.1	0.00011001100110011...
two thirds	0.666666...	0.10101010...

The Hexadecimal (Base Sixteen) Number System:

There are sixteen basic symbols, 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F; where 0,1,2,3,4,5,6,7,8,9 all represent their standard amounts, A represents ten, B represents eleven, C represents twelve, D represents thirteen, E represents fourteen, and F represents fifteen. These symbols are combined to denote larger numbers by combining them together in strings where each position in the string denotes a unit that is sixteen times as large as the unit of the position to the right of it (or said differently, each position in the string denotes a unit that is one-sixteenth as large as the unit to the left). For positive integers we assume that the right most position denotes a unit of one. For positive fractions we utilize place-value positions to the right of the ones place, with a '.' between the ones place and the largest fractional place, namely the one-sixteenth position. Let's look at some examples of numbers represented in the hexadecimal system:

Amount (in Words)	Amount (in base ten)	Amount (in base sixteen)
three	3	3
eighteen	18	12
seven hundred sixty three	763	2EB
sixty eight and one half	68.5	44.8
twenty nine and one ninth	29.9999...	$1D.1C71C7\ldots$
one tenth	0.1	0.1999999...
two thirds	0.666666...	$0.AAAAAAA\ldots$

<u>NOTE:</u> You Should Not Move On Before You Can Explain How The Expressions In The Last Column Of The Previous Tables Are Generated. This Is A Self-Check To Make Sure You Understand The Mechanics Of Positional Number Systems With Bases Other Than Ten.

In the explorations below we will study the concept of creating positional number systems with real number bases that are not positive integers. We will find that some work better than our decimal system, some do work but are quirky, and some don't work at all.

For each of the sets of numbers listed below choose a number in the set, use it as a base, and attempt to construct a positional number system with that base. As you work through the process of trying to devising a number system with a base from each set, make sure to formulate meaningful, detailed, mathematically complete answers to the following questions:

- How did you determine the collection of basic symbols you used?
- What amount does each of your basic symbols represent?
- Is your choice of basic symbols optimal, if so why, if not then what would be the optimal choice?
- Any real number can be represented as a decimal expansion, with a finite number of (non-zero) place value positions that are to the left of the decimal point, and either a finite, infinite and repeating, or infinite and non-repeating number of (non-zero) place value positions to the left of the decimal point. (This fact is shared by all positional number systems with a base that is a positive integer greater than or equal to two.) Can your number system represent all possible real numbers? Explain.

Examples within each category are expressed in base 10.

1. <u>Zero:</u>

$$0$$

2. <u>Negative Integers:</u>

   Whole numbers that are less than zero; for example -1, -2, -3, -789, -12780005349875249.

3. <u>Rational Non-Integer Numbers:</u>

   Fractions that are not whole numbers; for example 1/2, 2/3, -8/9, 67/80, 3/2, -800/77, 652781/98235823478.

4. <u>Non-Rational Algebraic Numbers:</u>

   A number is known as algebraic if it is a root of a non-zero polynomial in one variable with rational coefficients (or equivalently integer coefficients). Thus non-rational algebraic numbers are the algebraic numbers that are not whole numbers or fractions. For example $\sqrt{2}$, $7^{1/3}$, $1+\sqrt{5}/2$, -$\sqrt{2}-\sqrt{3}$.

5. <u>Trancendental Numbers:</u>

   A real number is transcendental if it is not algebraic. It turns out that there are many more transcendental numbers than there are algebraic numbers, but we have less examples of them. Some examples we do have are Pi (π), Euler's constant (e), and Chaitin's constant (Ω).

Test Your Understanding: The real test of your comprehension of concepts is the ability to apply them to new questions (in education circles we call this transferring the knowledge you've acquired). Use the questions below to help you determine the extent to which you have mastered the concepts and ideas we've explored above. Answers accompanied by detailed explanations are the only type of answers that count for anything.

(A) Let's focus on positional numeral systems with positive integer bases, W.L.O.G. suppose we have a base of N. What relationship (or relationships) are there between the prime factors of N and the fractions that will have terminating base N expansions versus those fractions that will have an infinite base N expansion.

(B) Deductively explain why the sum, difference, and product of any two rational numbers is a rational number. Also, for good measure, explain why the quotient of a rational number divided by a non-zero rational number is a rational number.

(C) Upon studying positional number systems, a student makes the following observation:

> Suppose we decide for a base 1/10 number system that the symbols 0,1,2,3,4,5,6,7,8,9 mean their usual amounts; then, if we want to find the expansion of any real number in base 1/10, all I have to do is flip the digits of the number's base 10 expansion over the ones digit. For example, if a number X has a base 10 expression of 4527.138, then the base 1/10 expression of X is 8317.254.

Explain why this observation is true. Further, let N represent a positive integer greater than or equal to 2; extend this students observation to a relationship between the base N representation of a number and the base $1/N$ expansion of the same number.

(D) (NOTE: This exploration will make more sense if you have already worked through the 'Dabbling With Instances Of Infinity' activity.)

(i) In the decimal number system it is the case that $0.999\ldots = 1$. There are many ways to argue why this relationship is true, but an algebraic argument is that if I let $s = 0.999\ldots$ then $10s = 9.999\ldots$. Thus $9s = 10s - s = 9.999\ldots - 0.999\ldots = 9$, so $9s = 9$ which implies that $s = 1$. Generalize this fact to a base N number system, where N is a positive integer greater than 2. Justify your generalization with rigorous calculation (i.e. arguing by analogy is not sufficient).

(ii) What is the amount represented by $\ldots 66666.523$ which is an expansion in the base 1/7, where the $0, 1, 2, 3, 4, 5, 6$ all represent the usual amounts.

(iii) What we saw in the first two parts of this question is that in a positional number system with base X, for $X = 2, 3, 4, 5, \ldots$ or $X = 1/2, 1/3, 1/4, 1/5, \ldots$, that there are amounts that can be represented with different base expansions, i.e. . Would this be true of the positional number system with base $\sqrt{2}$? What about a positional number system with base $\sqrt{N}$ where N is a non-perfect square positive integer?

(E) (Building a Number System with the Golden Ratio) The Golden Ratio, often represented as ϕ, is a number that has mystified and amazed mathematicians for millennia through it's many (purported) realizations in nature. Below we'll explore ϕ a little, and then think about how we might base a number system on the Golden Ratio.

In mathematics, two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. For a visualization of this concept, consider the picture below:

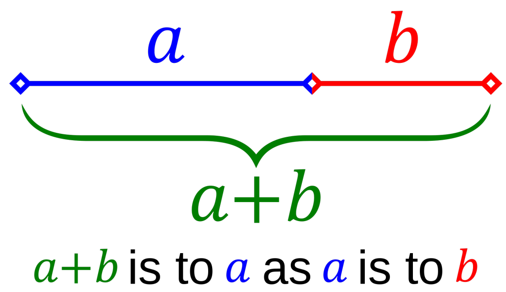

Figure 15: Three line segment, a, b, and $a+b$, that are in the Golden RationPicture from `https://commons.wikimedia.org/w/index.php?curid=1830029`

Expressed algebraically, for quantities a and b with $a > b > 0$, this would mean that when

$$\frac{a+b}{a} = \frac{a}{b}$$

we would say that $\frac{a}{b} = \phi$. We can solve for ϕ with the following calculation. Suppose that the above equation holds true, then

$$\begin{aligned}
\frac{a+b}{a} = \frac{a}{b} &\implies b(a+b) = a^2 \\
&\implies a^2 - ba - b^2 = 0 \\
&\implies a = \frac{b \pm \sqrt{(-b)^2 - 4(1)(-b^2)}}{2} \\
&\implies a = \frac{b \pm \sqrt{5b^2}}{2} \\
&\implies a = \frac{1 \pm \sqrt{5}}{2} b \\
&\implies \frac{a}{b} = \frac{1 \pm \sqrt{5}}{2}.
\end{aligned}$$

Of the two possible values we find for a/b we take the positive one, because we assumed that $a > b > 0$, so $a/b > 0$. Thus we find that the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$.

(i) One way of representing the Golden Ratio is with the infinite fraction, known as a continued fraction, below

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

Call this quantity x, and show that $x^2 - x - 1 = 0$, and so x equals the value for ϕ above.

(ii) Another way of representing the Golden Ratio is with the infinite square root, known as a continued square root (which is also known as an infinite surd, which I think is infinitely ab-surd), below

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$$

Call this quantity x, and show that $x^2 - x - 1 = 0$, and so x equals the value for ϕ above.

(iii) We have show in several ways that $\phi^2 = \phi + 1$, which is the same as saying that $\phi^2 = \phi^1 + \phi^0$.

(a) Explain why it is true that for any value of n that $\phi^{n+2} = \phi^{n+1} + \phi^n$.

(b) Use the result you just explained to show why for any even integer n that $\phi^n + \frac{1}{\phi^n}$ is a whole number. (HINT: Try this out for several values of n.)

(c) Use the result you just explained to show why for any odd integer n that $\phi^n - \frac{1}{\phi^n}$ is a whole number. (HINT: Try this out for several values of n.)

(iv) Use what you've done to try to create a positional number system involving the Golden Ratio (NOTE: I am not saying you need to use the Golden Ratio as the base, although you should try that.)

Reflect On Your Learning Experience: A fundamental skill to becoming (and being) a great teacher is to reflect on your experiences in order to learn from them and grow as an educator. Use the questions below to help guide your reflection. (NOTE: Specific answers to these questions indicate true and meaningful reflection, vague and/or nonspecific answers indicate no honest and valuable reflection.)

1. Put yourself in my position, and consider what you think my goals for you were for this exploration? What did I want you to take away from this experience?
2. What struggles did you experience during this exploration that pushed you to grow in your understanding of mathematical content?
3. What experiences did you have that helped you to better understand and appreciate the Standard of Mathematical Practice, and more generally the process of doing mathematics? About which standards did you learn the most?
4. Describe some experiences you had during this exploration with which you were impressed by how well you responded to struggle. (Examples of the sort of experiences I want you to consider could be that you found a creative way to approach a question, or that you recognized you were struggling and you actively worked past/around/through your negative feelings, or that you supported others in their work.)
5. Describe some experiences you had during this exploration where you recognize that you did not respond to struggle as well as you could. What lessons can you take away from this experience that will help you to grow into a better student, and ultimately a better educator?