DISCOVERING THE ART OF MATHEMATICS

GEOMETRY

MATHEMATICAL INQUIRY IN THE LIBERAL ARTS

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with Philip K. Hotchkiss and Christine von Renesse

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Geometry

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CHAPTER 1

Introduction

I have discovered such wonderful things that I was amazed... Out of nothing I have created a strange new universe.

Jànos Bolyai (Hungarian Mathematician; 1802 - 1860)

1. The Many Faces of Geometry

In general, much of what one learns in high school geometry does not transcend what was known over 2000 years ago. The high school canon includes important changes in language and approach, specifically trigonometry and the use of algebra, but little of true geometric substance moves beyond the ancient knowledge. It is as if nothing much had happened in our understanding of geometry since Euclid. This is a misleading perception and deprives us from learning about the wonderful areas in which geometry continues to flourish.

Our goal in this book is to help you to (re-)discover and explore some of the fundamental revolutions that have shaped geometry. In each of these revolutions the traditional way of thinking, the *dominant paradigm*, has been overturned by the insights of a few people who were able to break free of the prevailing views long enough to glimpse the possibilities of "strange new worlds." Once surprising and perhaps counterintuitive, these new views have become powerful tools that are central to our contemporary understanding of geometry.

You can discover whole new worlds simply by opening your mind - this is a very powerful lesson, one that we hope becomes part of your way of thinking and learning.

Speaking of learning, another knock on the typical high-school geometry experience is the way in which the subject is traditionally taught. As described in section 3 of the last chapter, this was the area in which structured logical reasoning and rigor were first introduced into the world of mathematics in an explicit way. Our long-held efforts to use high school geometry as "the arena in which students will finally get to engage in true mathematical reasoning," have generally not been successful. Instead, as Paul Lockhart describes, it has resulted in a course which is a "virus" which "attacks mathematics at its heart, destroying the very essence of creative rational argument, poisoning the students' enjoyment of this fascinating and beautiful subject, and permanently disabling them from thinking about math in a natural and intuitive way." [1]

Certainly you will have to think, to reason, and to use logic - one cannot do mathematics without working in this way - but we hope that your explorations here will seem more natural, intuitive, and relevant than two column proofs, the formal statements of the Propositions of Euclid's Elements, and the wholly impractical presentation of geometry as a fixed, formal, and ancient system.

[1] From A Mathematician's Lament, by Paul Lockhart, p. 67.

2. Dimension

The sculptures in Figure 1.1 by **Helaman Ferguson** (; -) and **George Hart** (; -) are these sculptors' rendition of a complex mathematical objects. They are just two of the many complex mathematical objects these artist have so beautifully rendered in sculpture.[2]

Like all sculptors, Ferguson and Hart use height, breadth and depth to convey the intricate relationships, connections, and attributes of various components of the work being modelled. The spatial freedom of the sculptor contrasts sharply to that of the painter, illustrator or photographer who are limited to the fields of their canvases which consist only of height and breadth - no depth. In mathematical terms, the canvases of the painter, illustrator, and photographer are *two dimensional* while the "spatial canvas" of the sculptor is *three dimensional.*

FIGURE 1.1. The sculptures "Compass Points" by George Hart and "Torus Cross Cap" by Helaman Ferguson.

The objects of our everyday existence are all three dimensional. Even the sheets of paper in that make up this book have length, width, and thickness. If they were two dimensional, with no thickness, the book, made up of several dozen pages would have no thickness either. Yet much of our art is portrayed two dimensionally - as paintings, as photographs, on computer monitors, in books, on televisions, etc. Isn't this a bit strange?

Stranger still is the analogous situation in contemporary geometry classrooms...

1. Take a few minutes to think about your school experiences with geometry. Write down the main themes and ideas. Then write down as many of the geometric objects you studied as you can remember.

2. The geometric objects that you have considered in your school geometry experience, are they generally two-dimensional or three-dimensional.

3. Have your experiences with school geometry helped enrich the way you interact with the three-dimensional world you live in in important ways? Explain.

Our experiences with school geometry limited us mainly to the study of fairly regular two-dimensional geometric objects - triangles, circles, regular polygons, along with a very few three-dimensional shapes like cubes and spheres. As we continued into our graduate education we studied much more sophisticated mathematical objects. Our experience with geometry was transformed when we began working with artists, builders, designers, geographers, medical specialists and craftspeople. We found rich and deep connections between these fields and our field of mathematics. As we were

[2] For more on Ferguson's work see his book Mathematics and Stone and Bronze. For more on Hart's work see his Internet site at `http://www.georgehart.com/`. Hart is the Chief of Content at the Museuem of Mathematics that will open in Manhattan, New York in 2012. For more on the growing field of mathematical art in general, see `http://www.bridgesmathart.org/` and `http://virtualmathmuseum.org/mathart/MathematicalArt.html`.

enriched by these connections we dreamed about finding ways to share them with others. After all, it seemed to us that these things were as much geometry as what is generally taught in the schools. In fact, maybe they are more.

The idea of *dimension* was a common theme for our enlightenment. We have put together this book as a guide so you can explore this theme. Please think of this book as a map of trail markers you might have for an extended hike. We want you to explore. We have one path for you. We indicated many beautiful routes that are less well-marked that you might travel. You might be drawn to some other area you glimpse along the path, striking out on your own. Whatever your path, it is likely to intersect with others that might surprise you - art, history, philsophy, geography, architecture, medicine, engineering, linguistics, literature, physical sciences, and biological sciences. For geometry is really all around us and informs much of what we do.

3. Getting Started

> Where there is matter there is geometry.
>
> **Johannes Kepler** (; -)

4. Do you agree with Kepler? Explain.

5. In grade school teachers often have geometry scavenger hunts; they ask students to find as many triangles, rectangles, and circles as they can. Spend a few minutes doing this. Write down basic shapes that you find.

6. Now write down things around you that do not contain basic shapes, but rather more complicated shapes, including things that do not have exact geometric names.

7. Compare your lists. What do you notice?

When we explore self-similar objects and fractals in 5 we will find deeper understanding of the words of **Benoit Mandelbrot** (; -) who said:

> Why is geometry often described as cold and dry? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

His implication is that these things are really parts of geometry.

8. Return to your list in Investigation **??**. Choose a few objects on your list that are interesting. Despite not containing basic geometric shapes, are there shapes (e.g. curves, arcs, contours, faces, portrusions, angles) that are interesting? What are some of the features of the surface of your objects? Describe the relative sizes and scales of your objects and/or various components of your objects.

9. Return to Investigation **4**. Is your opinion the same?

10. **Classroom Discussion: Geometry Show and Tell** Bring in a solid, three-dimensional object, or a likeness of it, that you find particularly interesting. Each person should share their object with the rest of the group. Together describe some of the geometric features (like those mentioned in Investigation **8** of the object that make it interesting. (Do not be concerned with names or basic shapes. Reach deeper. There is geometry in rainbows, in their shape, the configuration of colors, in the way light is defracted from the shapes of the water droplets, and many other areas. There is geometry in pineapples where spirals of *Fibonacci numbers* eminate from the base, where the nobs that form the surface fit together as a *tessellation*, in the surface of the leaves, and so on.)

> It [the Larkin Building in Buffalo, New York] was an essay in the third dimension.[3]
>
> **Frank Llyod Wright** (American Architect; -)

[3]From the Ken Burns documentary Frank Lloyd Wright.

Figure 1.2. Exterior and interior of the Larkin Building, designed by Frank Lloyd Wright, which was in Buffalo, New York.

11. Independent Investigation: Make a scale model of a significant three-dimensional structure.

Notes

- We will only use analogues of the ancient measuring tools: rulers and/or tape measures; yardsticks, poles, or other ***straightedges***; *clinometers* or protractors for measuring angles of elevation.
- To get a sense of the methods used by the ancients the structure you choose to model should be something which involves *indirect measurement.* The physical components of a car, shed or small room can all be made directly and should not be chosen as your structure. Instead, choose a house, large building, cell tower, water tower, bridge, or something of this sort.
- You may construct your model from cardboard. As it will be quite useful throughout this book, we encourage you to use the free *computer aided design*, or CAD, program Google SketchUp. A brief introduction, rudimentary instructions and reference material for this powerful tool are included in the Appendix.

Until fairly recently if one wanted to make a model of a building the would do so much like you did above. Now powerful CAD programs make it possible for all of us to make sophisticated models of three-dimensional objects.

Pictured in Figure 1.3 are Google SketchUp (GSU) models built by students in our introductory mathematics courses. None of these students had prior experience with GSU. It's not just our students who are making models. Amateurs the world over have joined with architects, regional planners, and modelers to help populate Google 3D Warehouse. When you use Google Earth or Google Maps to go to *any* major location in the world where there is human architecture, much of this architecture is available to you as virtual 3D models that you can explore! 3D Warehouse houses all of these models and hundreds of thousands of other models, including: cars, furniture, natural objects, proposed renovation plans, woodworking models, aircraft, and imaginary creatures of all sorts.

12. Find 10 structures, buildings, or monuments that have been modelled as part of the Google 3D Warehouse that you think are particularly beautiful, important, or artistic. Use the Earth mini-plugin on maps.google.com to Show 3D Images of an interesting view of each of the objects you have chosen. For each, print out your view, give the name and location, and

FIGURE 1.3. Original Google SketchUp models by Aaron Butler, Matt Pegorari, and Chris Fredette.

explain why you find it compelling. (Note: You may need to use screen captures to extract the images.)

Our goal his is to give you a warm-up before you embark on this exploration of geometry and dimensions. We hope that this has begun to open your eyes a bit to the broader role of geometry around you. As you explore going forward, please keep your eyes open for geometry in the world around you. Take a few minutes each day to ask yourself "Is there geometry here?" We believe that as you do your journey will be enriched and you will begin to agree with Kepler - where there is matter there is geometry.

CHAPTER 2

Navigating Between Dimensions

> Yet I exist in the hope that these memoirs, in some manner, I know not how, may find their way to the minds of humanity in Some Dimensions, and may stir up a race of rebels who shall refuse to be confined to limited Dimensionality.
>
> **A Square from Flatland** (; -)

1. Flatland – A Romance of Many Dimensions

Written by **Edwin Abbott Abbott** (English Teacher and Theologian; 1838 - 1926) the novella Flatland: A Romance of Many Dimensions[1] is the best-known piece of mathematical fiction of all time.[2] Flatland is both a powerful satire of the prejudices of Victorian England and a wonderful tour through the dimensions.

The setting for Flatland is a land which is a two-dimensional plane which is populated by geometric figures. Our hero, whose name is A Square, learns of the "prejudices of his own dimension" when visited by a sphere. Sphere's arrival in Flatland mystifies our hero who sees only variable circles - the *cross sections* of Sphere as he moves through Flatland. When descriptions of the third dimension do not convince A Square of a greater reality, Sphere literally lifts our hero out of Flatland and into the third dimension, providing enlightenment.

Abbott was a dedicated teacher, an important scholar of theology, and an outspoken critic of dogmatic beliefs. The prejudices and enlightenment that motivated Abbott to write Flatland went beyond the mathematical and metaphysical. As one of his important biographers **Thomas F. Banchoff** (American Mathematician; -) tells us:

> Abbott was a social reformer who criticized a great many aspects of the limitations of Victorian society. He was a firm believer in equality of educational opportunity, across social classes and in particular for women.[3]

His ultimate goal? To liberate us from prejudice and have our minds "opened to higher views of things."[4]

1. **Classroom Discussion:** Read §1 - 3 of Flatland. Talk about what life must be like in Flatland and consider Investigation **2** - Investigation **10**
2. In §2: Concerning the Inhabitants of Flatland, A Square describes in detail the different social classes of Flatland. In §7: Concerning Irregular Figures, A Square talks about irregular citizens of Flatland. He calls the "monsters" and tells us:

   > The irregular. . . is either destroyed, if he is found to exceed the fixed margin of deviation, or else immured in a Government Office as a clerk of the seventh class;

[1] The book Flatland is available online for free via Project Gutenberg at `http://www.gutenberg.org/ebooks/201` and at many other sites such as `http://www.ibiblio.org/eldritch/eaa/FL.HTM`.

[2] Mathematical fiction is not a small genre. See the Mathematical Fiction site at the URL `http://kasmana.people.cofc.edu/MATHFICT/` for information on almost 1,000 other works of mathematical fiction.

[3] From the "New Introduction" in the 1991 Princeton University Press edition of Flatland.

[4] Flatland, p. 4

> prevented from marriage; forced to drudge at an uninteresting occupation for a miserable stipend; obliged to live and board at the office...[5]

Why would Abbott have his characters portray the irregulars so dramatically?

3. Women in Flatland are straight lines. What is the dimension of a line? What is the dimension of the polygons which comprise the men of Flatland? What may be the purpose of these differing dimensions?

4. A Square describes women as "the Frail Sex" and says that because they have no angle "they are consequently wholly devoid of brain-power, and have neither reflection, judgement nor forethough, an hardly any memory."[6] If Abbott was really a social reformer interested in equality for women, why would he create a world where women were so inequitably treated?

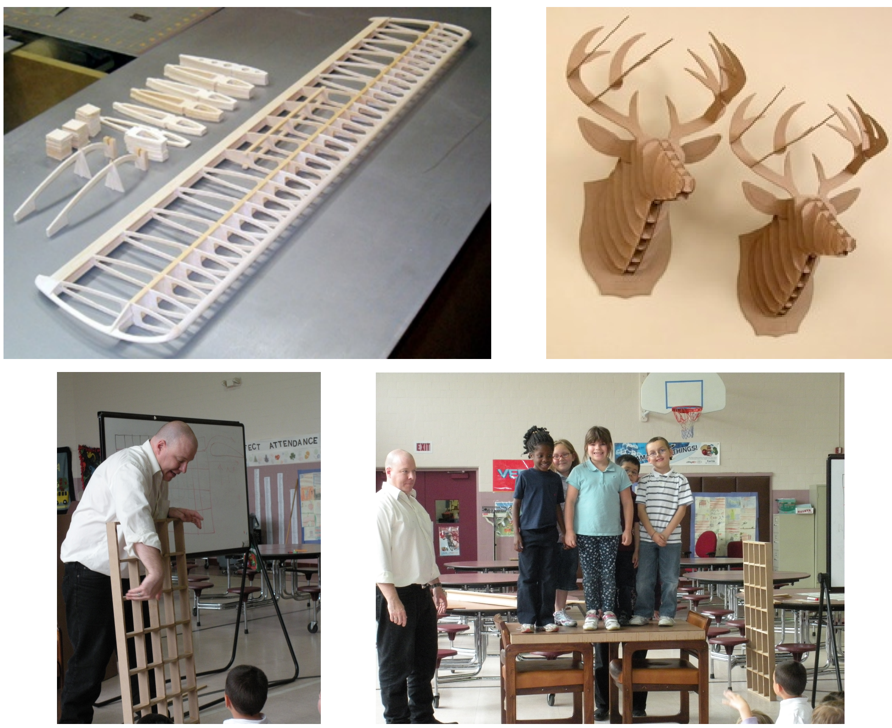


FIGURE 2.1. Three-dimensional objects constructed from two-dimensional components: airplane wing, Carboard Safari trophies, and the first author demonstrating the strength of *torsion boxes* with a little help from some of his friends.

Flatland, as we shall see in Chapter 4, is also notable in its vision of the importance of the understanding of the higher dimensions. This book predated Einstein's Theory of Relativity, CAT scans, 3D computer graphics, and many other profound advances that rest on an understanding of the higher dimensions. You'll get to explore several of these areas where interplay between the dimensions is so important in the next chapter.

5. Abbott uses a penny to help illustrate the trouble of visualization in Flatland. In your own words, with your own example, describe the visual challenge of recognizing objects in Flatland.

[5] Flatland, p. 30
[6] §4.

6. Suppose you were a Flatlander. If object in front of you was A Square, what would he look like? Would it matter how the square was "standing"?

7. Can you think of any easy we to distinguish different men in Flatland - other than moving back into three-dimensional space so you can see them from above?

8. If the object in front of you was a woman, what would she look like? Would it depend on how she was "standing"?

9. Capable of "invisibly... inflicting instantaneous death... no female is suffered to stand in any public place without swaying her back from right to left... [This is] in every respectable female, a natural instict. The rhythimical and, if I may so say, well-modulated undulation of the back in our ladies of Circular rank is envied and imitated by the wife of a common Equilateral."[7] What are the social aspects of this passage? Describe the geometrical implications.

10. Watch the trailer of the Flatland movie at `http://www.flatlandthemovie.com/`. Describe a few aspects of this trailer that you found useful, surprising, comical, or otherwise noteworthy.

In §5: Of Our Methods of Recognizing One Another of Flatland Abbott tells us:

> YOU, WHO are blessed with shade as well as light, you, who are gifted with two eyes, endowed with a knowledge of perspective, and charmed with the enjoyment of various colours, you, who can actually see an angle, and contemplate the complete circumference of a Circle in the happy region of the Three Dimensions - how shall I make clear to you the extreme difficulty which we in Flatland experience in recognizing one another's configuration?
> Recall what I told you above. All beings in Flatland, animate or inanimate, no matter what their form, present to our view the same, or nearly the same, appearance, viz. that of a straight Line. How then can one be distinguished from another, where all appear the same?

Let us experiment a bit with the difficulties one faces in visualizing in a flat world like this.

Pictured below is a construction created using the six different blocks that make up **pattern blocks**, a set of manipulatives often used in elementary mathematics classrooms. The blocks are made out of thin pieces of wood, foam, or plastic. Notice that the edges are all the same length, except the one edge of the trapezoid which is twice the length of all other edges. You can discern certain characteristics of the pattern blocks from this figure.

For the following investigations it will be helpful to have a set of pattern blocks to use. If you do not have access, a template is included in the appendix which will help you make your own set out of cardboard, foamcore, or other material.

Let us suppose we live in a flat world where the inhabitants are shaped identically with the pattern blocks, but are, as in Flatland, perfectly two-dimensional and devoid of color.

11. Suppose you came upon citizen shaped like the tan rhombus. Describe your view of this citizen as you move around him in a circular path.

12. Suppose now you came upon two citizens, one shaped like the tan rhombus the other shaped like the blue rhombus. Could you distinguish which is which? Explain.

13. Could you distinguish a citizen shaped like a hexagon from a citizen shaped like a triangle? Explain.

14. Could you distinguish a citizen shaped like a triangle from a citizen shaped like a square?

15. Can you distinguish all citizens with different shapes (of these six) or are there some that cannot be distinguished. Explain in *rigorous* detail.

16. Create a new, unique shape that is indistinguishible from a hexagon no matter what (Flatland) angle you view it from. (Hint: There are infinitely many.)

[7] §4.

17. If the shape you created in Investigation **16** was not *symmetric*, make on that is highly symmetric.
18. Repeat, if possible Investigation **16** and Investigation **17** for the square.
19. Could you repeat Investigation **16** and Investigation **17** for the pattern block shapes? Explain.
20. Choose a standard shape or create one of your own. Find a way to describe *all* shapes that would be indistinguishible from your shape no matter what angle you view it from.
21. Is there a way we can imagine what our three-dimensional world would be like if we lost our ability to see in perspective, see color, and have depth perception? Explain.
22. By naming the hero of Flatland A Square, Abbott has used a mathematical and *self-referential* pun. Explain.
23. What is the literary value of this pun and how does it fit with Abbott's theme?

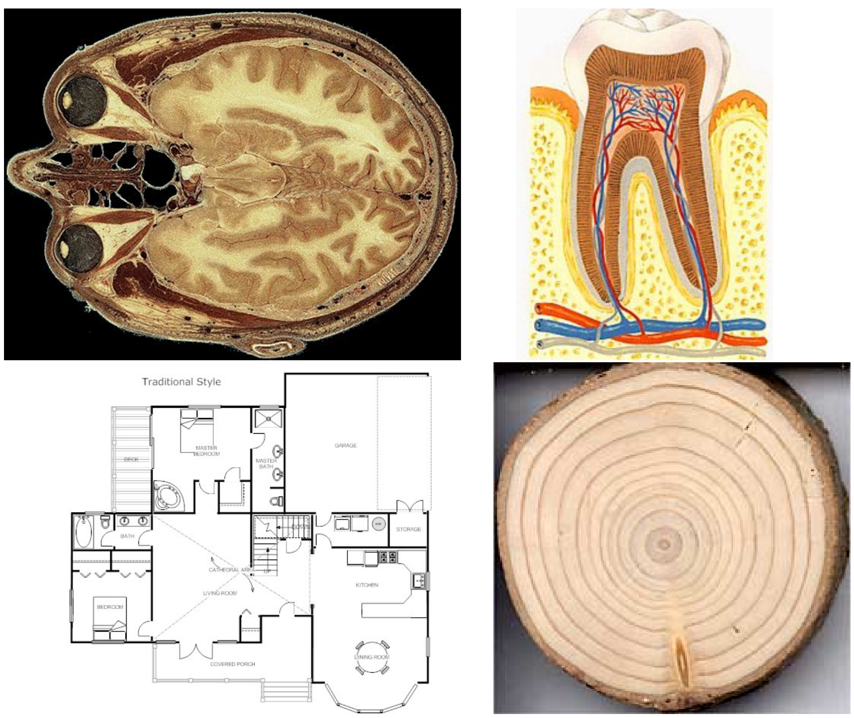


FIGURE 2.2. Cross sections of a human skull, a tooth, a house, and a tree trunk.

2. Cross Sections

Pictured in Figure 2.2 and Figure 2.3 are several ***cross sections***.[8] The word cross section can have somewhat different meanings in different contexts. When mathematicians use the phrase they generally use it to refer to the two-dimensional figures that are formed when a solid is sliced by a knife travelling along a fix plane. So when we say that Figure 2.4 shows cross sections of an apple, we are thinking of these slices as infinitely thin. Alternatively, the cross sections are what you would get if you dipped each slice in ink and stamped it to get a two-dimnensional image.

[8]A formal definition uses the notion of sets; a cross section of an object is the intersection of the object with a given plane.

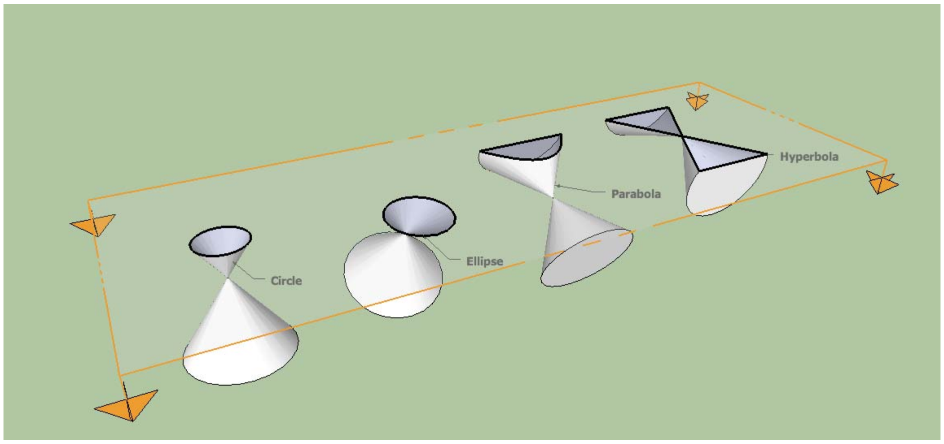


FIGURE 2.3. Cross sections of cones. These cross sections give rise to the important mathematical curves called, naturally, the **conic sections**: *circle*, *ellipse*, *parabola*, and *hyperbola*.

> **Activity Preparation** Collect together several dense, dry objects that can be easily cut by a knife or thin piece of wire. (E.g. apples, zuchinni, bagles, sponge cake, Play-Doh, cheese,...) Line a table with a large sheet of paper. Using a knife or piece of wire, you are going to slice your objects into thin, parallel, and equally spaced slices which mimic cross sections of these solids, as pictured in Figure 2.4.

24. Choose an object to slice. Choose a direction to slice this object - vertically, horizontally, or along some fixed angle. Before you do any slicing, think about what the resulting cross sections of your solid will look like. Draw a sketch of these cross sections in your notebook.
25. Make a number of equally spaced, parallel slices through your object. Arrange the slices linearly and sketch them in your notebook.
26. How do the actual cross sections compare to what you predicted? Is there anything that was surprising?
27. How do the cross sections change as you move through the object? Describe these changes in detail.
28. Take another copy of the same solid object and choose a different direction to slice in. Now repeat Investigation **24** - Investigation **27**, slicing in this new direction.
29. Choose a different, second object and repeat Investigation **24** - Investigation **28**.
30. Choose a different, third object and repeat Investigation **24** - Investigation **28**.
31. What are all of the different shapes cross sections of a cube can take?
32. What are all of the different shapes cross sections of a soda bottle can take?
33. Find, draw, or describe a solid object whose cross sections include a triangle, a circle and a square.

34. In our description of Flatland above we noted that the climax begins when A Square is visited Sphere. Describe what A Square sees and what he imagines has visited him as Sphere passes through Flatland.
35. Does it matter how Sphere was oriented as he passed through Flatland? Explain.
36. Are there any (or any other, as the case may be) objects who would appear identical to Flatlanders no matter what orientation they passed through Flatland?

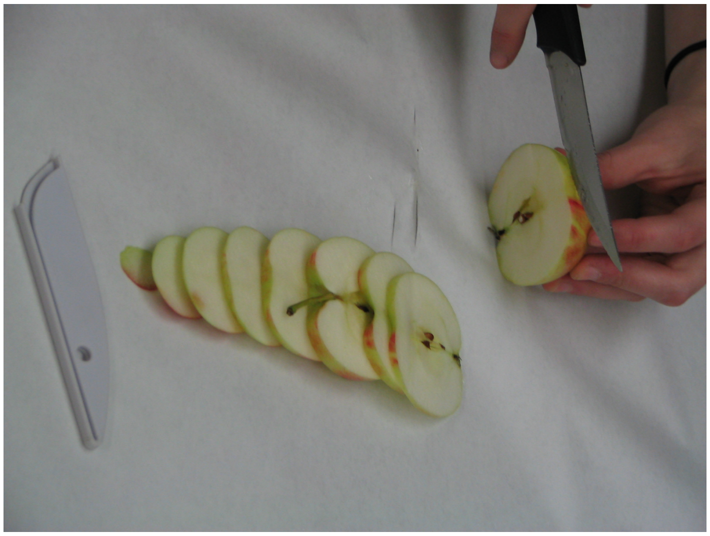


Figure 2.4. Parallel cross sections of an apple.

37. Find several objects whose cross sections in one orientation are all identical. How are these objects similar?
38. Find several objects whose cross sections in one orientation are all identical to each other *and* are also identical to those in another direction.
39. Can you find an object whose cross sections in one orientation are all identical to each and *and* the cross sections in a different orientation are all identical to each other but are *not* equal to the cross sections in the original orientation? Either describe these objects or explain why none can exist.
40. Is it possible for all of the cross sections of two objects to be the same even though the objects are different? Justify your answer.

3. The Flatland Game

Sphere's visit to A Square in Flatland began a journey of enlightenment. We would like to put ourself in A Square's position and see how we can be enlightened. We'll do so via the **Flatland game**.

Flatland Game Goal Determine the identity of a solid object from a series of parallel cross sections taken at regular intervals.

41. Can you guess what secret solid makes the cross sections shown in the first series of clues in Figure 2.5 as it passes through Flatland? If so, explain what the solid is and how you know its identity. If not, describe what you can ascertain about the solid from its cross sections.
42. Can you guess what secret solid makes the cross sections shown in the second series of clues in Figure 2.5 as it passes through Flatland? If so, explain what the solid is and how you know its identity. If not, describe what you can ascertain about the solid from its cross sections.
43. Can you guess what secret solid makes the cross sections shown in the third series of clues in Figure 2.5 as it passes through Flatland? If so, explain what the solid is and how you know its identity. If not, describe what you can ascertain about the solid from its cross sections.

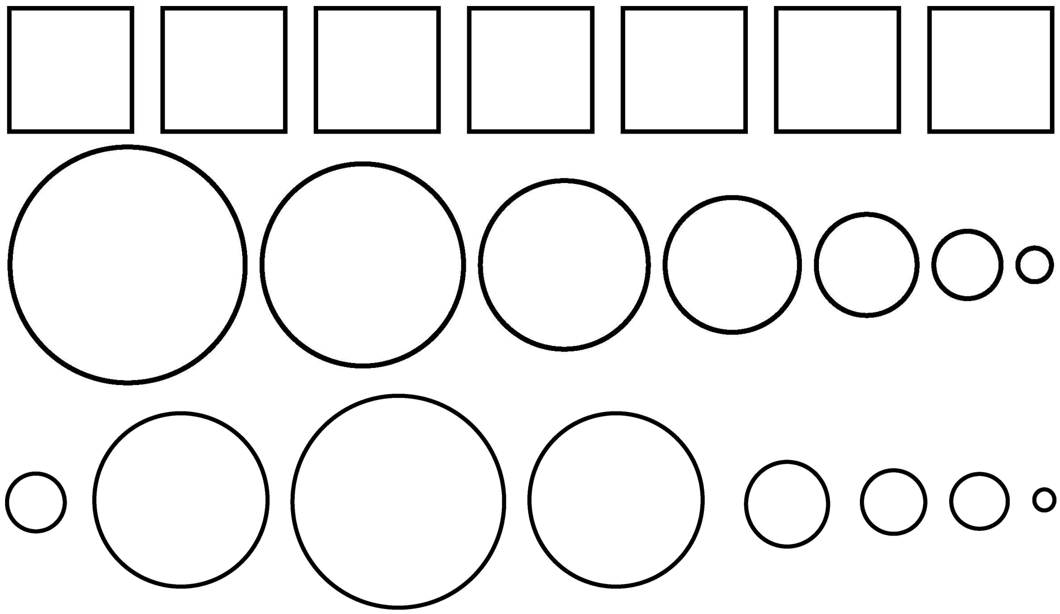


FIGURE 2.5. Three sets of clues for Flatland Games.

Determining what a solid is from a sequence of clues is the *inverse problem* of finding the cross sections of a solid. Both are important problems. The Flatland game will give you practice with both - allowing you to move from Spaceland to Flatland and back.

Rules and Roles for the Flatland Game

1. Choose a team of **radiographers**. This can be a single person or a small team where an illustrator has been elected.
2. Choose teams of **builders**. Teams can consist of a single person if necessary; it is best if a few small teams compete.
3. The game starts with the radiographers secretly determining a solid object from Spaceland whose identity will be the focus of the game.
4. The illustrator for the radiographers then begins play by drawing a single cross section, as viewed from above, of the secret solid.
5. The builders attempt to guess the identity of secret solid.
6. The illustrator for the radiographers then draws another cross section of the secret solid. This cross section must be parallel to earlier cross sections and the cross sections must be revealed consecutively as they would if the object actually passed through Flatland.
7. Steps 5 and 6 are repeated until:
   a. a team of builders correctly guesses the identity of the secret solid, in which case they are declared the winner, or,
   b. there are no more cross sections to draw and the radiographers are declared the winner.

44. Play the Flatland game. Draw the clues in your notebook and describe how the game went, noting any interesting geometrical issues that arose.

45. Play the Flatland game again with the same teams, recording the clues and any observations in your notebook.

46. Now switch roles. Let one of the teams of builders become the radiologists. Play two more games, recording the clues and any observations in your notebook.

47. Play a few more Flatland games, letting each team have a chance to be the radiographers. Each time record the clues and any observations you have in your notebook.

48. Compare your roles as radiographer and builder. Were there similar skills you needed? Were there similar challenges? In which did you learn the most about geometry?

49. Hopefully the radiographers weren't too tough. Name some objects whose identity would be really, really hard to determine in the Flatland game. Explain why they would be so hard.

4. Making Your Own Flatland Movie

As Sphere passed through Flatland A Square saw him growing and shrinking continuously, not just as a few discrete cross sections. You can mimic what it would look like as an object moves through Flatland by making a **Flatland flip book**.

50. Independent Investigation: You will need approximately two dozen small pieces of rectangular cardstock. Anything between a business card and a 3" by 5" index card will work.

- Choose a solid object that you would like to see pass through Flatland as a movie.
- Draw successive parallel cross sections (like the clues in the Flatland game) of the solid, one on each piece of cardstock. Leave some blank space on the left (this is where you'll hold the flip book) and be sure to draw the images in consistent locations on each page.
- Assemble the pictures into a flip book, binding it with a binder clip or heavy duty staple.
- Holding it on the left, thumb through the individual pictures and you will have a movie of your object passing through Flatland!

5. Sliceforms

Box dividers are found in many packaging situations, protecting bottles, glasses, ornaments or other breakables, as shown in Figure 2.6. If you have not seen box dividers, see if you can locate one and figure out how it works. Mechanically they are very interesting. Simply by making regular slots in lengths of cardboard and then lacing them together to form a grid, one arrives at a very inexpensive and sturdy way to safely package a variety of objects.

In the 1870's **Olaus Henrici** (Danish Mathematician; -) discovered these same mechanical principles could be used to make beautiful, dynamic mathematical models. We say dynamic because while box dividers fold nicely for storage, Olaus' *sliceform* models deform gracefully to show varied mathematical properties of the oject - in addition to the natural interest in having three-dimensional mathematical "solids" that can be folded flat to be carried in a pocket.

Sliceforms were resqued from relative obscurity[9] by **John Sharp** (English Teacher; -) who published a wonderful book of templates called Sliceforms: Mathematical Models from Paper Sections in 1995. Subsequently he published full-lenght book on the topic: Surfaces: Explorations with Sliceforms in 2004.

Subsequently, sliceforms have seen quite a resurgence as many of the figures below help illustrate.

[9] They do make a brief appearance on p. 19 of Geometry and the Imagination by **David Hilbert** (; -) and **Cohn-Vossen** (; -).

Figure 2.6. Cardboard box dividers.

Figure 2.7. Sliceforms as art; the sculpture of Richard Sweeney and the award-winning greeting card by Up With Paper.

Sliceforms are related in fundamental ways to the geometric ideas that you investigated via the Flatland game. In this section you will explore sliceforms. The ultimate goal is for you to design and build your own original sliceform.

How do we go about designing and building a sliceform of a non-trivial three-dimensional object? Well, like most things you will need to explore, experiment, think, and plan. Your notebook is a perfect place to do this. The investigations below should provide some starting points.

51. Look at the sliceforms pictured in Figure 2.7, Figure 2.8, and Figure 2.10. Imagine viewing them from the top. How are the slices positioned? What shape is formed in the openings between any four slices that meet at intersecting pairs?

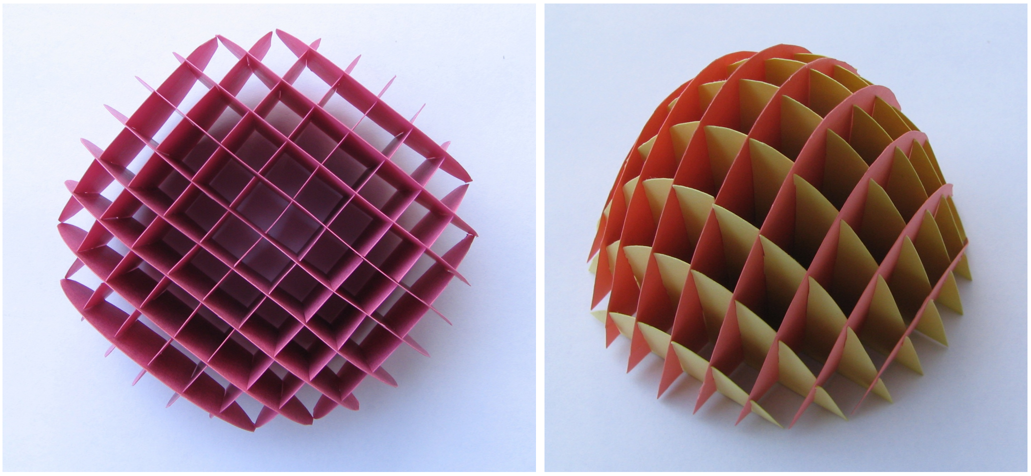


Figure 2.8. Three sets of clues for Flatland Games.

52. Sliceforms are constructed so they easily fold down to be flat. As they are folded, how will the shape you described in Investigation **51** change? Explain.

53. Is there any other layout of slices that can be used to make a sliceform where the slices are all "straight" and the resulting sliceform can fold flat? I.e. will any shape other than those considered in Investigation **51** - Investigation **52** make a working sliceform?

54. In the materials list graph paper is listed. Based on Investigation **51** - Investigation **53**, describe how this graph paper can be used to help build an original sliceform.

Sliceform Materials Medium to heavy-weight cardstock. Sharp scissors. Ruler and possibly other measuring instruments such as a contour or profile gauge. Graph paper.

55. Cut out two small squares, say 3" on each side, from cardstock. In each cut a slit from the center of an edge directly to the center of the square. Insert one slit into the other. Is it easy to make the squares cross at right angles without holding them in place? Do the slits act as a smooth hinge so the object can be flattened down, easily opened back up, and then flattened down the other way?

56. Cut out two more small squares. Instead of cutting slits from the edge to the center, cut ***slots*** where a small amount of cardstock is actually removed by making two parallel cuts very close to each other. Now assemble the two pieces by mating their slots. Is it easier to make the squares cross at right angles without holding them in place? Do the slots act as a better hinge?

In Figure 2.9 are templates for a stock sliceform. Larger versions of these templates are included in the appendix.

57. Can you guess what object this sliceform models when it is completed? Explain.

58. The individual pieces are labelled. What do you think the labels tell you? How were the names for these labels chosen?

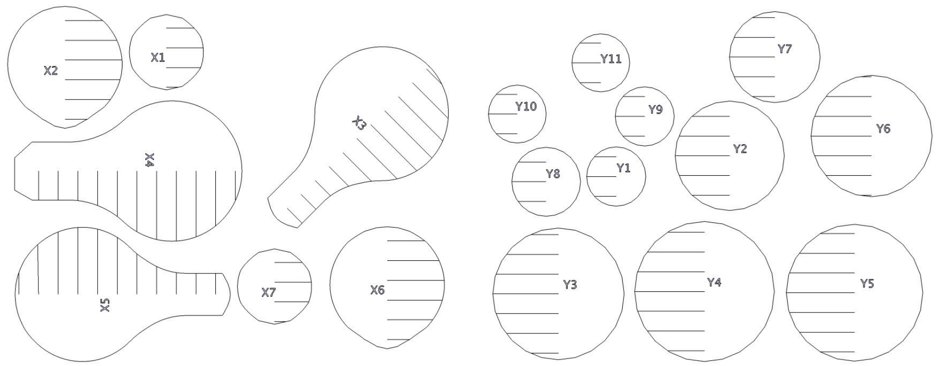


FIGURE 2.9. Templates for a sliceform model.

59. INDEPENDENT INVESTIGATION: Cut out the templates. Cut slots along each of the indicated lines. Then assemble the pieces into a completed sliceform. Describe the artistic, physical, and mechanical aspects of your sliceform.

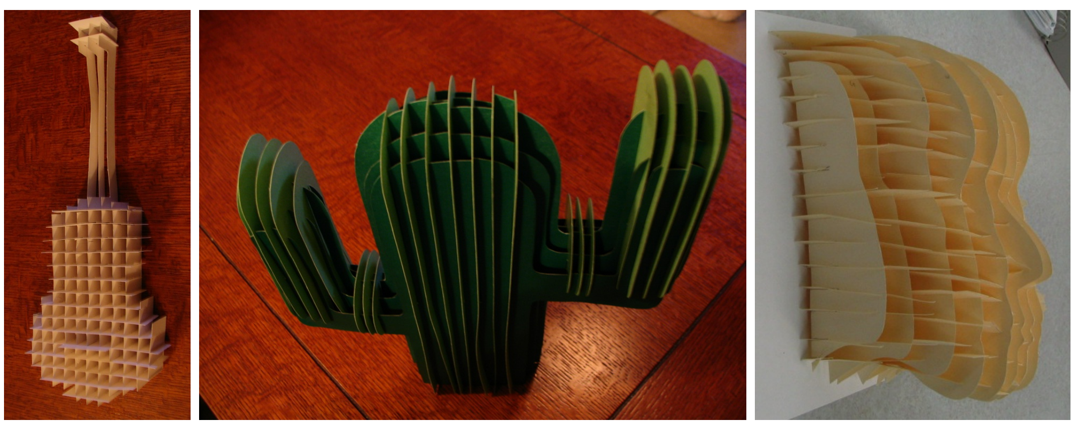


FIGURE 2.10. Original student sliceforms; "Guitar" by Katherine Cota, "Cactus" by Sharon Kubik-Boucher, and "The Face" the Lydia Lucia.

60. Look at your sliceform and those pictured in the figures in this chapter. When vertical and horizontal slices instersect - and are in fact joined at - a slot, how are the heights of the two slices related?

61. When you make your original sliceform you will have to figure out how deep to make each slot. Does Investigation **60** provide a good rule of thumb for doing this? Explain.

62. Suppose you loose a single slice of your model. How could you use your model to recreate this missing piece?

63. Does this give you an idea how you can create horizontal slices once you have all of the veritcal slices of your sliceform created appopriately? Explain.

64. Independent Investigation: Design and build your own original sliceform.

This process may take you several hours over a few sittings. The design phase is quite important. As you design you should consider what you learned in the Flatland game and supplement it with tools like Play-Doh to make mock models, measuring tools like profile or contour gauges (see Figure **??**), profiles cast by shadows of light, actual cross sectioning of the object, or the use of Computer Aided Design software like the free Google SketchUp.

Figure 2.11. A profile gauge; used regularly by carpenters and machinists.

6. More General Cross Sections

End with Biesty books, general cross section, body worlds, etc. Now the possibilities are limitless. How do you want to explore our 3D world and 3D objects therein?

Body Worlds intro. There have been ethical issues related to this and similar exhibits. There are also religous and personal decisions that individuals must make. But as this exhibit travels the world thousands of people see the human body in entirely different ways - learning to see this miraculous machine in entirely new ways. Each time the authors have visited the exhibit the audience has included professionals - including doctors, nurses, athletic trainers, physical therapists - seeing new things. It is regular to hear things like "Wow, now I really see how the position of the —- impacts the motion of the —-."

FIGURE 2.12. Images from the Body Worlds exhibit.

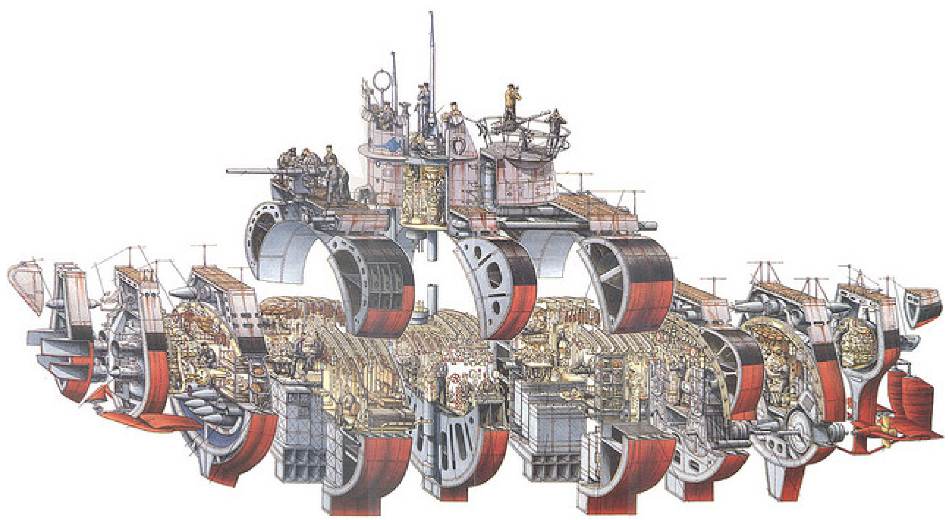


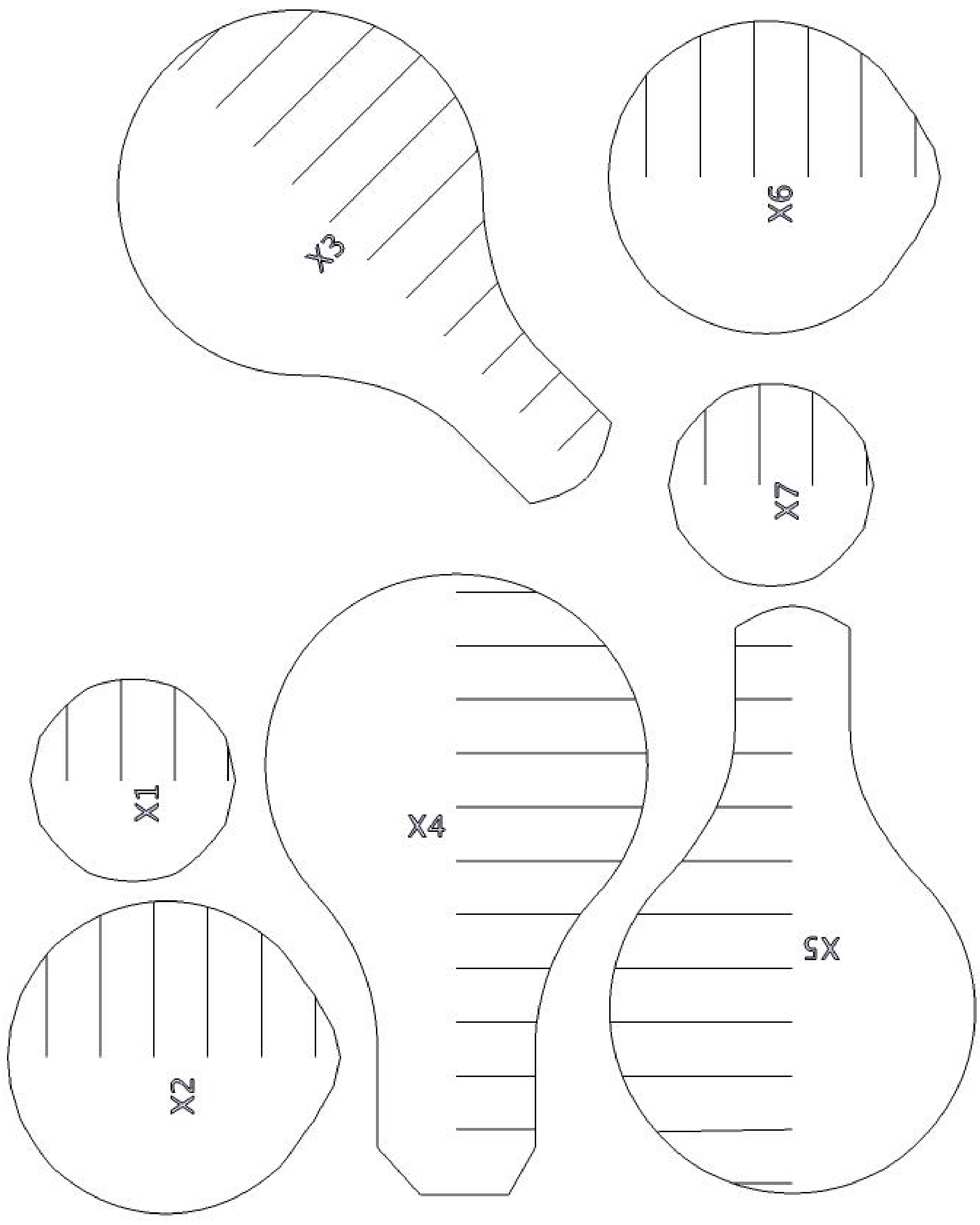
FIGURE 2.13. UBoat Cross Section by Stephen Biesty.

7. Alternate Approach: Projections

8. Connections

Needs to be something about Plato's "Cave" giving a context where we might think about a two-dimensional world. The context is quite different - but both are getting at our ability to think about the limits of our perception providing notable limits to knowledge.

Needs to have something about Google SketchUp and its ability to do cross sections. Mention Google 3D Warehouse and let them know that they can take cross sections of any of the models that are available in here. This might also be good for near the conclusion.


X3
X6
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X1
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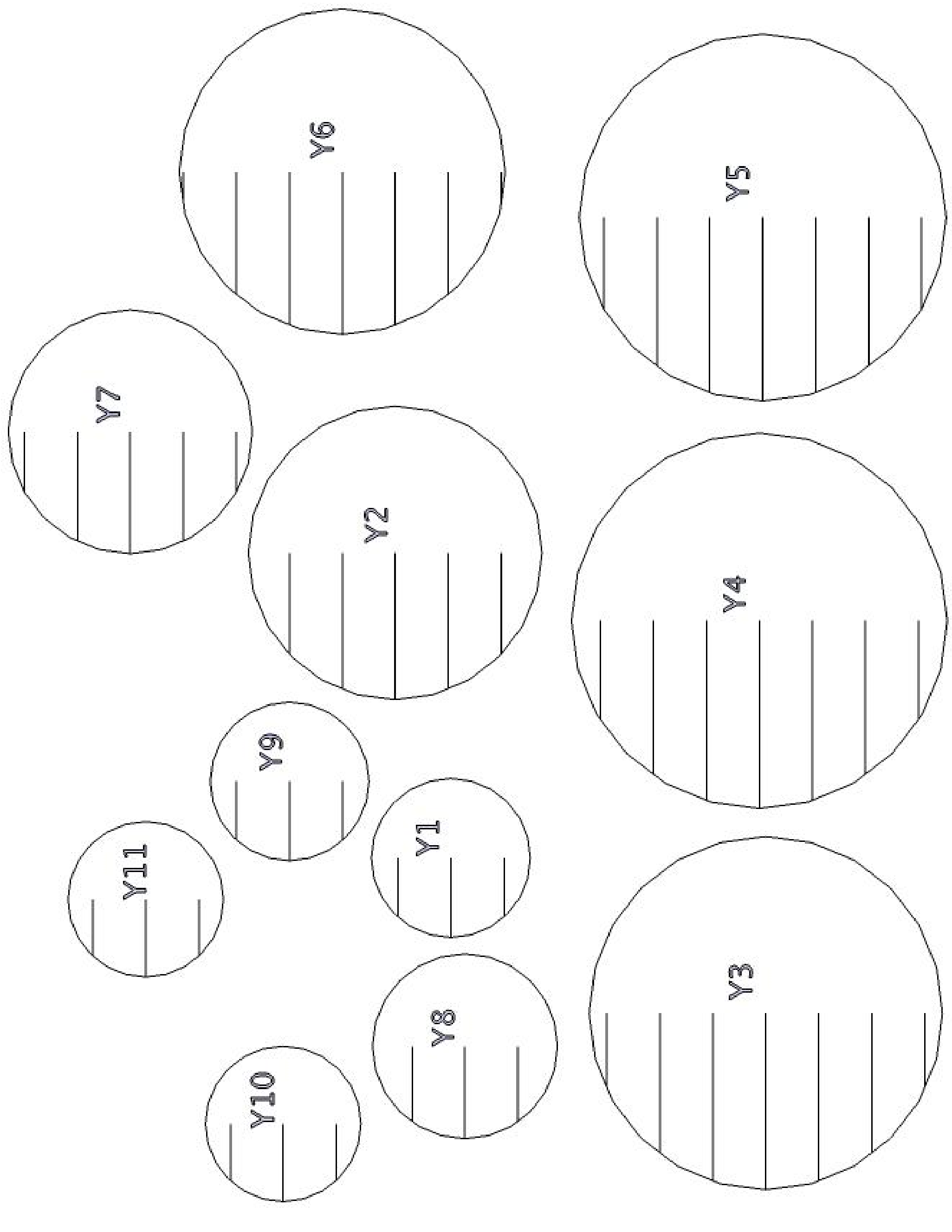
X5
X2


Y6
Y5
Y7
Y2
Y4
Y9
Y1
Y11
Y3
Y8
Y10

CHAPTER 3

Dimensional Interplay in Other Fields

In 1953 I realized that the straight line leads to the downfall of mankind. But the straight line has become an absolute tyranny. The straight line is something cowardly drawn with a rule, without thought or feeling; it is the line which does not exist in nature... Any design undertaken with the straight line will be stillborn. Today we are witnessing the triumph of rationalist knowhow and yet, at the same time, we find ourselves confronted with emptiness. An esthetic void, desert of uniformity, criminal sterility, loss of creative power. Even creativity is prefabricated. We have become impotent. We are no longer able to create. That is our real illiteracy.

Friedensreich Regentag Dunkelbunt Hundertwasser (Austrian Artist and Architect; 1928 - 2000)

The whole science of geometry may be said to owe its being to the exorbitant interest which the human mind takes in lines. We cut up space in every direction in order to manufacture them.

William James (American Psychologist; 1842 - 1910)

1. Blueprints, building, and architecture

1.1. Computer Aided Design (CAD). Architects, designers, and engineers routinely use Computer Aided Design (CAD) software to flexibly navigate the dimensional ladder. With this software customers can take a virtual walk through designs of their new house, the architect can explore how the lighting inside the building might change in the course of the day, and a builder can determine the total amount of materials needed.

Google SketchUp is a free CAD tool you can use to create, view, and modify three-dimensional designs yourself. Download the most recent version from `http://sketchup.google.com/` and explore some of its basic tools. This software was designed to allow users to provide a three-dimensional rendering of their house for inclusion in Google Earth. It is sophisticated enough to create realistic models of houses; see Figures 3.1-3.2.

Basic SketchUp operations can lift us from two into three dimensions. After drawing a two-dimensional figure (such as a circle, a rectangle, or a polygon), we can pull the two-dimensional figure into the third dimension: starting with a circle we could create a cylinder, for example.

Similarly, we can also step *down* the dimension ladder: placing a "section plane" into our design will cut it open and reveal a particular cross-section; moving the plane will reveal a radiographers movie. We can produce x-ray views by making surfaces opaque; shadows provide realistic views. A particular fly-through can be captured as a slide show.

1. SketchUp Introduction: Use the short series of self-paced tutorials included in SketchUp to learn about basic tools (Intro tutorial). Follow the basic steps in creating a simple house (look for three short tutorials called "Start A Drawing," downloaded through the Help menu).
2. SketchUp Building: Open SketchUp and create a closed, planar shape using Rectangle, Circle, or Polygon. Select this shape and then activate the Push/Pull tool. You have created

Figure 3.1. Wire diagram of a house. Source: SketchUp Gallery at `http://picasaweb.google.com/gallery.sketchup`.

a right prism. Every horizontal cross section of this solid is congruent. In other words, if it passed vertically through Flatland, a Flatlander would not see the shape changing at all.

3. SketchUp Shaping: Create another right prism as in Investigation **1**. Choose the Rotate tool and rotate the top surface of the prism. Choose the Orbit tool to closely inspect the object you created. What would a Flatlander observe? Experiment with rotating by different angles.

4. SketchUp Radiography: Create a solid in SketchUp, as in Investigation **1** or using GetModel to get a more sophisticated model such as the Statue of Liberty from the Internet. Activate Section Plane and create a section plane. Select this plane, activate Move, and then move the section plane by clicking and dragging on the plane. As this plane moves you will various cross sections of your SketchUp solid.

Figure 3.2. Town model created by Eighth Grader Andrew, using SketchUp. Source: SketchUp Gallery at `http://picasaweb.google.com/gallery.sketchup`.

2. CAT scans, MRI, PET scans, and other medical imaging.

In 1895, Wilhelm Conrad Röntgen created the first plates of X-ray photographs, taking shadowy images of his wifes hand; see Figure 3.3. This profound medical advance allowed us for the first time

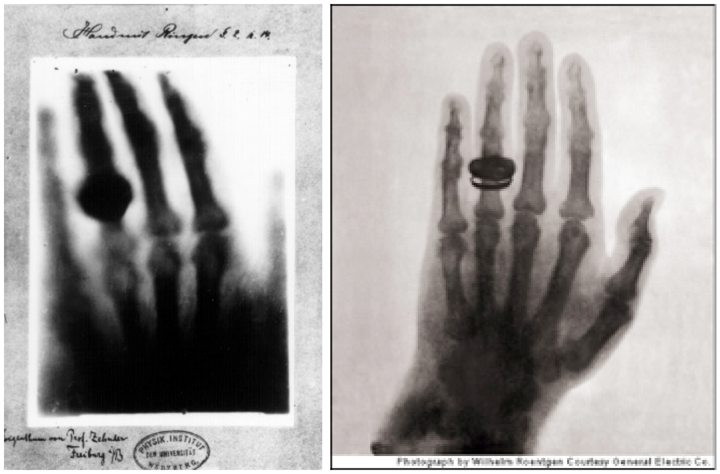


FIGURE 3.3. Photographs by Wilhelm Röntgen: Berta Röntgen's hand with wedding band. Courtesy General Electric Co.

to see internal body structures in a non-invasive manner. Röntgen's discovery earned him the very first Nobel Prize (for Physics in 1901), but also contributed to his death in 1923 from Leukemia due to X-ray exposure. On November 8 2010, Google celebrated the 115-year anniversary of this discovery by showing the logo in Figure 3.4 on their web page.

FIGURE 3.4. Google celebrates the 115-year anniversary of Röntgen's discovery of X-rays.

Important as X-Rays are, they offer only a single planar image of even the most complex anatomical structures. Extending imaging techniques to capture 3-dimensional structures in their entirety via tools like CAT scans, MRI, and PET scans accomplished over the last 35 years or so has revolutionized modern medicine by allowing us to visualize every bodily organ with tremendous resolution as a real 3-dimensional object; see Figure 3.5.

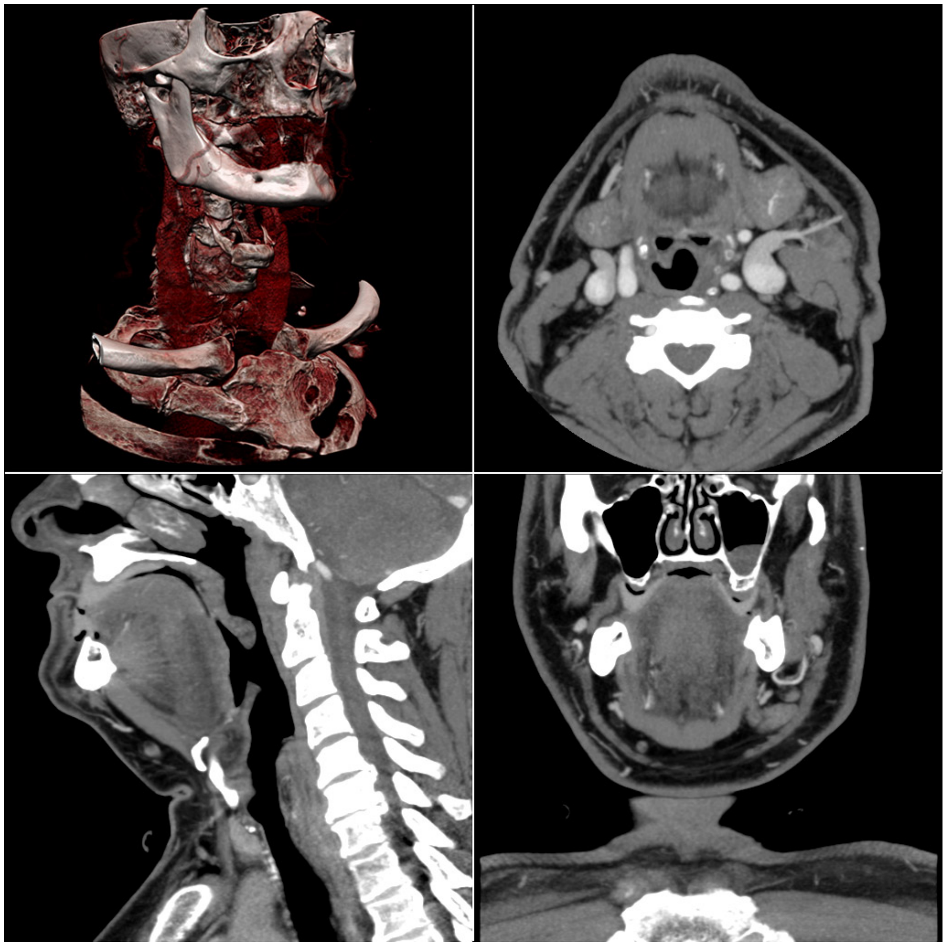


Figure 3.5. Typical screen layout of workstation software used for reviewing multi-detector CT studies. Clockwise from top-left: Volume rendering overview, axial slices, coronal slices, sagittal slices. Source: Wikimedia.

CAT scans–like the other imaging technology–work by little more than a combination of the slicing and projection techniques we have considered so far. Figure 3.6 shows a modern (2006) CT scanner with the cover removed, demonstrating the principle of operation. The X-ray tube (labeled "T") and the detectors (labeled "D") are mounted on a ring shaped gantry. The patient lies in the center of the gantry while the gantry rotates around him as a broad fan-shaped X-ray beam (labeled "X") passes through the body.

Using this circular array of X-Rays and detectors, numerous images from different perspectives along an axial slice (the A in CAT) are relayed to a computer (the C in CAT). Using mathematical algorithms known as tomographic reconstruction (the T in CAT), the computer obtains a high-resolution image of an axial slice. This process is repeated to obtain hundreds of slices and the computer integrates these much like we did in constructing sliceforms to provide a true 3-dimensional map of the solid being analyzed. The solid can be seen using interactive computer graphics or even through stereolithographic models like those that are made for hip replacement surgery. Those who make this possible are Flatland game experts with real life rewards. For accessible further information, see

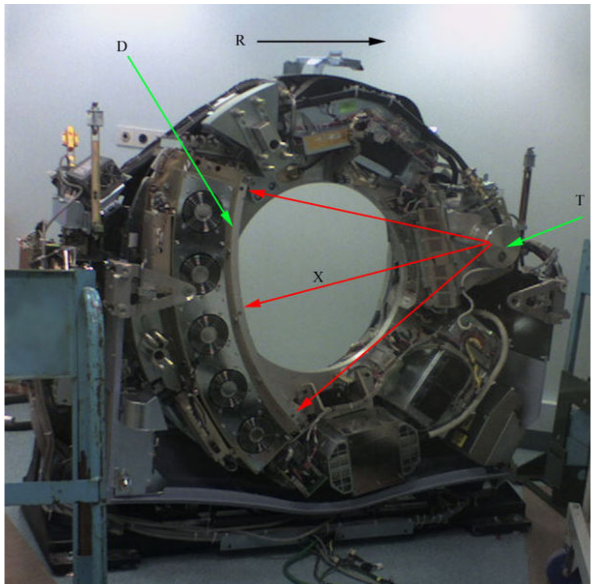


FIGURE 3.6. A modern (2006) CT scanner with the cover removed, demonstrating the principle of operation. Source: Wikimedia.

Sochurek's book "Medicines New Vision" [?] or the Physics 2000 Internet site materials on X-rays and CAT scans [?].

5. Working in small groups, draw axial cross sections of different regions of your bodies. (This is the view one would see as if they were passing through Flatland standing up.) Check your accuracy using one of the many available Visible Human Program viewers available online (e.g. the NPAC/OLDA Visible Human Viewer at `http://www.dhpc.adelaide.edu.au/projects/vishuman2/VisibleHuman.html` or the viewer at the Center for Human Simulation at `http://www.uchsc.edu/sm/chs/browse/browse.htm`.)

6. Working in small groups, play the Flatland game using human body parts as the mystery objects.

7. Working in small groups, play the Flatland game using a variety of different animals.

3. Geography, maps, and topography

3.1. Topographic Maps:

8. Consider the five landscape photographs in Figure 3.7. For each of these, draw a series of Flatland Game clues.

9. What features of the landscape do you see reflected in your clues?

(a) Landscape 1

(b) Landscape 2

(c) Landscape 3

(d) Landscape 4

FIGURE 3.7. Investigation **8**: Draw a series of Flatland clues for these landscapes.

10. Consider the topographic map shown in Figure 3.8 and focus on the contour lines. What relationship do you notice between these contour lines and the clues you drew for the Flatland Game in Investigation **8**? Be specifc.

11. Which of the images in Figure 3.7 do you think corresponds to the landscape shown in the topographic map in Figure 3.8? Explain your reasoning.

12. Find a topographic map of a hilly hiking area, perhaps close to where you live. The US Geological Survey is a good source. Lower resolution versions of some of their maps are available for free on the web via services like `MyTopo.com` or `http://www.digital-topo-maps.com/`. Figure 3.9 was obtained in this manner; it shows the area around Mount Tom, the Whiting Street Reservoir, and the Connecticut River north of Holyoke, MA.

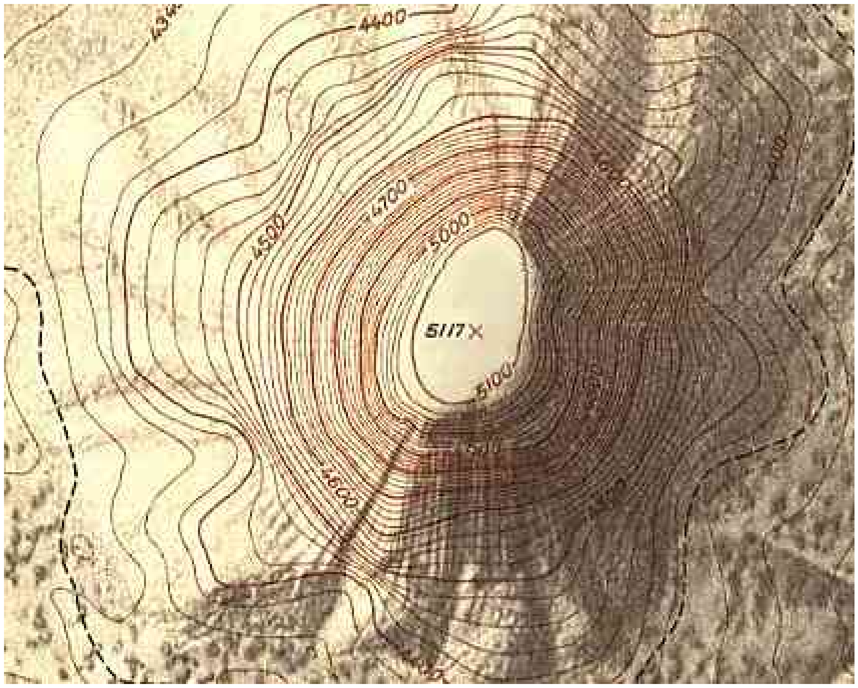


FIGURE 3.8. Source: Compass Dude.

13. In what ways do the contour lines on the map provide information about landscape features? How can you tell where is it flat; where to find gullys or canyons; where to find steep cliffs, etc.?

14. Can you find a location on the map where different contour lines touch or cross? What would that mean in the landscape?

15. Using the map, design hiking trails for people with varying abilities: a mostly flat trail for wheelchair access, a gentle walk, a trail with strenuous climbs, or even a rock-climbing path?

16. Build a model of the area using Play-Doh.

3.2. Maps and Sliceforms: Find a contour map of a mountain with longitude and latitude grid overlay. The map shown in Figure 3.9 shows an example of this.

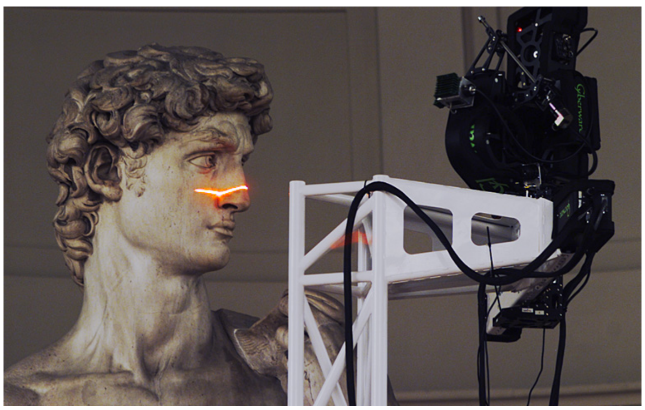
17. Draw a straight line through the top of the mountain at whatever angle you like. Imagine you cut the mountain along that line. Create a drawing of what the surface along the cut would look. You can do this by laying a piece of paper along the line, marking its intersection with each of the contour lines, drawing perpendicular lines of appropriate height above these intersections, and then connecting the tops of the lines. Geographers call this a ***topographical profile***.

18. Make a sliceform model of the mountain whose contour map is give above. You will need to make topographical profiles along each longitude and latitude first. Then you cut slots at the intersection of the longitude and latitude lines, cutting the slots down on the topographical profiles in one direction and up in the other direction, splitting the difference each time. (For more details see Sharp [**?**], pp. 54 - 60.)

FIGURE 3.9. Mount Tom and Connecticut River in Western Massachusetts, north of Holyoke. Source: `MyTopo.com`.

4. 3D Laser Scanning and 3D Printing

We have grown accustomed to scanning and printing photographs and images using paper documents. Imagine if we could actually "scan" entire three-dimensional objects, such as a sculpture, a piece of furniture, or a new hip joint? In fact, the technology already exists, and will likely find wider use as the price of the instruments will come down. As an example, consider the "Digital Michelangelo" project at Stanford University which had as its goal to create a 3D scan of the famous statue "David," created by the artist **Michelangelo** (Italian Renaissance painter, artist, sculptor; 6 March 1475 - 18 February 1564); see Figure 3.10.



(a) Laser scanning Michelangelo's sculpture of "David."

(b) 3D replica of "David" sculpture created from the scanned data.

Figure 3.10. © Stanford University's "Digital Michelangelo" Project. Source: `http://graphics.stanford.edu/projects/mich/`. Notice: The images of Michelangelo's statues that appear on this web page are the property of the Digital Michelangelo Project and the Soprintendenza ai beni artistici e storici per le province di Firenze, Pistoia, e Prato. They may not be copied, downloaded and stored, forwarded, or reproduced in any form, including electronic forms such as email or the web, by any persons, regardless of purpose, without express written permission from the project director Marc Levoy. Any commerical use also requires written permission from the Soprintendenza.

4.1. Three-dimensional laser scanning and stereolithographic building. Researchers from Stanford University have used laser rangefinders to scan the surface of famous sculptures, such as Michelangelos David, and create high resolution digital models; see [**?**]. This is done using a carefully calibrated laser which takes detailed measurements of the location of points in one horizontal slice after another, capturing a radiographers outline at each stage; see Figure 3.10(a). A computer collects all the data to build a digital model of the statue; see Figure **??**. This digital model can be used, for example, to determine the volume and weight of the statue, or to determine the center of gravity for a structural analysis (As we might expect by now, this slicing process can be reversed. Using the digital models, builders can recreate actual three-dimensional replicas; see Figure 3.10(b). Important not only to artists and historians, this process is critical in medical, manufacturing, moviemaking, and video game industries. Many different kinds of precision machining tools can be used to make these replicas.

One example is called ***stereolithography***. This widely used ***rapid prototyping*** technology. Digital models created by CAD programs drive an ultraviolet laser which solidifies liquid plastic as it moves along a two dimensional cross section. Once a layer is complete, a platform holding the model drops a the specified layer thickness, covering the top layer again with liquid plastic for the next layer to be built. Fine layer by fine layer, the machine proceeds – just like our builders – until a complete 3D model is completed.

Need a picture here for doing a laser scan by hand:

with a horizontal laser range finder illuminating one slice,

one person holds a ruler to a certain line, while another person

measures the length/depth. See investigation \ref{inv:measure}
s

Figure 3.11. Investigation **22**: One group member aligns a ruler perpendicular to the axis with its tip touching the object where the laser meets it. Another group member measures the distance from the axis to the solid.

4.2. 3D Laser Scanning: In small groups, you will use a laser level to collect scan data for a modestly complicated but reasonably sized solid object.

19. On a sheet of paper, draw a straight line whose length is the width of the solid and is divided into about a dozen equally spaced intervals.
20. Set the object on the paper so it does not obscure the axis drawn in Investigation **19**.
21. Set a laser level to "horizontal" and position it in front of the solid. The laser now provides the boundary of a cross sectional slice. (If you do not have a laser level handy, you can also use a carpenter's gauge. See Investigation **24** and Figure 3.12(a).)
22. As shown in Figure 3.11 (STILL NEED THIS), have one group member align a ruler perpendicular to the axis with its tip touching the object where the laser meets it. Have another group member measure the distance from the axis to the solid.
23. When all measurements are taken, make a cross section much as you did with the topographical profiles in Investigation **18**.
24. Alternatively, each group should use a carpenters profile gauge (Figure 3.12(a)) to quickly copy the boundary of different cross sections of the solid.
25. Taking horizontal and vertical cross sections at regular intervals, make precise cross sections to build a sliceform such as the human face in Figure 3.12(b).

4.3. 3D Laser Building: While creating sliceforms gives us a rough skeleton of the three-dimensional object, say a mountain, the following investigations aim at creating a **solid** copy of this object, using layering techniques similar to those used in stereolithography.

26. Find an interesting contour map (available in Gazetteers in your library) that has a range of at least 20 contour lines separating the highest and lowest elevation.
27. Prominently mark two points on a copy of the map: the highest elevation and another point a significant distance away, choosing a point of highest local elevation if possible.
28. Make as many copies of the marked map as there are contour lines.

(a) Carpenter's Gauge Tool

(b) Sliceform of a face, by Lydia Lucia, Westfield State University.

Figure 3.12. 3D Laser Scanning and Building.

29. Each class member is responsible for a specific contour line. Cut along this contour to obtain a horizontal cross section of the regions topography.

30. Affix or trace the outline of each cross section to a piece of 1/4" foam-core or double thickness of corrugated cardboard. Then cut out each cross section, carefully labeling it, and poking two holes at the marked locations of the map.

31. Beginning with the lowest elevation cross section, successively stack up the cross sections, using pencils or short pieces of dowel to thread through the holes to keep the cross sections aligned.

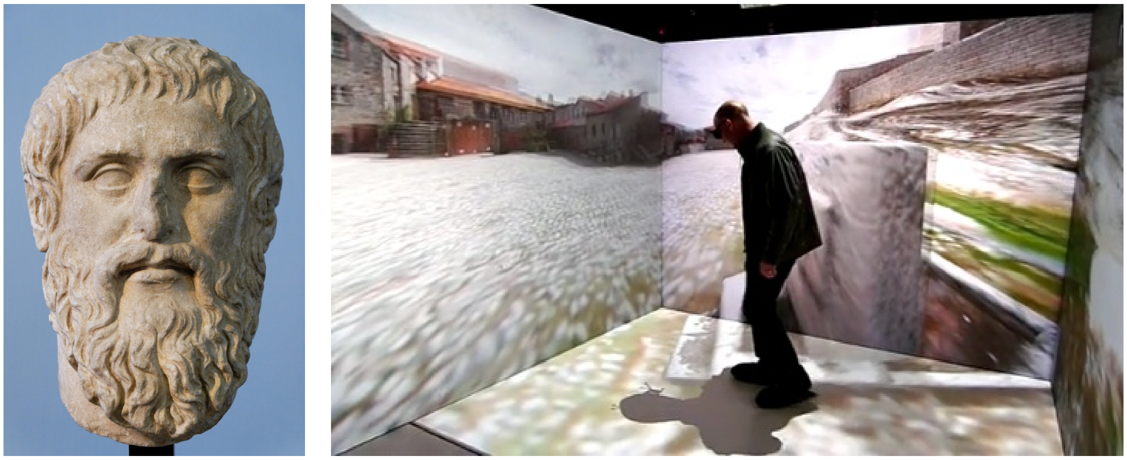
Your completed object is a raised relief map of the region constructed in much the same way stereolithography makes prototypes.

5. CAVES and Virtual Reality Theaters

In his "Allegory of the Cave," **Plato** (Classical Greek Philosopher; 428/427 BC - 348/347 BC) (Figure 3.13(a)) explores the ideas of perception, reality, and illusion by using the analogy of a person with no three-dimensional experiences; their experiences are limited to two-dimensional shadows projected on the back of a cave.

Many of us have experienced the thrill of feeling completely immersed watching a movie in the cave of an IMAX theater. In similar ways, virtual reality games, computer-generated animation, and flight simulators draw us deep into their reality – with sophisticated use of geometry.

Now, what would it be like to freely explore a three-dimensional world with a similarly rich sense of immersion? Imagine meeting Michelangelos David face-to-face, or peeking around in a CAT scan model of your own body? The CAVE, a virtual reality theater, is one step towards realizing this vision. "CAVE," the name selected for the virtual reality theater, is both a recursive acronym (CAVE Automatic Virtual Environment) and a reference to honor Platos "Allegory of the Cave." It consists of a multi-person, room-sized, high-resolution, 3D video and audio environment. Computer-generated graphics are projected in stereo onto three walls and viewed with 3D stereo glasses. A viewer wearing



(a) Copy of a portrait bust showing Plato created by Silanion.

(b) CAVE Automatic Virtual Environment

FIGURE 3.13. Plato and a Modern CAVE.

a position sensor moves freely within its display boundaries. Instantly, the correct perspective and stereo projections of the environment are updated by a powerful computer: the images move with and surround the viewer. To the viewer with 3D stereo glasses the projection screens become seemingly transparent: three-dimensional objects such as tables and chairs appear to be present both inside and outside this projection-room. To a viewer these objects are really there until they try to touch them or walk beyond the boundaries of the projection-room. There are many rips and tears on projections screens where viewers have forgotten to be careful when walking within these invisible boundaries.

32. Search online for videos of people moving inside a CAVE.

5.1. CAVE Game Activity. This game is a variant of the Flatland Game, where the goal is now to recognize an object from the shape of one or more of its shadows. Hide an object behind a blind in such a way that an overhead projector casts its shadow onto a wall for all to see.

33. Break into groups: one group of radiographers and several groups of builders. The radiographers secretly select a solid mystery object.

34. The radiographers project an image of their object and teams of builders try to guess the object.

35. The radiographers then project a different image of their object and the builders guess again.

36. Step 3 continues until

- **a.** a team of builders correctly guesses the identity of the mystery object, in which case they are declared the winner, or,
- **b.** after a fixed number of projections has been surpassed the radiographers are declared the winners and the object is revealed.

37. The game can be made easier by showing the projections as the object is turned (using a lazy susan) or harder by simply showing the final projections.

6. Longitude, Latitude, and other types of Coordinates (section 1)

7. Blueprints, building, and architecture

7.1. Blueprints. Like contour maps, traditional blue-prints, represent an amalgamation of information based on slices of the object along major directions. A builder synthesizes the information from such a collection of two-dimensional prints to create a fully-featured three-dimensional object, be it a porch, a roof, or an entire house. These are the "builders" of the Flatland game in real life.

Consider the set of blue-prints in Figure **??**.

38. What is this? What information can you read off the blueprint?

39. Build a model using manila envelopes and tape. It is easiest if your model is the same size as the blue-print.

CHAPTER 4

Climbing a Ladder to the Higher Dimensions

> This is because you think of space only in three dimensions... We travel in the fifth dimension. This is something you can understand, Meg. Don't be afraid to try. Was your mother able to explain a tesseract to you?[1]
>
> **Madeleine L'Engle** (American Writer; 1918 - 2007)

.

> Today the major reason for our interest in Flatland is that for the first time we can achieve some of the dreams of our ancestors a century ago and obtain direct visual experience of phenomena in a dimension higher than our own.[2]
>
> **Thomas Banchoff** (American Mathematician; -)

1. Perspectives on Art

All artists who work in two dimensions - those who paint on canvas, those who draw on paper, those who do graphic design on computers - are faced with the challenge of artistically representing the three-dimsional world within the confines of two dimensions.

> Every child is an artist. The problem is how to remain an artist after he(she) grows up.
>
> **Pablo Picasso** (French Artist; -)

1. Think back to your childhood years prior to middle school. What types of art did you create? Please think broadly - a Lego building is a sculpture as much as Micheangelo's David.

2. The art that you created - was it mostly two-dimensional or mostly three-dimensional? Determine some potential reasons why the balance of your art was this way.

3. Find - in your memory, with the help of family, or by finding actual examples - actual examples of your early artwork. Are there ways - as a child - that you tried to represent the three dimensional world in two-dimensions? Explain how.

4. At some point - generally in the middle grades - most of us are taught how to draw a cube. Carefully draw a cube. Explain how you have drawn it and how it is that your figure represents a cube.

5. How has your involvement with art - again, broadly defined - changed since you started middle school? Do you do more or less art? Is the art of different types than previously? What has happened to the balance between two-dimensional and three-dimensional art that you use? Describe possible reasons for this change and whether Picasso's quote has any meaning for you.

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[1] Spoken by the fictional character Mrs. Whatsit from A Wrinkle in Time.

[2] From the Introduction to the 1991 Princeton University Press edition of Flatland.

2. Perspective Drawing

As early as 25 B.C. **Vitruvius** (; -) wrote "perspective is the method of sketching a front with sides withdrawing into the background.[3]" Evidence of perspective drawings were found in murals and frescoes in the Roman city of Pompei - preserved only because the entire city was buried by the volcanic eruption of Mount Vesuvius in 79 A.D. There is also evidence of perspective drawing in ninth century Chinese art.[4]

The more modern and permanent (re)discovery of perspective drawing is widely attributed to **Fillippo Brunelleschi** (Italian architect and artist; 1377 - 1446) at the height of the Rennaissance; his first drawing said to be of the cathedral in Piazza del Duomo in Florence, Italy. In a time when most of us have high resolution, digital cameras built into our cell phones, creating a life-like image may not seem to be important. But the development of perspective drawing fundamentally changed the world of art and, as a result, the way we view the world.[5]

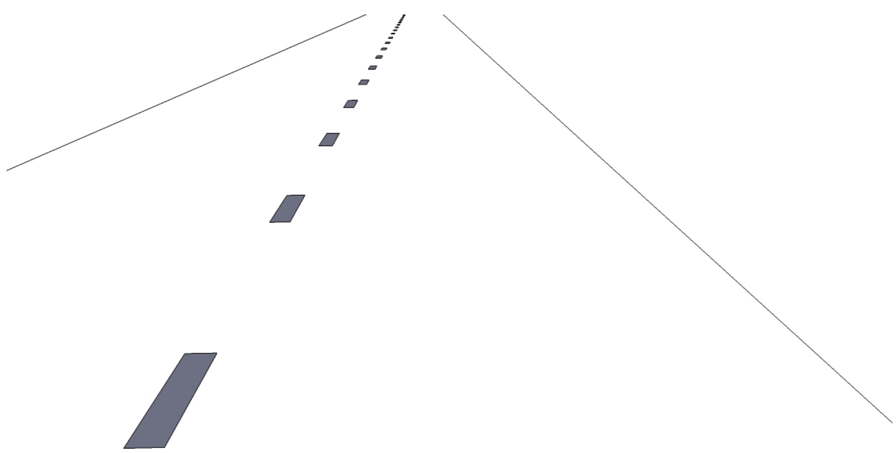

FIGURE 4.1. A portion of a one point perspective drawing of a road.

2.1. One-Point Perspective.

6. Figure 4.1 shows a portion of a *one-point perspective* drawing of a rudimentary road. Explain precisely how you can find the *vanishing point*, also known as the *point at infinity*, used to create this drawing. Explain how you can locate the *horizon line* for this drawing.

7. Draw your own one-point perspective drawing of railroad tracks vanishing into the horizon. Line the railroad tracks with a row of telephone polls on one side and a path/road on the other. Breifly explain how you created your drawing and any difficulties you faced.

8. Explain - geometrically - how you knew how to orient the railroad ties, telephone polls, and other important features in your drawing.

9. On the left of Figure 4.2 is a square and a point at infinity. Draw the one-point perspective drawing of the solid object whose front face is this square and which recedes indefinitely

[3]Quoted in Squaring the Circle: Geometry in Art and Architecture by Paul A. Calter, p. 373.

[4]See e.g. p. 3 of Perspective Drawing: Freehand and Mechanical by Joseph William Hull.

[5]For much more on this fascinating topic, at a level that is similar to the book you are currently working through, see the wonderful book Mathematical Perspective and Fractal Drawing by **Annalisa Crannell** (American Mathematician; -) and **Marc Franz** (American Mathematician; -).

towards the point at infinity. What does this object resemble? What does it serve as a model for? Can you name it?

10. Repeat Investigation **9** for the figure in the center of Figure 4.2.

11. Repeat Investigation **9** for the figure on the right of Figure 4.2.

We would now like to draw cubes in one-point perspective.

12. Using the larger version of Figure 4.2 in the Appendix, draw the one-point perspective drawing of a cube whose front face is this square. Do you know exactly how long to make the side edges? Explain.

13. Repeat Investigation **12** with the square and point at infinity for the figure in the center of the larger version of Figure 4.2 in the Appendix.

14. Repeat Investigation **12** with the square and point at infinity for the figure in the center of the larger version of Figure 4.2 in the Appendix. The figure you have drawn is called the **foreshortened cube**.

15. Carefully compare all of these perspective drawings to the cube you drew in Investigation **4**. Are any of these drawings the same as your cube? If not, how do they differ? Can you move the point at infinity to a location so the one-point perspective drawing will exactly correspond to your cube? Either do so or explain why it cannot be done.

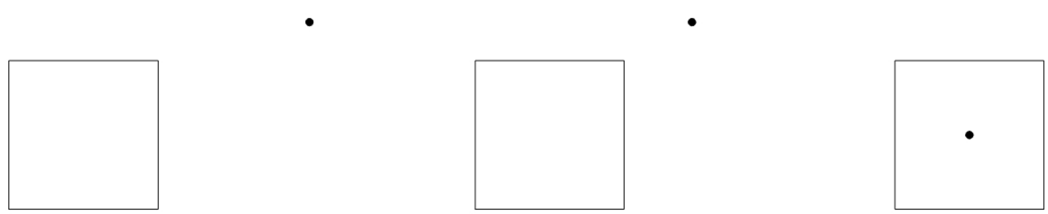

FIGURE 4.2. Set-ups for creating different one point perspective drawings with square front faces.

Figure 4.3 and Figure **??** show images by **Albrecht Dürer** (; -). A critical Renaissance artist, Dürer was the first to write systematically about the use of perspective in art - his instructions first appearing in The Painter's Manual.

16. The images in Figure 4.3 and Figure **??** show *perspective machines* that could be used to make realistic perspective drawings. Study these drawings and describe how these machines seem to work.

2.2. Two-Point Perspective. Figure 4.5 shows students using tape and window panes to create perspective drawings, like that in Figure 4.6, of nearby buildings. For more information on this is a wonderful activity please see Mathematical Perspective and Fractal Drawing.

17. There is a vanishing point somewhere to the left of the "drawing" in Figure 4.6. Explain how this vanishing point can be located.

18. None of the faces of Courtney Hall, the building pictured, are parallel to the plane of the window pane, there must be a vanishing point to the right as well. Explain why, in this case, it will be somewhat harder to locate.

The Flatiron Building in New York City, pictured in Figure 4.7, is remarkable building. As it is pictured here you can clearly see the two streets that frame it, Fifth Avenue and Broadway, receeding away from it. Looking at a building edge on, with two street receeding away from it in perpendicular directions, is a typical topic for *two-point perspective* drawings.

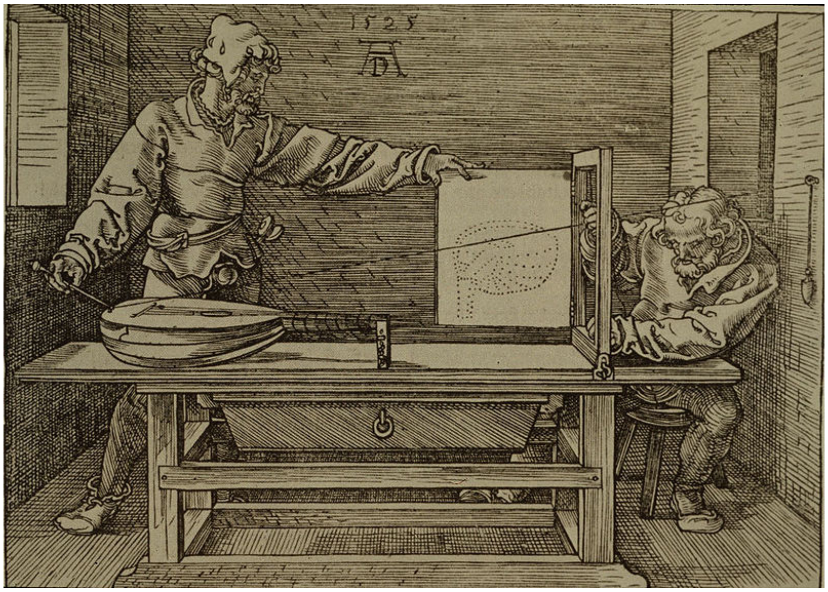

Figure 4.3. Dürer's "Man Drawing a Lute" from 1525.

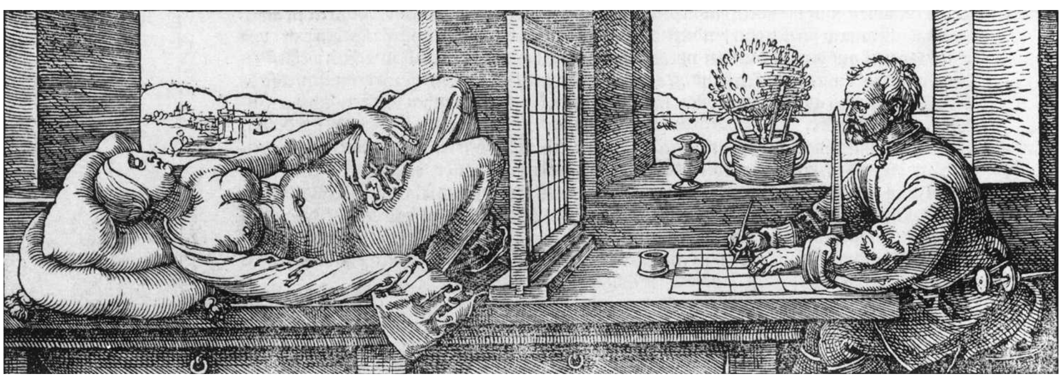

Figure 4.4. Dürer's "Draughtsman Making a Perspective of a Woman" from 1525.

19. Draw a building edge on, with two streets receeding away from it, in two-point perspective. Carefully explain how you have drawn key elements: curbs, windows, doors, and rooflines.

20. Use the set-up in Figure 4.8 to draw a cube in two-point perspective. (A larger version of this image is included in the Appendix.) Do you know how long to make the edges? Do you know where the rear, vertical edge should (approximately) be? If you need to, redraw your drawing until it really looks like a cube.

21. Now compare your cube drawn in two-point perspective to those that you draw in one-point perspective and the one that you drew in Investigation **4**. Are any of these identical, up to their physical scale? Explain in detail how you know certain drawings are identical or what exactly it is that distinguishes them.

JF Note - Location of perspective drawings to look correct.

Figure 4.5. Creating a two-point perspective model.

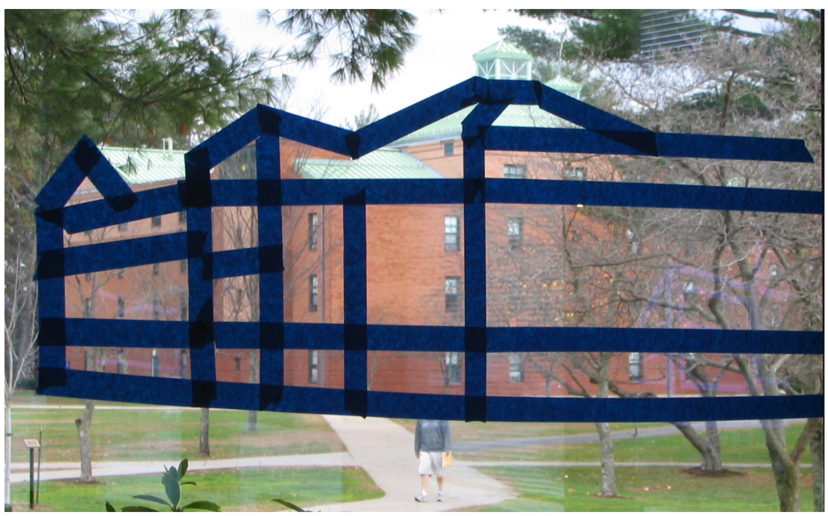

Figure 4.6. A finished two-point perspective taping.

3. Projection

Since I found that one could make a case shadow from a three-dimensional thing, any object whatsoever – just as the projecting of the sun on the earth makes two dimensions – I thought that by simple intellectual analogy, the fourth dimension could project an object of three dimensions, or, to put it another way, any three-dimensional object, which we see dispassionately, is a projection of something four-dimensional, something we are not familiar with.[6]

Marcel Duchamp (French Artist; 1887 - 1968)

[6]Quoted in The Fourth Dimension and non-Euclidean Geometry in Modern Art by Linda Dalrymple Henderson.

FIGURE 4.7. The Flatiron Building.

Below we will consider Duchamp's challenge more directly - the fourth dimension. But first, let us think about shadows. In the previous chapter you considered the difference between slicing and projection; CAT scans and XRays; life in Flatland versus life in Plato's cave. (JF Notes - Did Volker do this? If not, do a bit of it here. Make the distinction.)

22. Build two ***wire frame cubes***, cubes with solid vertices and edges but empty faces. The edges of one should be 8 - 16 inches. The edges for the other should be less than three inches. Suggestions for materials include: Zome; stiff wire; marshmallows/gumdrops/Play Doh and skewers/toothpicks; straws with paperclips to join them at vertices.

We would like to consider the different types of shadows that are cast by this cube.

3.1. Parallel Projections. The first shadows you should make are those called ***parallel projections***. These are made by a light source that is so far away that the rays are essentially parallel. The sun works very nicely for this. If you need to use artificial light, just about any type of light will do. A flashlight or bare lightbuld works fine as long as the distance between the light source and the cube is quite significantly greater than the distance from the cube to the surface it is projected on.

FIGURE 4.8. Set-up for creating a cube in two-point perspective.

The relative location of the light source and cube is also important. To begin, we would like to insure that the line from the light source to the cube is perpendicular to the surface the shadow is projected onto. If you are using a lightbulb or flashlight as a source, this is easy to arrange. As the sun is generally not directly overhead, if you use the sun you will have to bring a large piece of cardboard with you so that it can be tilted to be perpendicular to the rays of the sun.

23. Do you think parallel projections of the cube will differ from the cube drawing and perspective drawings that you created previously? If so, how. If not, why not?

24. Now create a parallel projection of the cube. You should be as exact as possible in duplicating the projection. If possible, trace over the actual shadow as it is projected onto a large sheet of paper, cardboard, chalkboard, or whiteboard. Then copy a scale version into your notebook.

25. Now rotate the cube to see different parallel projections. When you find a projection you particularly like, stop rotating and carefully draw it. Do this five times so you have completed Investigation **33** a total of six times.

26. Carefully compare all of these parallel projections to the cube you drew in Investigation **4**. Are any of these drawings the same as your cube? If not, how do they differ? Can you rotate the cube so the parallel projection will correspond to your cube? Either do so or explain why it cannot be done.

The shadows created will be quite different if the line from the light source through the cube is *not* perpendicular to the surface the shadow is projected onto. Projections of this type are called ***oblique parallel projections***.

27. Do you think oblique parallel projections of the cube will differ from the cube drawing, perspective drawings, and parallel projections that you have created previously? If so, how. If not, why not?

28. Now create an oblique parallel projection of the cube. You should be as exact as possible in duplicating the projection. If possible, trace over the actual shadow as it is projected onto a large sheet of paper, cardboard, chalkboard, or whiteboard. Then copy a scale version into your notebook.

29. As above, rotate the cube to see different oblique parallel projections. And perhaps change the angle at which the light source hits the surface of projection. When you find a projection you particularly like, stop rotating and carefully draw it. Do this five times so you have completed Investigation **33** a total of six times.

30. Carefully compare all of these oblique parallel projections to the cube you drew in Investigation **4**. Are any of these drawings the same as your cube? If not, how do they differ? Can you rotate the cube so the parallel projection will correspond to your cube? Either do so or explain why it cannot be done.

3.2. Point Projections. Now we are going to change the type of light source. Now you need a ***point light source*** where light is coming from (essentially) a single point in space with no focussing, dispersion, etc. Regular lightbulbs are designed to cast lots of ambient light; they do not work here. Most flashlights are designed with reflective surfaces that help direct a beam that is wide enough to be useful without losing its power to illuminate. These don't work here. What is best is a small, incandescent flashlight bulbs. Sometimes the lens and reflective surface of a flashlight can be removed to leave the working bare bulb. If not, such bulbs can often be found in school Science Departments or are inexpensive to purchase from Radio Shack.

Point projections are created using point light sources which are quite significantly closer to the object being projected than the object is from the surface it is being projected onto. As above, it is generally assumed that the line from the source to the object is perpendicular to the surface. If this is *not* the case then the projection is called an ***oblique point projection***.

31. Explain why the projections of your cube will change as you rotate the cube.
32. Carefully draw what you think of few of the projections will look like.
33. Now actually create a number of different projections of the cube as you rotate. When you find a projection you particularly like, carefully draw it - perhaps tracing over the actual shadow on your piece of paper.
34. Repeat Investigation **33** five more times to find five more interesting projections of the cube.
35. Carefully compare all of these projections to the cube you drew in Investigation **4**. Are any of these drawings the same as your cube? If not, how do they differ? Can you rotate the cube so the projection will correspond to your cube? Either do so or explain why it cannot be done.
36. Create three different parallel projections where the cube is oriented in the same direction relative to the sun as it was to the point-source of line in three of your projections above. How do the parallel projections differ from the projections?

4. The Necker Cube

As you have seen, there are many different ways to draw a cube. One of the most common is created by drawing a square, drawing another square partially superimposed over the first, and then connecting corresponding corners, as shown in Figure 4.9. Perhaps this is the type of cube you drew in Investigation **4**.

The object that many recognize as a drawing of a cube in Figure 4.9 is called a **Necker cube** after **Louis Albert Necker** (Swiss Geologist; 1786 - 1861). Necker provided the first known mention of this object, in a letter to **David Brewster** (Scottish Physicist, Writer, and Inventor; 1781 - 1868) - inventor of the kaleidoscope - in a letter in 1832.

(??Notes - Links to psychology here. To perception. Blind dude reference. Because of these things, the Necker cube also plays a role in quite a bit of literature. It plays a starring role in the wonderful science fiction Factoring Humanity - a wonderful tour de force of mathematics, ethics, and Jungian-like psychology.)

In our experience when asked to draw a cube, students universally draw a Necker cube. Each of the authors did even after through they earned Ph.D.s in mathematics. If you search around a bit you will find that almost all mentions of the Necker cube that you find will describe it as a projection or perspective drawing of a cube. Analogously, the most common pictures of four-dimensional cubes in "projection" or "perspective" are Necker-like images.

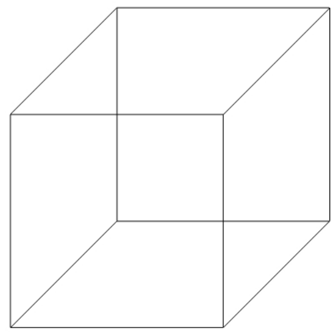

Figure 4.9. The Necker cube.

Mathematically this is *incorrect*. The Necker cube is not a perspective drawing. It is not a projection. It is not a parallel projection. It does not correspond in any way to standard mathematical or artistic ways of creating two dimensional objects that represent three dimensional objects. It is a forgery. It is simply a optical illusion that has been widely accepted as a representation of a cube.

5. 3D Drawing

Above you considered a wealth of different ways to draw cubes. Each had its own benefits and drawbacks - as well as its many applications. We consider one last one here that is widely used in technical drawings, in engineering drawings, in computer graphics, and to help children as a first experience in drawing three-dimensional representations. It is called ***isometric drawing*** and is a special type of parallel projection. The main tools in creating isometric drawings are ***isometric dot paper***, shown in Figure **??**, and shading.

The word isometric is from the Greek roots iso, for same, and metron, for measure. In isometric drawings all lengths along the three coordinate axes are preserved. For a cube this means that the edge lengths are all preserved.

37. Draw a cube - whose edge lengths are equal to the length between adjacent dots - on isometric dot paper. Compare your cube to those of others. (Extra copies are included in the appendix.)

38. Experiment with shading some of the faces of your cube to see if you can make it appear more "cubish."

39. Figure 4.11 shows an absurd attempt to draw a cube using isometric dot paper. What is with using isometric dot paper to make this representation of a cube?[7]

40. Draw another cube on your dot paper. Now draw another cube adjacent to the first.

41. Continue by drawing a whole row of adjacent cubes. Is the isometric dot paper helpful?

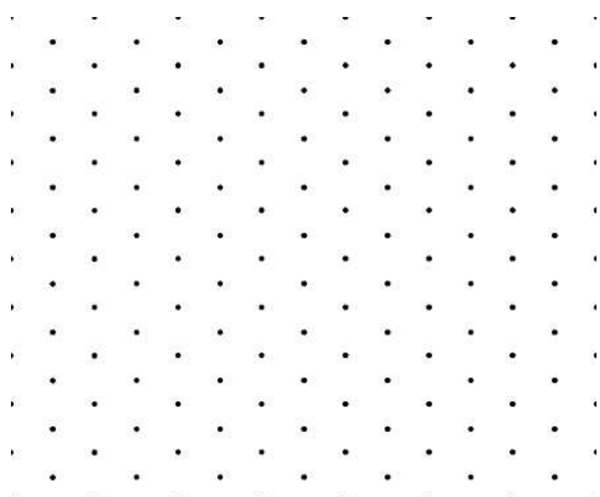

Figure 4.10. Isometric dot paper.

Figure 4.12 shows a block structure rendered isometrically. Figure **??** shows another such block structure as well as its top, front, and right views.

42. Figure 4.14 shows top, front and right views of a *block building* that has been created from cubes. In other words, you are looking at a blueprint of this building. Using actual blocks if you wish, determine how this building was constructed. Draw the result on isometric dot paper.

43. Compare your result with others. Is it the same?

[7]It is quite sad that this was from an "educational" wwwebsite.

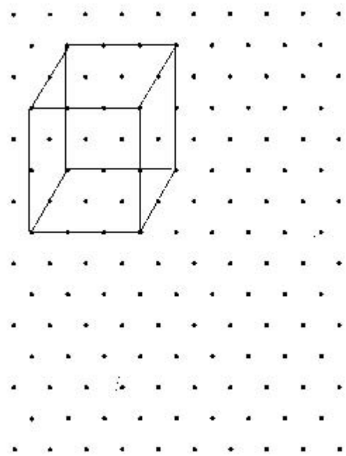

FIGURE 4.11. A failing attempt at drawing a cube using isometric dot paper.

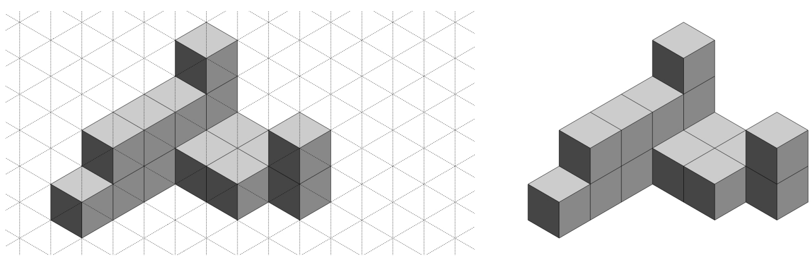

FIGURE 4.12. A block structure rendered on isomoetric graph paper; with and without grid lines.

44. Figure 4.15 shows top, front and right views of a *block building* that has been created from cubes. In other words, you are looking at a blueprint of this building. Using actual blocks if you wish, determine how this building was constructed. Draw the result on isometric dot paper.
45. Compare your result with others. Is it the same?

46. Figure 4.16 shows top, front and right views of a different *block building*. Determine how this building was constructed. Draw the result on isometric dot paper as if your view was from the right, front corner.
47. Compare your result with others. Is it the same? Does this make sense?

48. Figure 4.16 shows top, front and right views of a third building *block building*. Determine how this building was constructed.
49. Draw the result on isometric dot paper as if your viewpoint was from the left, front corner.
50. Compare your result with others. Is it the same? Does this make sense?
51. Now draw the result on isometric dot paper as if your viewpoint was from the right front corner. Does this make sense?
52. Suppose you showed your projection drawing in Investigation **51** to somebody who had not determined how to build this object first. What would they think?

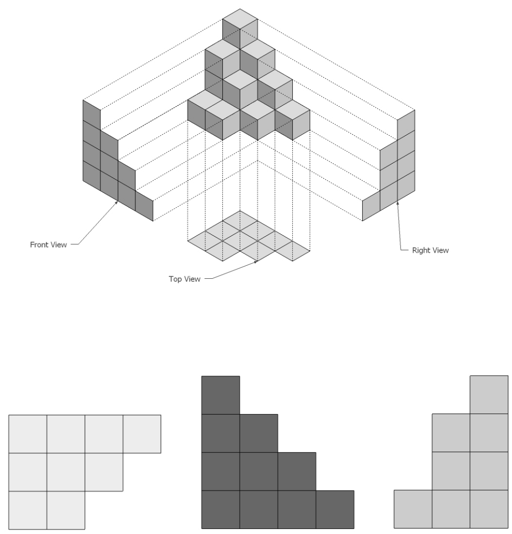

Figure 4.13. A block structure rendered isometrically (left) with top, front and right views (right). Note orientation of the views; this will be used consistently.

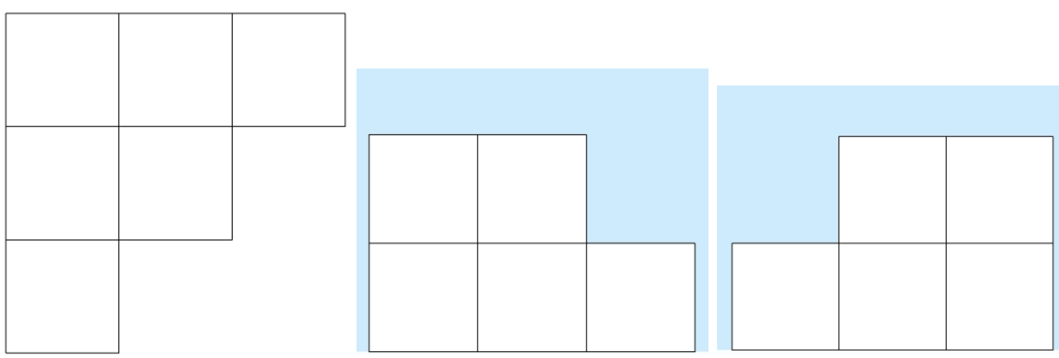

Figure 4.14. Top, front, and right views of a block building.

53. Figure 4.17 shows a famous engraving by **M.C. Escher** (?? artist; -). Does the example in Investigation **51** help you understand how Escher made this engraving?

54. If drawing with isometric dot paper only creates optical illusions, how can one actually make "Ascending and Descending" out of Legos?

55. Find some other optical illusions that deal with perspective and projection.[8]

Two wonderful pieces of art by **Shigeo Fukuda** (; 1932 -) are shown in Figure 4.18.

[8] Masters of Deception: Escher, Dali & Artists of the Optical Illusion by **Al Steckel** (; -) is a wonderful book of optical illusions.

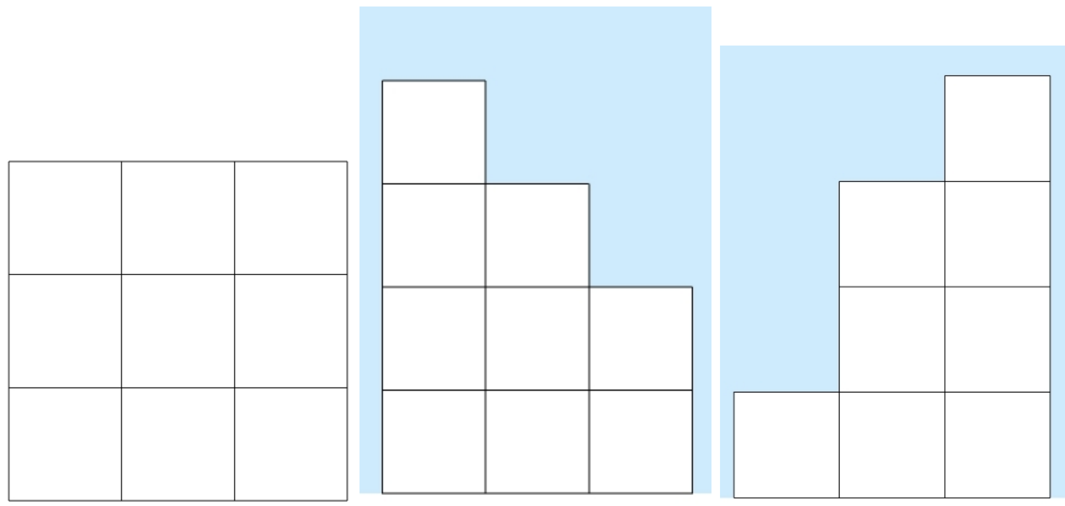

Figure 4.15. Top, front, and right views of a block building.

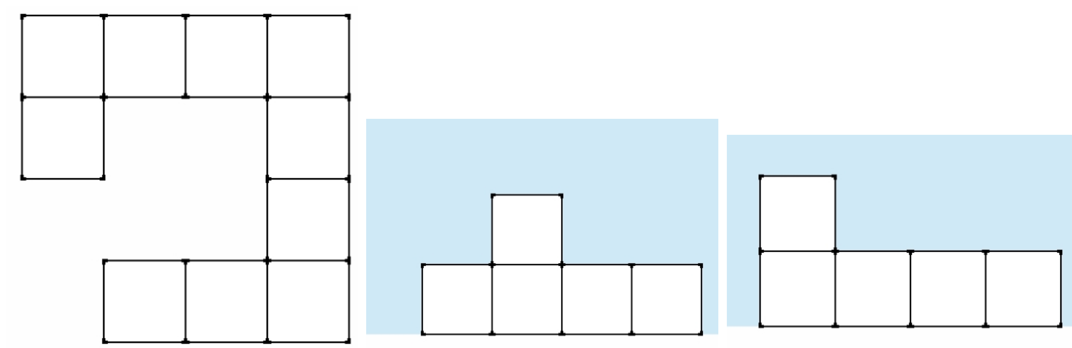

Figure 4.16. Top, front, and right views of a different block building.

You might try your hand at some optical illusion artwork. We also highly suggest the game ***Rumis*** (aka 3D Blockus) which is a mutliplayer game where players must place three-dimensional Tetris-like pieces in arrangements that maximize the number of pieces that they can use.

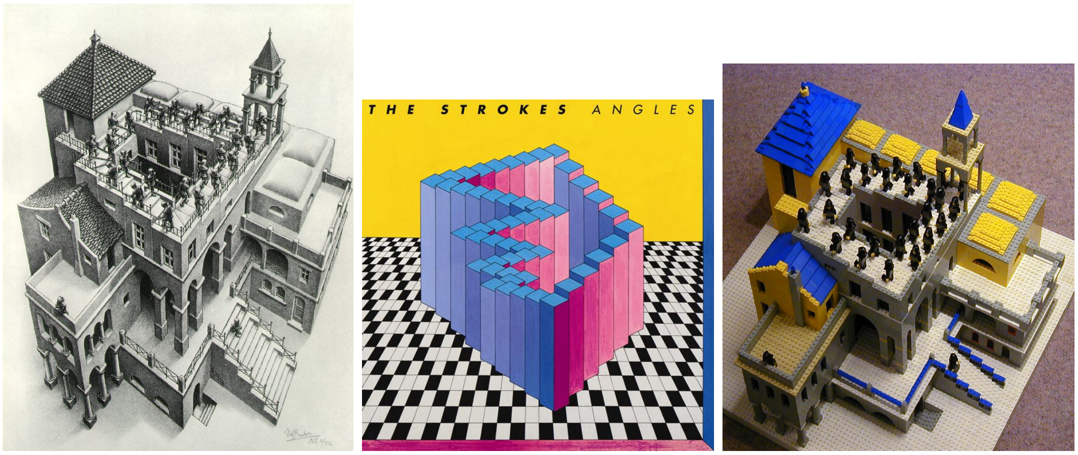

FIGURE 4.17. M.C. Escher's "Ascending and Descending," The Strokes album cover artwork from "Angles" and "Ascending and Descending in Lego."

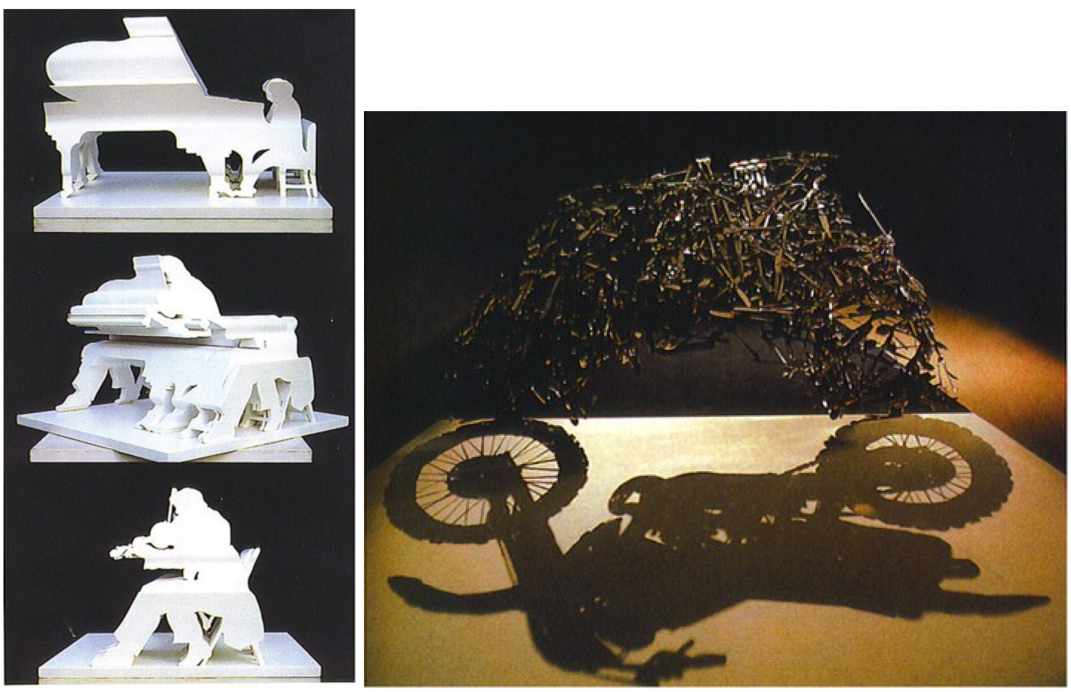

FIGURE 4.18. Encore and Lunch with a Helmut On by Shigeo Fukuda. The former is from wood, the latter from 848 steel spoons, knives and forks!

6. The Dimensional Ladder

The figure below is commonly referred to as the dimensional ladder by mathematicians. You have learned how to climb up and down this dimensional ladder by using slices and building. Of course, these things are complicated in the 2D 3D dimensional cycle. However, you can move up the ladder as far as you want in the dimensions if you restrict yourself to the hypercube family. (Note: Nice connection to the discussion between A Cube and his hexagonal grandson in section 15 (pp. 65-6) and when A Square tries to point out the next analog to A Sphere's disgust near the end of section 19 (pp. 88-9).)

We shall specify more precisely what we mean when we use the term *dimension* below. But for now, the intuitive notion that a line is one-dimensional, a plane is two-dimensional, and the world of space with surrounds us is (apparently) three-dimensional. Hopefully a bit of thought will make it natural to call a point 0-dimensional.

We shall now see one way to navigate the dimensional ladder.

7. The Fourth Dimension in Art

> M. Poincet [sic] read Henri Poincaré in the text. M. Princet has studied at length non-Euclidean geometry and the theorems of Riemann, of which Gleizes and Metzinger speak rather carelessly. Now then, M. Princet one day met M. Max Jacob and confided him one or two of his discoveries relating to the fourth dimension. M. Jacob informed the ingenious M. Picasso of it, and M. Picasso saw there a possibility of new ornamental schemes. M. Picasso explained his intentions to M. Apollinaire, who hastened to write them up in formularies and codify them. The thing spread and propagated... Cubism, the child of M. Princet, was born.[9]
>
> **Loius Vauxcelles** (; -)

.

Vauxcelles coined the term *cubism* - one of the major movements in the history of art. Figure 4.19 shows one of the more iconic cubist paintings, "Nude Descending Staircase" by **Marcel Duchamp** (French Artist; 1887 - 1968). As described in Discovering the Art of Mathematics - Patterns this painting was inspired not just by the cubist movement, but also the ***chronophotography*** begun by **Étienne-Jules Marey** (French scientist and photographer; 1830 - 1904) and **Eadweard J. Muybridge** (English photographer; 1830 - 1904) in the late 1800's.

Marey and Muybridge used trip wires to trigger multiple cameras to take a sequence of pictures one after another. Put together next to each other, you then saw a sequence of pictures that showed stages in the motion. So we saw, for the first time, a horse running "frame by frame" as we might now call it. These were precursors to moving pictures. One particularly well-known chronophotograph is of a nude woman descending a staircase.

Famous *Life Magazine* photographer **Gjon Mili** (Albanian photographer; 1904 - 1984) was one of the first artists to prominently explore the use of *stroboscopic light* to show motion in still photographs. Inspired in turn by Duchamp, Mili created his own "Nude Descending Staircase," shown in Figure 4.20.

Mili's most well-known photograph is of Picasso and is particularly relevant here as we talk about the fourth and higher dimensions. Mili told Picasso, "I would like you to draw in space while I photograph you." Picasso replied, "That would ammuse me.[10]" The result is the photograph in Figure 4.21.

8. The Fourth Dimension in Physics

Coming...

9. Drawing Shadows of the Higher Dimensions

> The point is the result of the initial collision of the tool with the material plane. Paper, wood, canvas, stucco, metal - may serve as this basic plane. The tool may be pencil, burin, brush, pen, etching-point, etc. The basic plane is impregnated by this first collision... The geometric line is an invisible thing. It is the track made by the moving point; that is, its product. It is created by movement - specifically through the destruction of the intense self-contained repose of the point. Here, the leap out of the static into the dynamic occurs.[11]
>
> **Wassily Kandinsky** (Russian Artist; 1866 - 1944)

56. With a red pen draw a point. Imagine this point moving directly to the right one unit. Draw another red point which marks where the point ends its movement. Then connect these two points by a blue or black line segment which represents the path of the moving point.

[9] Quoted in The Fourth Dimension and non-Euclidean Geometry in Modern Art by **Linda Dalrymple Henderson** (; -)

[10] Quoted in the 60^{th} anniversary issue of *Life*, October, 1996, p. 64.

[11] From Point and Line to Plane, pp. 28, 57

Figure 4.19. Nude Descending Staircase by Marcel Duchamp.

57. How do the dimensions of the red components compare to the dimension of the entire object created in Investigation **56**?

58. Draw a red line segment of unit length. Imagine this line segment moving one unit perpendicular to its length. Draw another red line segment which marks where the line ends its movement. Then connect corresponding endpoints of the two red line segments by blue or black lines. You should have a square whose parallel sides are colored in matching colors.

59. How do the dimensions of the red components compare with dimensions of the entire object created in Investigation **58**?

FIGURE 4.20. Nude Descending Staircase by Gjon Mili.

60. Mimic Investigation **56** and Investigation **58** by drawing the two square faces in red and connecting corresponding edges in blue or black to form a Necker cube.
61. How do the dimensions of the red components compare with the dimensions of the entire object created in Investigation **60**?
62. Use Google SketchUp to illustrate the process in Investigation **60** dynamically. Namely, use the Rectangle Tool to draw a square. Then use the Push/Pull Tool to transform it into a cube - *extruding* your square into a three-dimensional cube. Describe relationships between this computer generated approach, your hands-on approach above, and the process described above by the artist Kandinsky?
63. Now extend this drawing process to the *fourth dimension.* That is, draw a cube in red and then another also in red. Now connect corresponding vertices of the cubes in blue or black.

The object you have just created is a projection of a *rectangular 4-parallelopiped*; i.e. a 4-dimensional box. What type of projection you have drawn - as you may have guessed - depends

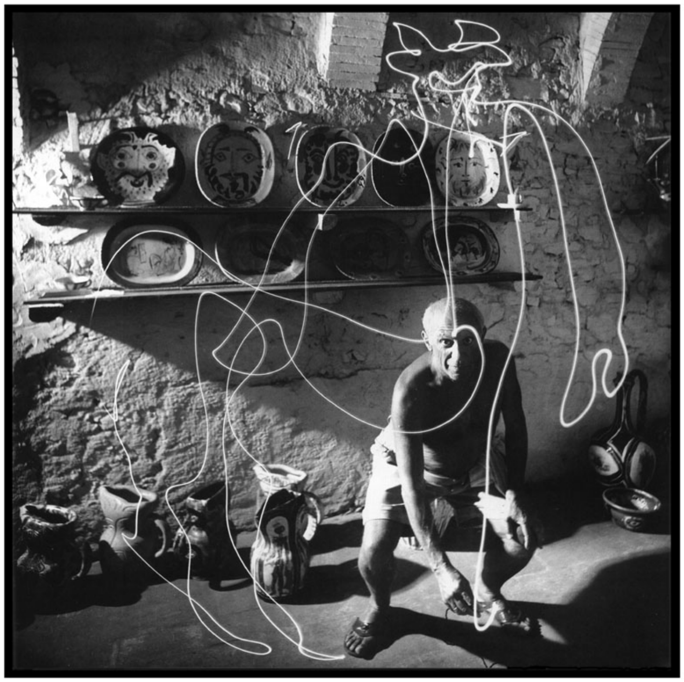

FIGURE 4.21. Pablo Picasso "painting" in time; Photograph by Gjon Mili, January 30, 1950.

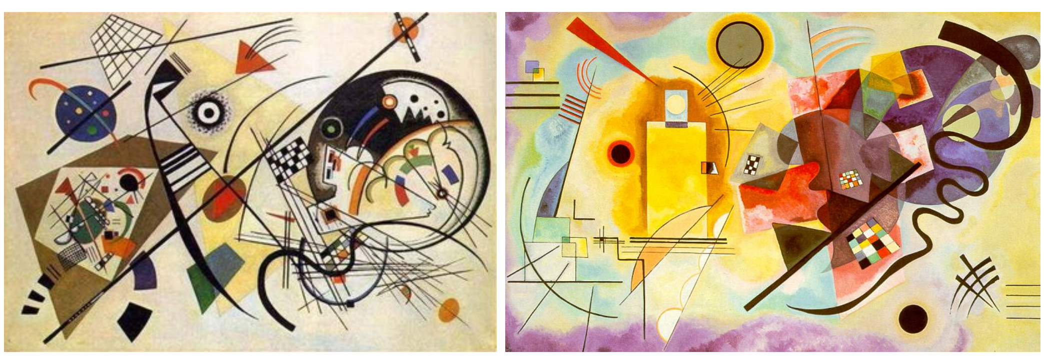

FIGURE 4.22. Wassily Kandinsky's Unbroken Line from 1923 and Yellow-Red-Blue from 1925.

on the types of projections you used to draw your cubes *and* the way you have located these cubes. Is yours a projection of a 4-dimensional cube, which is generally called a **tesseract**, a **4-hypercube**, or simply **hypercube** if the context is clear? This depends on where you view it from.

64. Compare your figure in Investigation **63** with those of other peers. What similarities and differences do they share? Are you surprised that there are so many figures that are so very different? Explain.

65. Following Investigation **63**, draw four more, different, projections of a 4-hypercube. Experiment with a number of different locations and projection types.

66. There are many different online scripts where you can interact with and animate projections of 4-hypercubes in an attempt to visualize them better. Go to one of these sites and play. [12] Do you see projections that look like some of those you drew in Investigation **65**? Explain.

67. Explain whether these dynamic scripts help you get a sense of what the 4-hypercube looks like and behaves under rotations?

68. Some of these scripts employ 3-D glasses to help you see the 4-hypecube as if it were three dimensional. Does this help? What might some implications of this technology be?

69. Describe *how* you would make a 5-hypercube. Could you also make a 6-hypercube? Can this process continue indefinitely? Explain.

10. Other Perspectives of the 4-Hypercube

At this point you might be somewhat dubious about the shadows of the hypercube that we've asked you to create. Sure, the drawing trick is an interesting process. But we hear you thinking, "How do you know this says anything about the existence of objects in higher dimensions?" This is a legitimate criticism.

In this section you will explore some additional evidence for the appropriateness of our claim that these really are projections of objects from the higher dimensions.

10.1. Patterns. The table below contains information about some of the geometric components that make up our ***family of hypercubes*** where we think of a point as a 0-hypercube, a line segment as a 1-hypercube, etc.

Dimension	Object	#Vertices	#Edges	#Faces	#Cubes	#4-Hypercubes	Boundary
0	Point	1	0	0	0	0	Empty
1	Line	2					
2	Square						4 Lines
3	Cube						
4	4-hypercube						

70. Make a copy of this table in your notebook. In your table, fill in the number of vertices in a square and in a cube. Do you see a pattern in the data of the vertices? If so, what does this pattern predict for the number of vertices a 4-hypercube should have? A 5-hypercube? A 6-hypercube?

71. Count the number of vertices in a 4-hypercube. Does this number agree with your prediction in Investigation **70**?

72. Above you were asked how you would draw a 5-hypercube and a 6-hypercube. Based on your instructions above, will the number of vertices in these hypercubes agree with your predictions in Investigation **70**? Explain. Will your prediction hold after this? Explain.

73. Return to the table and fill in the number of edges in a square and a cube. Do you see a pattern in the data for the number of edges? If so, what does this pattern predict for the number of edges a 4-hypercube should have?

74. If you pick a given edge in a square, how many edges (including the original one) are parallel to this edge? Repeat for a cube. Repeat for a 4-hypercube. Describe a pattern that allows you to predict the number of edges in a 4-hypercube and beyond.

75. Count the number of edges in a 4-hypercube. Does this number agree with you prediction in Investigation **74**?

[12] At this printing two recommended sites are http://dogfeathers.com/java/hyprcube.html http://darkwing.uoregon.edu/~koch/java/FourD.html

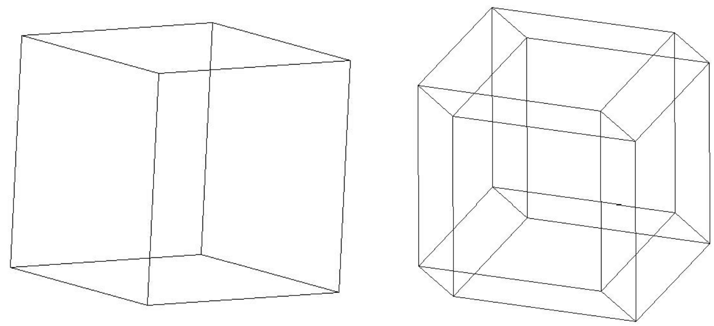

FIGURE 4.23. Orthogonal parallel projections of the cube and 4-hypercube. Notice that the front and back faces of the cube are not square in the projection.

What can we find when we try to fill in the rest of the table above?

76. In Figure 4.23 are projections of the cube and the hypercube. There are multiple copies of these images in the appendix/online. Using a marker or highlighter, highlight all of the different faces of the cube - organizing your collection by using several different images to record your findings.

77. Repeat Investigation **76** to highlight all of the faces in a hypercube.

78. Repeat Investigation **76** to highlight all of the cubes in a hypercube.

79. Consider what we find when we find the sum of the individual rows in the table above:

- **a.** Line: 2 0-hypercubes + 1 1-hypercube = 3 total hypercubes
- **b.** Square: 4 0-hypercubes + 4 1-hypercubes + 1 2-hypercube = 9 total hypercubes

Extend these calculations to a standard cube (i.e. 3-hypercube) and then to a 4-hypercube. What pattern do you see?

80. Does the consistency of all of these patterns help address some of your concerns about the legitimacy of our projection of the 4-hypercube and beyond? Explain.

11. Coordinate Geometry

As noted in the introduction "What is Geometry?", one of the major changes in geometry over the past half-millenium that you have seen as part of your middle and high-school education is the idea of using coordinates in geometry. The use of coordinate geometry is an essential tool that underlies many of the applications in the previous chapter as well as applets like those used to visualize the 4-hypercube above.

On the left of Figure 4.24 is a unit line segment - our line segment that we have been working with throughout. Notice that the vertices need only one coordinate to be labelled on our number line: 0 and 1.

81. In the center of Figure 4.24 we have drawn a unit square. Describe how the coordinates of the vertices are named.

82. Similarly, on the right of Figure 4.24 we have drawn in a unit cube and the three coordinate axes as indicated. Find the coordinates of each vertex of the cube.

83. You should see a pattern forming. Explain it in detail.

84. Use Investigation **83** to write down the coordinates of all of the vertices of a standard, unit 4-hypercube. Show that your answer agrees with your results above.

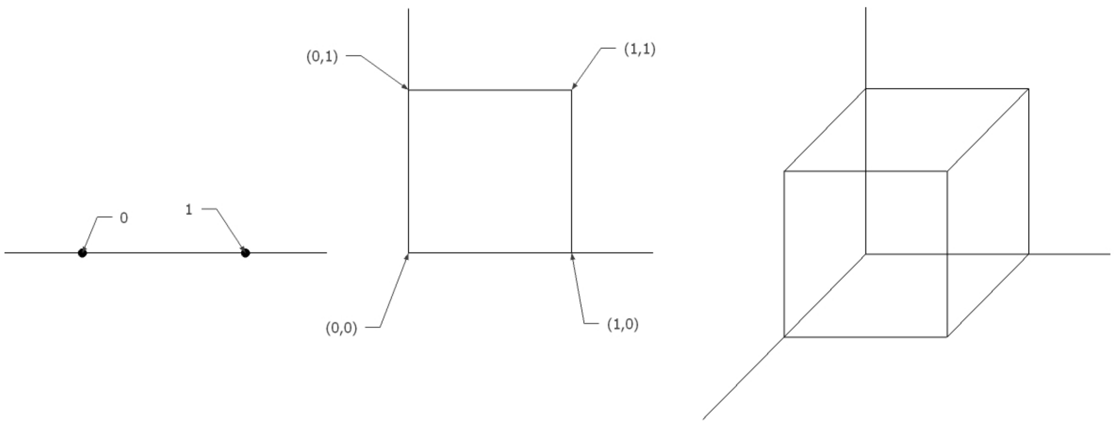

FIGURE 4.24. Line, plane, and space - the dimensional ladder.

In Discovering the Art of Mathematics: Patterns and Discovering the Art of Mathematics: Proof, Truth, and Certainty there are sections on ***Euler's formula***, an important formula relating the number of vertices (v), edges (e), and faces (f) in many polyhedra. This formula, discovered inductively by **Leonard Euler** (Swiss Mathematician; 1707 - 1783), states

$$v - e + f = 2.$$

85. Show that Euler's formula holds for the cube.

If we let c denote the number of cubes, for the cube itself we have $c = 1$ and we can write:

$$v - e + f - c = 1.$$

86. Does this formula hold for a point? For a line? For a square?

87. The formula relates the number of 0, 1, 2, and 3 dimensional components for 0, 1, 2, and 3-dimensional objects. To extend it to 4-dimensional objects, one has to include a term for the number of hypercubes. Conjecture an apporpirate formula and use it to predict the number of cubes in a 4-hypercube.

88. How might Euler's formula be extended to arbitrary dimensions to unify the numbers of the different dimensional components of ***hypersolids*** in that dimension?

12. Defining Dimension

We say that space is 3-dimensional because the walls of a prison are 2-dimensional.

Hermann Weyl (German mathematician and physicist; 1885 - 1955)

The goal of this section is to develop a more formal definition of dimension. There are many ways to define dimension - you'll see another in the next chapter. Not only are they very different in their construction, but they often give different results!

We start with our intuitive notion of dimension that has been implicit throughout. This is most easily seen via the use of Cartesian coordinates as in the last section. A line is one-dimensional because there is one *degree of freedom* - you can move left and right from a fixed point and the distance moved is denoted by a single variable x. A plane is two-dimensional because you have two degrees of freedom. You can move left-right and up-down and we can specify any point in the plane using two coordinates, written (x, y). Similarly, points in three-dimensional Euclidean space are specified by three coordinates, (x, y, z) as we have three degrees of freedom. Theoretically there is no limit to the number of coordinates that can be specified.

We are used to other specialized surfaces being described in similar ways. For example, we describe locations on the surface of the earth by longitude and latitude which make up a two-dimensional coordinate system for this surface. For pilots and divers this can be extended to space by specifying longitude, latitude, and height above sea level.

Mathematicians work with idealized objects which approximate real objects. As we work with the intuitive notion of dimension, let us agree to do think of real objects by their idealized versions. For example, string will be considered infinitely thin as will cell membranes and sheets of paper.

For each of the objects below, determine the dimension of the object and describe, intuitively, why you believe this is the appropriate dimension:

89. Find a dozen everyday objects that, in their idealized versions, are one-dimensional. Explain how you know they are one-dimensional.

90. Find a dozen everyday objects that, in their idealized versions, are two-dimensional. Explain how you know they are two-dimensional.

91. Find a dozen everyday object that, in their idealized versions, are three-dimensional. Explain how you know they are three-dimensional.

Mathematicians have discovered how to piece together objects that look locally like Euclidean spaces, calling the results *manifolds*. The notion of dimension that is suggested by the investigations below is that of *small inductive dimension*, which is a topological notion of dimension.[13] This definition is less technical than the machinery necessary to define maninfolds, is both historically and mathematically important, and is fully formalizable.

We must make one caveat. Fundamental to these investigations is the notion of *boundary* which we will not define formally - to do so would require developing a number of non-trivial notions that make up the first several weeks of an introductory topology course for mathematics majors. *However*, we do use the appropriate terms below and the investigations below parallel the more formal development.[14]

Mathematicians name terms in accordance with their everyday meanings, but on more precise levels. So, in mathematics, as in everyday usage, a ***boundary*** is the periphery of a region, a bound or frontier, or a limiting extent of a region.

Using this terminology, describe the boundary of each of the mathematically idealized[15] versions of the everyday objects below.

92. ...a country.

93. ...a pear.

94. ...a piece of property.

95. ...a stop sign.

96. ...a house.

97. ...a drinking glass.

The boundary has several important properties. Figure 4.25 shows the boundary of a square, which ***separates*** the *interior* from the *exterior*. In constructing a stop sign, large sheet metal is stamped with a punch which sheers the metal and separates the sign from the larger sheet. The bark of a tree separates the wood which makes up the tree from its surroundings. Conceptualized differently, if you are in a country and walk in a specific direction you will eventually get to a point where you cannot walk any further and remain in the country. This *limit point* is a *boundary point*

[13] *Topology* is a critcal area of mathematics that arises in many books in this series, including: Discovering the Art of Mathematics - Knot Theory and Discovering the Art of Mathematics - Art and Sculpture.

[14] If you wish to see more of the precise development, Chapter 3 - Topological Dimension of the beautiful book Measure, Topology, and Fractal Geometry by Gerald A. Edgar is recommended.

[15] For example, the drinking glass is - like an idealized mathematical plane - infinitely thin.

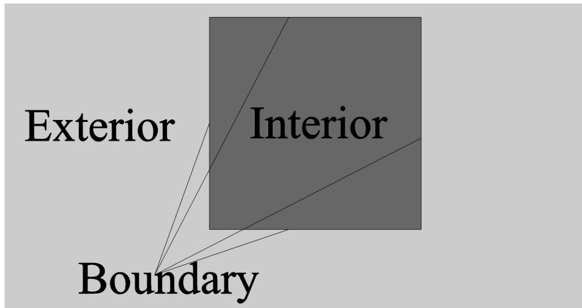

FIGURE 4.25. The interior (dark grey), exterior (light grey), and boundary of a square.

and the entire boundary of the country can be determined by finding limits like this. Figure 4.26 shows a drinking glass which has been given a coordinate system. As we follow each *geodesic* on the glass from the center of the bottom we see that each stops at a point on the rim, a boundary point.

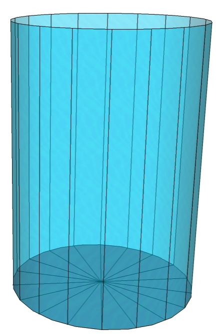

FIGURE 4.26. A ruled drinking glass.

98. Return to Investigation **92** - Investigation **97** and show how the notion of separation and/or limit points can be used to check that your previous answers are correct.

99. What is the boundary of a cube?

100. What is the boundary of a line segment?

101. Return to the table 10.1 and fill in the boundaries to complete the last column.

102. As we move up the dimensional ladder point → line segment → square → cube → 4-hypercube, how do the dimensions of the object and its boundary appear to be related? Explain.

103. Investigation **102** should suggest a definition of dimension:

A point is zero-dimensional. A topological object has dimension $\boxed{}$ if its boundary $\boxed{}$.

Of course, in making a definition we need to insure that it correctly applies to those examples we hope to use it to describe. Let's check.

104. What is the dimension of a solid ball?

105. What is the boundary of a solid ball?

106. What is the dimension of the boundary of a solid ball?

107. What is the dimension of a disc?

108. What is the boundary of a disc?
109. What is the dimension of the boundary of a disc?
110. What is the dimension of a solid cylinder?
111. What is the boundary of a solid cylinder?
112. What is the dimension of the boundary of a solid cylinder?
113. What is the dimension of a solid torus (doughnut)?
114. What is the boundary of a solid torus?
115. What is the dimension of the boundary of a solid torus?

116. Use your definition to show that a 1-hypercube (i.e. line segment) is 1-dimensional.
117. Use your definition to show that a 2-hypercube (i.e. square) is 2-dimensional.
118. Use your definition to show that a 3-hypercube (i.e. cube) is 3-dimensional.
119. Use the definition to show that a 4-hypercube is 4-dimensional.
120. Compare your observations thus far with Weyl's quote that opens this section.

So far, so good. This seems a useful, compelling definition.

121. What is the boundary of a sphere?
122. What is the boundary of a circle?
123. What is the boundary of a surface of a torus?
124. What happens if you try to apply your definition of dimension to a sphere, circle, or surface of a torus? Can you think of a way to circumvent this problem?

We hope you feel your definition is compelling enough that we should attempt to adapt it to suit this new evidence that it may not yet be complete.

A circle can naturally be decomposed into two semicircles.

125. What is the boundary of a semicirle?
126. Can you use your definition dimension to determine the dimension of a semicircle? Explain.
127. Does this help to determine the dimension of a circle? How?
128. Can you decompose a sphere so your definition of dimension can be applied to each of the components?
129. Can you decompose the surface of a torus so your definition of dimension can be applied to each of the components?
130. Is there are limit on the number of pieces that an object can be decomposed into before this process breaks down? Either describe why their is no limit or provide an example which illustrates the limit.
131. Adapt your definition of dimension in light of the examples just considered.

For typical mathematical objects[16] this is the definition of ***small inductive dimension***.

13. Implications of the Fourth Dimension

All of us are slaves to the prejudices of our own dimension.

Thomas Banchoff (American Mathematician; -)

The greatest advantage to be derived from the study of geometry of more than three dimensions is a real understanding of the great science of geometry. Our plane and solid geometries are but the beginning of this science. The four-dimensional geometry is far more extensive than the three-dimensional, and all the higher geometries are more extensive than the lower.[17]

Henry Parker Manning (American Mathematician; 1859 - 1956)

[16] E.g. *compact*, *separable*, *metric spaces.*
[17] From Geometry of Four Dimensions.

> One's mind, once stretched by a new idea, never regains its original dimensions.
>
> **Oliver Wendell Holmes** (; -)

> Artists who are interested in four dimensional space are not motivated by a desire to illustrate new physical theories, nor by a desire to solve mathematical problems. We are motivated by a desire to complete our subjective experience by inventing new aesthetic and conceptual capabilities. We are not in the least surprised, however, to find physicists and mathematicians working simultaneously on a metaphor for space in which paradoxical three dimensional experience are resolved only by a four dimensional space. Our reading of the history of culture has shown us that in the development of new metaphors for space artists, physicists, and mathematicians are usually in step.[18]
>
> **Tony Robbin** (; -)

Figure 4.27 shows what is called a *net* of a dodecahedron.

132. Cut out the net, fold along the lines, and tape edges together to create a dodecahedron.

133. Create a net of a square-based pyramid.

134. Create several different nets of a cube. Cut two of them out and check to make sure they appropriately form a cube.

135. Can you create a net for a cube that resembles a cross? Either do so or explain why it cannot be done.

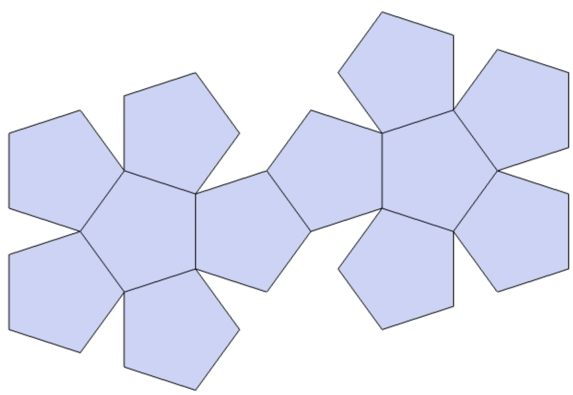

FIGURE 4.27. Net of a dodecahedron

136. Figure 4.28 shows the famous 1954 painting Crucifixion (Corpus Hypercubus) by **Salvidore Dali** (Spanish artist; 1904 - 1989). How is the cross related to topics we have been discussing? Explain in detail.

137. Can you think of some reasons Dali may have utilized this type of cross? I.e. what might his artistic, cultural, and/religious message have been?

When Sphere visits Flatland, the hero A Square eventually becomes exasperated and in his confusion/fear he assaults Sphere, exclaiming, "Monster, be thou juggler, enchanter, drea, or devil, no more will I endure they mockeries. Either thou or I must perish." A Square then collides into the Sphere, seeking to destroy him. Yet he finds, "I could feel him slowly and unarrestbly slipping from

[18]Quoted in The Fourth Dimension and non-Euclidean Geometry in Modern Art by Linda Dalrymple Henderson.

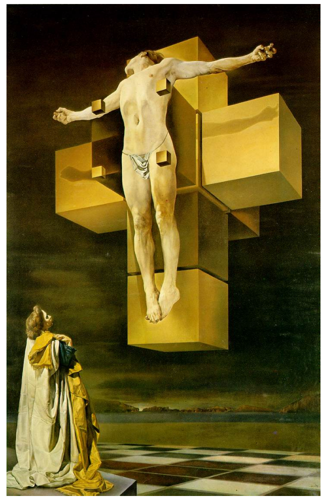

FIGURE 4.28. The Crucifixion (Corpus Hypercubus) by Salvidore Dali.

my contact; no edging to the rigt nor to the left, but moving somehow out of the world and vanishing to nothing.[19]"

Frustrated that A Square "will not listen to reason," Sphere decides to resort to deeds.

138. Sphere's first deed is to demonstrate how he "can see from my position in Space the inside of all things that you consider closed." He goes on to describe the contents of A Square's cubbard, including money hidden in closed boxes. How is it that Sphere can see these things and so amaze A Square with his power to see through Flatland's boundaries?

139. Sphere's final deed, "as crowning proof," is to giving A Square "a touch, just the least touch" in his stomach. One that "will not seriously injure... and the slight pain... cannot be compared with the mental benefit" which knowledge of a third dimension will impart. A Sphere reports of a "shooting pain in my inside, and a demoniacal laugh seemed to issue from within me." How did Sphere touch A Square in his stomach? If A Square was ill and needed some sort of surgery, what advantage would Sphere have over Flatland doctors?

140. Extend the analogy now. Suppose you had cash locked in a safe. If you were visited a being from the fourth dimension, would they have access to the contents of your safe? Explain precisely.

141. If you were ill and needed surgery, would a doctor from the fourth dimension have any advantages over Spaceland doctors? Explain precisely.

All sorts of "paradoxes" arise when we think about the fourth dimension. Here is another.

When you played the Flatland game, it is likely that some of your cross sections were *disconnnected*. For example, an axial cross section of a hand shows a thumb and four independent fingers. They look entirely disconnected from one another from the view of a Flatlander. Only by seeing into Spaceland can one ascertain that these independent objects are actually connected together as part of a larger whole.

142. **Carl Gustav Jung** (Swiss Psychiatrist; 1875 - 1961) wrote of the *collective unconscious*. Might this be part of a larger physical collective? E.g. might each of us be part of a larger physical whole - humanity - that only seems independent because our views are limited to three dimensions? Explain.

[19]P. 77.

14. The Fourth Dimension in Literature

The fourth dimension, and beyond, appears in literature of many different sorts:

- Time Machine by **H.G. Wells** (; -) - An 1895 science fiction novella that spurred our interest in time travel and thereby spawned hundreds of other books, stories, and movies.
- "Mimzy were the Borogoves" by **Lewis Padgett** (Pseudonym for **Henry Kuttner** (; -) and **C. L. Moore** (; -); -) - a 1943 short story which was later turned into the hit 2007 movie The Last Mimzy.
- A Wrinkle in Time by **Madeline L'Engle** (American writer; 1918 - 2007) - Award-winning young adult science fiction novel written in 1962.
- Factoring Humanity by **Robert J. Sawyer** (Canadian writer; 1960 -) - Outstanding science-fiction novel from which the analogy in Investigation **142** was borrowed.
- Slaughterhouse Five by **Kurt Vonnegut** (American writer; -) - Ranked # 19 on the Modern Library's 100 Best English Novels, this famous science fiction includes as fundamental characters the *Tralfalmadorians* who see in four dimensions and have thus seen every moment of their lives.

143. Choose two of the pieces of literature above. Find out more about these pieces of literature, focussing particulary on the role that the fourth dimension plays as a literary vehicle for the piece. Describe this role and its relationship with what you have learned about the higher dimensions here.

144. Find two examples of literature that are not included on the list above where the fourth, or higher, dimensions plays a critical role. Describe the role of the higher dimensionsa and its relationship with what you have learned about the higher dimensions here.

15. Connections

Factoring Humanity and projections of collective unconscious into 3D being us!! What is the question? Maybe set up the issue so they can descibe it!!

Frederick Taylor, the Gilbreths, and Time and Motion Studies in production.

16. Further Investigations

F1. In analogy with our determination that the number of edges formed a pattern based on the dimension, investigate the number of faces.

F2. (Note: For those with some experience with basic combinatorics including the binomial coefficients. Adapted from Beyond the Third Dimension: Geometry, Computer Graphics, and Higher Dimensions by Thomas Banchoff, p. 76.)

Extend the reasoning in Investigation **74** and Investigation **1** to find the number of k-hypercubes in an n-hypercube as follows:

a. Determine the number of k-cubes there are which include a given vertex as follows:
  (i) Explain why there are n edges eminating from the given vertex.
  (ii) Explain why choosing to include any k of the edges eminating from the given vertex completely determines a unique k-hypercube.
  (iii) Explain why the total number of k-cubes which include a the given vertex is $C(n,k) = \frac{n!}{(n-k)!k!}$.

b. Since there are 2^n vertices (explain) and $C(n,k)$ k-hypercubes at each vertex, explain why, noting that we might be counting each k-multiple times, this gives a total of $2^n C(n,k)$ k-hypercubes.

c. Explain why the counting in the last step counts every individual k-cube 2^k times instead of once.

d. Conclude that the total number of k-hypercubes in an n-cube is given by

$$Q(n,k) = 2^{n-k} C(n,k).$$

F3. Show that the formula in the previous problem agrees with all of the data in your table from the Investigations section.

F4. (Note: For students who have seen induction or the binomial formula.) Prove the result in Investigation **??**.

Do parameterization of n-hypercubes? Show how this gives a parameterization of "non-invasive surgery" of 3D objects?

17. Teacher's Manual

We find the actual physical creation of the perspective drawings and parallel projections to be quite informative. If you do not have the ability to do these experiements physically one can use computer technology to help. For example, Google SketchUp, the free computer aided design software, allows you to easily make a cube, display it as a wireframe model, and toggle between perspective views and parallel projections as you view the cube from different vantage points.

Hint for Investigation **1** For cubes, the parallel faces come in pairs and each pair can exist in one of three coordinate planes. Hence, $2 \times 3 = 6$ is as expected. Does this line of reasoning work in dimension 4? Explain.

CHAPTER 5

Inbetween the Dimensions - Fractals

The existence of these patterns [fractals] challenges us to study forms that Euclid leaves aside as being formless, to investigate the morphology of the amorphous. Mathematicians have disdained this challenge, however, and have increasingly chosen to flee from nature by devising theories unrelated to anything we can see or feel.

Benoit Mandelbrot (; -)

Why is geometry often described as cold and dry? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line... Nature exhibits not simply a higher degree but an altogether different level of complexity.

Benoit Mandelbrot (; -)

Dimension is not easy to understand. At the turn of the century it was one of the major problems in mathematics to determine what dimension means and which properties it has. And since then the situation has become somewhat worse because mathematicians have come up with some ten different notions of dimension: topological dimension, Hausdoff dimension, fractal dimension, self-similarity dimension, box-counting dimension, capacity dimension, information dimension, Euclidean dimension, and more. They are all related. Some of them, however, make sense in certain situations, but not at all in others, where alternative definitions are more helpful. Sometimes they all make sense and are the same. Sometimes several make sense but do not agreee. The detials can be confusing even for a research mathematician.[1]

Heinz-Otto Peitgen (; -)
Hartmut Jurgens (; -)
Dietmar Saupe (; -)

1. Proportion

The notion of proportion is fundamental in mathematics. Its role in geometry will be highlighted in the concluding chapter of this volume.

Here we consider a basic example.

??Note - Make the scale on the full-page to be one inch. Change width to 110". Then the scale will be 110 to 1.

The main entrance of Westfield State College's Courtney Hall is shown in the image in Figure 5.6. (A larger version appears in the Appendix.) Direct measurement shows that the distance between the two brick pillars that form the entryway (roughly vertically below the space between the "O" and "U" to the center of the "A") is 110 inches.

1. Maps have scales on them, things like ??Put box here 1" = 25 miles. Create a scale for this photo.
2. Another way to describe how things are scaled is by a magnification factor. Zoom on a camera, Reduce/Enlarge on a photocopy machine, or even the magnification factor on a magnifying

[1]From Fractals in the Classroom.

FIGURE 5.1. Diagram for Investigation **??**.

glass, microscope, binoculars, or telescope. How would we have to scale, or magnify, the photo for it to be life-sized?

3. Measure the height of Courtney Hall on the photo, from the ground to the top of the *cupola*, in inches.

4. Determine the approximate[2] height of Courtney Hall in inches.

5. Using your data, record the following two ratios:

$$\frac{\text{Door Width in Courtney Photo}}{\text{Door Width in Courtney}} = \underline{\qquad} \text{ and } \frac{\text{Height of Courtney in Photo}}{\text{Height of Courtney}} = \underline{\qquad}$$

6. Reduce both of the ratios in Investigation **5** until they are both simple fractions. Are they equal? Why?

7. Once we have fixed the scale from the two measurement for the doors, one in the photo and one in real life, what other quantities are proportional? Explain.

8. Suppose we doubled the door scale, i.e. we told you the doors were 220” apart. How would this change the proportion in the previous problem? I.e. how would the heights scale?

While these are likely not new ideas to you, they provide a starting point that will lead to quite surprising discoveries.

2. Zipf’s Law

Just before his death, **George Kingsley Zipf** (American Linguist and Statistician; 1902 - 1950) self published the book <u>Human Behavior and the Principle of Least Effort</u>. Benoit Mandelbrot says:

> I know very few books. . . in which so many flashes of genius, projected in so many directions, are lost in so thick a gangue of wild notions and extravgence. on the one

[2]We say approximate because the cupola is not directly above the entranceway.

> hand it includes a chapter dealing with the shape of sexual organs and another in which the Anschuluss of Austria into [Nazi] Germany is justifiable by means of a mathematical formula. On the other hand... [it has] been of considerable historical importance... and has not yet been exhausted.

The centerpiece of its importance is what has become known as *Zipf's law*.

Do it for language. Do the city thing later? City thing is easier to understand.

Rank order things. The population is then inversely proportional to the rank. Do it first as inversely proportional.

Give references to Pareto. Both his importance in economics and how his ideas foreshadow what we are thinking about below.

9. Determine an algebraic formula for p as a function of r.

10. This is well and good. But for our purposes we would like to envision this a bit differently. Now do the scaling thing. If you double the population, what can you expect to happen for the rank? Does it matter where you start? Make sure that the numbers come out well. Maybe this is where we need to have a hypothetical example. Certainly things at least need to work out.

Introduce the term **scale invariance**.

3. Scaling

Euclid and circles:

$$\frac{A_{C_1}}{A_{C_2}} = \left(\frac{r_1}{r_2}\right)$$

We will call the quantity on the left the area scale. We will call the quantity on the right the radii scale.

Notice that the radii scale is a linear scale - it is a length that is to be measured.

4. Kleiber's Law

With cubes to make things work out more nicely. It is cool that the major point is that the exact constants that make up the shape do not really matter.

Have them derive the surface area of a cube as a function of the volume. Have them see that it is scale invariant.

As a function of mass, metabolic rate is scale invariant. BUT, the exponent is 3/4, not 2/3. The biological, chemical, and physical issues interact with the geometry.

Discuss how remarkable it is that Kleiber's law holds so universally.

Get them to do an example to show how things scale up and down - mouse and elephant.

5. Perimeters, Areas and Volumes

Have them discover a conjecture about perimeters empirically:

Theorem 1. *Let R be a region in the plane whose perimeter is P. If R is scaled by a factor of m then the perimeter of the scaled region is* $\boxed{}\times P$.

Have them discover areas empirically as well:

Theorem 2. *Let R be a region in the plane whose area is A. If R is scaled by a factor of m then the area of the scaled region is* $\boxed{}\times A$.

Ask them if they can generalize this for volumes:

Theorem 3. *Let R be a region in the plane whose volume is V. If R is scaled by a factor of m then the volume of the scaled region is* $\boxed{} \times V$.

Links back to scaling section.

We have seen a number of examples from different fields. From this point forward we will be considering only geometric objects. So let us see how our results specialize.

In each of the investigations ???? the result has the form:

$$\left(\frac{\text{Length}_0}{\text{Length}_1}\right)^d = \frac{\text{Measure}_0}{\text{Measure}_1} \tag{1}$$

Here the ratio in parenthesis on the left is the **linear scaling factor**. It is the degree in which we have magnified the linear scale. The ratio on the right is the measured result of this scaling - be it perimeter, area, volume or any other measure.

Again, this language is more than two centuries old. The *only* shift in thinking that is needed to arrive at fractals is to reconceptualize the ratio when considering *self-similar* geometric objects.

11. Divide a square each of whose sides is $\text{Length}_0 = 1$ into four smaller, congruent squares. What is the length, Length_1, of each of these smaller squares? What are the areas, Measure_0 and Measure_1, of the larger and smaller squares?
12. What is the linear scaling factor in this situation? How is the area scaled?
13. What exponent d will make the equation in 1 hold?
14. Repeat Investigation **11** when the unit square is divided into nine smaller, congruent squares.
15. What is the linear scaling factor? How is the area scaled?
16. What exponent is needed now?
17. Can you repeat Investigation **11** with different linear scaling factors? If so, explain by describing how the area has been scaled and what happens to the exponent d. If not, explain why not.

What about cubes?

18. Show how a cube, each of whose sides is $\text{Length}_0 = 1$ can be divided into eight smaller, congruent cubes. What is the linear scaling factor? How has the volume been scaled?
19. What exponent d will make the equation in ?? hold?
20. Repeat Investigation **18** with linear scaling factor $\frac{1}{3}$. How has the volume been scaled?
21. What exponent d will make the equation in ?? hold?
22. Can you repeat Investigation **18** with different linear scaling factors? If so, explain by describing how the volume has been scaled and what happens to the exponent d. If not, explain why not.
23. Can the process in Investigation **11** and Investigation **18** be carried out for lines? If so, describe how and what the results are. If not, explain why not.
24. In these examples, how much the length, area, and volume have been scaled can be determined without actually calculating any of the lengths, areas, or volumes because of self-similarity. Explain how.

A square is, of course, two dimensional and a cube three dimensional. You've known this for years. In the previous chapter we observed how certain patterns allowed us to codify what we meant by topological dimension. We saw this extended our notion of dimension, allowing us to investigate higher dimensions that the third in a rigorous way.

The situation is similar here. (Please excuse the pun.) The following can be taken as the formal definition of dimension. Lines will have dimension 1, squares dimension 2, and cubes dimension 3, just as they should.

The **self-similarity dimension** of a self-similar object is the unique number d which satisfies

(2) $$\left(\frac{\text{Length}_0}{\text{Length}_1}\right)^d = \frac{n_0}{n_1}$$

where $\frac{\text{Length}_0}{\text{Length}_1}$ is the linear scaling factor and n_1 is the number of scaled copies it takes to make the original object which consists of n_0 components.

$\frac{n_0}{n_1}$ comes from your observations in Investigation **24**. In each of the examples above $n_0 = 1$. This is generally the case.

6. Fractals

> Fractal geometry will make you see everything differently. There is danger in reading further. You risk the loss of your childhood vision of clouds, forests, flowers, galaxies, leaves, feathers, rocks, mountains, torrents of water, carpets, bricks, and much else besides. Never again will your interpreation of these things be quite the same.
>
> **Michael F. Barnsley** (; -)

All of what has come before can be described using *power functions* of the form $y = a \cdot x^p$. We have used the notion of scaling because it provides a more natural way to explore fractals.

The centerpiece is the notion of self-similarity dimension. As we saw in the last section, this is "just" a reconceptualization of scaling for self-similar objects.

The word *fractal* comes from the Latin fractus which means fractured or broken. The term was coined in the mid-1970's by Benoit Mandelbrot who was the first to exploit advances in modern computers and computer graphics to explore many fabulously complex objects that had been discovered by famous mathematicians such as **Pierre Joseph Louis Fatou** (French mathematician; 1878 - 1929), **Gaston Maurice Julia** (French mathematician; 1893 - 1978), **Niels Fabian Helge von Koch** (Swedish mathematician; 1870 - 1924), and **Waclaw Franciszek Sierpinski** (Polish mathematician; 1882 - 1969) some 70 years earlier.

The first fractal we will consider is due to von Koch after whom it is named. It begins with the curve in Figure **??** which is known as the **Koch curve generator**. All four line segments that make up the generator have equal lengths. If the left and right horizontal segments were connected with a line the resulting triangle would be equilateral.

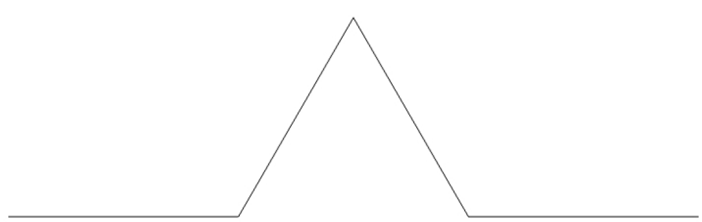

FIGURE 5.2. Koch curve generator.

25. Draw the figure that results when each of the four line segments that make up the generator are replaced by scaled copies of the generator.

26. What is the linear scaling factor that is required in Investigation **25**?

27. Draw the figure that results when each of the line segments that make up the object that resulted from Investigation **25** is replaced by a scaled copy of the generator.

28. Explain how the process described in Investigation **25** and Investigation **27** can be continued indefinitely, yielding the **Koch curve** pictured in Figure 5.3.

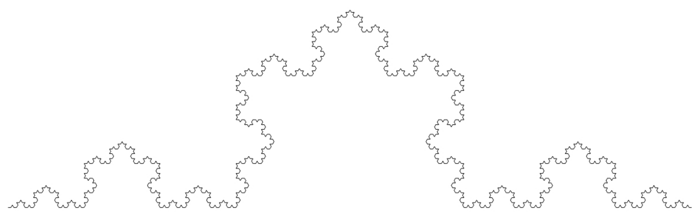

FIGURE 5.3. The Koch curve.

29. When the Koch curve is scaled by the scaling factor in Investigation **26**, how many scaled copies of the Koch curve are required to recreate the entire curve?
30. Use your observation to write down the dimension formula for self-similarity that describes the Koch curve, with the only variable remaining undertermined being the dimension d.
31. Can you easily determine the value of d from this equation? Explain.
32. Using a calculator, show that the number $d = 1.2619$ very nearly solves the equation from Investigation **30**. Conclude that this number approximates the dimension of the Koch curve. (Note: The exact value of d that solves the equation is $d = \frac{log(4)}{log(3)}$.)

Conclusion? The Koch curve has dimension $d \approx 1.2619$! We have a mathematical object whose dimension is *not* a whole number!

33. Connect three copies of the Koch curve into the shape of an equilateral triangle to form the **Koch snowflake**.

The next fractal we shall meet is named after its inventor (discoverer?) Sierpinski. Sierpinski was held in such high regard that one of the craters on the moon is named in his honor.

34. Carefully draw an equilateral triangle.
35. Find the midpoints of each of the sides of your triangle. Connect each of these midpoints to one another with a line. Describe the figure that these lines create.
36. Remove the figure described in Investigation **35** and draw the remaining object.
37. What is the linear scaling factor between the triangles remaing in the object from Investigation **36** and the original triangle?
38. Draw the resulting object created when each of the triangles making up the object in Investigation **36** is replaced with a scaled copy of the object itself.
39. Draw the resulting object created when each of the triangles making up the object in Investigation **38** is replaced with a scaled copy of the object in Investigation **36**.
40. Explain how the process described in Investigations **38-39** can be continued indefinitely, to yield the **Sierpinski gasket** as shown in Figure 5.4.

41. hen the Sierpinski gasket is scaled by the scaling factor in Investigation **37**, how many scaled copies of the Sierpinski gasket are required to recreate the entire gasket?
42. Use your observation to write down the dimension formula for self-similarity that describes the Sierpinski gasket, with the only variable remaining undertermined being the dimension d.
43. Can you easily determine the value of d from this equation? Explain.

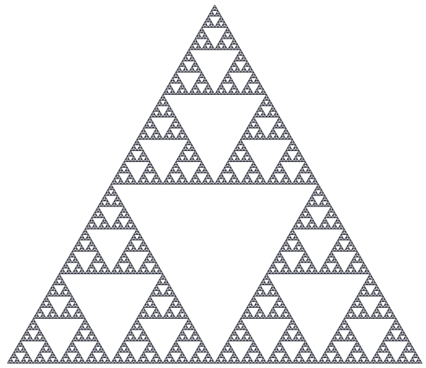

Figure 5.4. Koch curve generator.

44. Using a calculator, find an approximate dimension d that very nearly solves the equation from Investigation **42**.

At this point some people object that this is all a clever ploy to describe complex mathematical objects. Dimensions that are whole numbers don't really exist. (See Further Investigation **F7** to consider other potential objections.)

In fact, they exist. Barnsley's warning above was echoed many years before when Mandelbrot first began to champion the notion of fractals in his groundbreaking book The Fractal Geometry of Nature:

> Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

45. Pictured in Figure 5.5 is bolt of lightning. Explain how the lightning bolt exhibits signs of self similarity.
46. Similarly, explain how clouds exhibit signs of self-similarity.
47. Similarly, explain how the fern leaf exhibits signs of self-similarity.
48. The fern leaf in Figure 5.5 was generated electronically on a computer by Barnsley. If it had a realistic background, would you think it was a forgery or would you think it was real?
49. What about mountains, rivers, and tree bark? Are they self-similar? Explain.
50. Find a dozen other examples of real-world objects that exhibit self-similarity.

A wonderful statement of the role of fractals in the universe is from **H.W. Smith** (Artist and mathemtician; -) who founded Art Matrix in the late 1970's to help share the beauty of fractals with others:

> The physical universe is basically an iterated system, so actually it is surprising we have made the progress we have, using only simple evaluation. The equations have been around forever. The physical universe has been USING them almost forever. The equations have as part of their very nature things like fixed points, period cycles, chaotic cycles, basins of attraction, etc., so you can be sure all these things are manifested in the physical universe INCLUDING FRACTALNESS.
> To say therefore that fractals have nothing to do with anything and have not explained or proven useful in our understanding of the universe is more a statement about the people who are working with fractals rather than a statement about the pertinence of fractals to the world at large. Fractals are so pertinent to the universe

FIGURE 5.5. Lightning bolt, clouds, and a fern leaf.

> no one can see it yet. Long time ago, they thought math did not pertain either. The "why" was God. The "why" might still be God, but if it is, then clearly God is a mathematician of significant merit, and no doubt a fractal enthusiast.

Images of fractals abound on the Internet. We highly recommend Viewpoints: Mathematical Perspective and Fractal Geometry by Annalis Crannell and Marc Franz which could serve as a text for a course very much like the one you are currently enrolled in. Books considered the classics in the area include The Fractal Geometry of Nature by Mandelbrot, The Beauty of Fractals: Images of Complex

Dynamical Systems by **Heinz-Otto Peitgen** (; -) and **P.H. Richter** (; -), and Fractals Everywhere by Barnsley. We also recommend Chaos Under Control: The Art and Science of Complexity by **Michael Frame** (; -) and **David Peak** (; -).

So fractals are part of nature. When did humans catch on? In terms of the history European/American mathematics one of the oldest fractals is the *Cantor set* which you will investigate below. Funny thing, while the set is named after **Georg Cantor** (German mathematician; -), widely considered the father of the infinite in mathematics, it was discovered a decade earlier by the lesser known **H.J.S. Smith** (; -).[3] As noted, discoveries and investigations by von Koch, Sierpinski, Fatou and Julia followed. Also notable was the work by **Lewis Fry Richardson** (American meteorologist, physicist, mathematician and teacher; -) who measured coastlines on smaller and smaller scales. Looking back on his work now we would say that he discovered that many coastlines are fractals *and* he even found the fractal dimensions of these coastlines significantly before Mandelbrot and others brought things into a systematic organization.[4]

We should note that like many histories, that related here is narrow and one-sided. In his book African Fractals: Modern Computing and Indigenous Design **Ron Eglash** (; -) provides significant evidence that fractal-like design has been used in Africa for decades. Since fractals occur in nature, it is likely that many other indigenous cultures used them as well. Which makes one wonder why the European/Americans took so long to catch on. Aren't we supposed to be more sophisticated?

7. A Few More Fractals

To construct the Cantor set begin by drawing a line segment of unit length.

51. Divide this segment into thirds. You are to remove the middle third, leaving the endpoints $\frac{1}{3}$ and $\frac{2}{3}$. Draw a picture of what remains after this first step.

52. Divide each of the line segments that remain from Investigation **51** into thirds. Delete each of the middle thirds, again leaving the endpoints. Draw a picture of what remains after this second step.

53. Divide each of the line segments that remain from Investigation **52** into thirds. Delete each of the middle thirds, again leaving the endpoints. Draw a picture of what remains after this third step.

54. Draw a picture of what remains after the fourth step in this construction.

The **Cantor set** is the set of points that are never removed from the unit interval when this process is continued indefinitely.

55. Find a number of points that are in the Cantor set. How many such points are there?

56. One can show that the point $\frac{1}{4}$ is not an endpoint of any of the removed intervals. Nonetheless, it is a member of the Cantor set. Does this surprise you? Why?

57. What is the linear scaling factor for the Cantor set?

58. Determine the self-similarity dimension of the Cantor set.

Do Menger Sponge.

8. Conclusion

How to end? Do the bread, balled paper and other do-it-yourself fractals examples. Video feedback is good too. Get student to think about the fact that these things really should be intermediaries in the dimensional ladder.

[3]See "A history of the Cantor set and Cantor function" by Julian F. Fleron in ???

[4]See the discussion in the chapter "How long is the coast of Britian" in Mandelbrot's Fractals: Form, Chance and Dimension for more details.

You should really think of my pictures [of fractals] as a metaphor for all living things.

John Hubbard (French mathematician; -)

[5]

[5] From the videotape "The Beauty and Complexity of the Mandelbrot Set."

9. Further Investigation

F1. Have them do Zipf's theorem for several countries, using at least a dozen of the biggest cities. How well does it seem to work?

F2. Now show them the log-log data and have them comment on how nicely this fits a line.

> When a straight line, the mathematical evidence of a law, failed ao appear on the paper, I suggested different kinds of logarithmic paper. If you cannot simplify the curve on one kind of paper, simplify the paper.[6]
>
> **Theodore von Karman** (; -)

F3. Now let the cat out of the bag and talk to them about the real percentage error in these things.

F4. Get them to reflect on believing everything that you read.

F5. Analyzing Zipf's theorem leads directly to finite sums of the harmonic series. Have them develop one or two of these so they see the pattern.

F6. Then make notes about the non-convergence of the harmonic series. Because it diverges so slowly, this is not an issue in any particular instance. However, it does raise some important theoretical considerations. Among them, that the exponent need not be $s = 1$, but rather $s > 1$. All of a sudden we are in the realm of *p-series* and then *Riemann hypothesis* and the *Riemann Zeta function*!!

F7. Give excerpts from Krantz article. Is it available online? Have them comment on it. Maybe compare it to other scientific fads? String theory?

Need to do the length of the Cantor set somewhere. Some discusson about infinitude of measures for the lower, whole dimensions.

10. Teacher Notes

> I think that it is time that teachers of geometry become a little more ambitious. . . It seems to me regrettalbe that students are not given the opportunity, while still at school, of learning a good deal more about the real subject matter out of which modern geometrical systems are built. It is probably easier, and certainly vastly more instructive than a, great deal of what they are actually taught. . . I have never not yet encountered a student who finds difficulty with such ideas [projective geometry, the nature of axiom systems, and perspective]. . . [We must] widen the horizon of knowledge, reconising, as regards the niceites of logic, sequence, and exposition, that the elementary geometry of schools is a fundamentally and inevitably illogical subject.
>
> **G.H. Hardy** (English mathematician; -)

[7]

Somewhere in here there needs to be a note that a function is a power function if and only if it scales. This is important to note. We have chosen to approach the topic via the perspective of scaling for several reasons:

- It is least algebra intensive, which is appropriate for this audience.
- It mimics the language in which many in the social sciences, natural sciences, and even humanities talk about growth. Relative growth is critical here. There is no presupposition that exact, closed term formulas are possible or even appropriate. Relative growth and scaling are more natural tools.
- We believe that the development of the definition of self-similarity dimension is then much more natural.

[6] Quoted in Fractals: Form, Chance and Dimension by Benoit Mandelbrot, p. 273.

[7] From "What is Geometry?" Hardy's Presidential Address to The Mathematical Association in 1925. Reprinted in The Changing Shape of Geometry edited by Bhris Pritchard, Cambridge University Press, 2003.

11. JF Notes

I have some notes written up in Word about the introduction and conclusion. There is also the issue of Dimension paralleling the history of the development of number. Don't forget to look at these later.

11.1. Things still to get in there. Area and perimeter of the Koch snowflake. Very good self-similarity argument in Sandefur, p. 113.

Frame and Peak thing on Chernobyl.

Jackson Pollack fractal dimension thing.

Picture of Newton's method for $z^4 = 1$.

MUST get them to get to the computer and zoom in on these things. Actually, this should go with the stuff about self-similarity.

Certainly the Mandelbrot set must be in there somewhere.

Stephen Smale: The Mathematician Who Broke the Dimension Barrier. What does this refer to? Check this out. Might be good to have something about Smale in here. The links to the now-solved Poincare conjecture. Etc.

Scaling up/down and the link to Penrose tilings. MUST be mentioned.

Cut from first draft:

The complexity of these marvelously beautiful objects, several of which you will construct below, is captured by their dimension. Although their infinitely repeating complexity might suggest higher, or infinite, dimensions for these objects, their dimensions capture the infinite in another, finite way. Namely, their dimensions are represented by numbers other than whole numbers! Their definitions are given by infinite decimals which are of finite size. Numbers such as 0.6309... and 1.5850... They are objects that are neither 0-dimensional, 1-dimensional, nor 2-dimensional. They are objects that "live" between our familiar dimensions.

The history of number saw first the natural numbers 1, 2, 3, ... and then, over the millenia, increasingly complex and surprising expansion of our understanding of number. Our study of dimension is beginning to look similar. We have managed to move beyond the 0th, 1st, 2nd, and 3rd dimensions - to the 4th, 5th, ..., and even to infinite dimensions. Now we have made the step to decimal dimensions.

Perhaps this sounds fabulously complicated. But in fact, much of what we need to fill out the claims above is simply a reinterpretation of our notion of the scaling exponent from the previous lesson. Let us start with the unit cube. Instead of thinking of the cube in relation to some larger cube, like we did in the Delian problem, let us think of the unit cube on its own. In analogy with problem ?? we can decompose this cube into eight smaller cubes, as shown in the figure below.

Notice that the length of any portion of one of these smaller cubes (edge, diagonal, etc.) is exactly half a long as the corresponding length in the original cube. We say that the scaling factor between the larger and smaller cubes is s = 1/2. The unit cube is composed of 8 smaller cubes, and cubes have dimension 3. How are the numbers 1/2, 3 and 8 related? By the formula sd = 1/n, where d is the dimension, n is the number of pieces, and s is the scaling factor. For certainly we have sd = (1/2)3 = (1/2)(1/2)(1/2) = 1/8 =1/n.

Following Mandelbrot, we'll call objects whose self-similarity dimension is not a natural number a fractal. You'll meet several of these objects below. Like the Cantor set which has a similarity with 2 components and scaling factor 1/3, meaning that its dimension d must satisfy (1/3)d = 1/2. The number d that satisfies this equation is the irrational number d = (log 2)/(log 3) = 0.6309... We have an entire new species of object, those with dimensions that are not whole numbers! The Pythagoreans would be truly shocked.

FIGURE 5.6. Diagram for Investigation ??.

CHAPTER 6

So, What is Geometry?

1. Introduction

At the outset of this book we urged you to think about entire new worlds simply by opening your minds. We hope that your exploration through dimension has been mind opening. We hope that you have broken free of some of the "prejudices of your dimensions." We hope that you are more comfortable in recognizing all sorts of geometry in the three-dimensional world around you. And we certainly hope that your exploration of the higher dimension and non-integer dimensions was compelling.

If these things have happened, or if other things of value have happened, then this is success. Wonderful!

Should we just end here? You may if you choose. But our hope is that you will continue on a bit more - looking forward to other areas of geometry to explore. We wanted to open your eyes to areas of geometry that are often left out of the typical school geometry. As we worked to decide what areas of geometry to include we had a very hard task. For we could have written a several books about wonderful, accessible areas of geometry that most people are wholly unaware of. It was tough to pick.

As we progressed through the book we have tried to include links to some of these areas so you can find out more. In some cases others have already written wonderful books that we have mentioned: Mathematical Perspective and Fractal Drawing by Crannell and Franz; Groups and Symmetry by Farmer; Symmetry, Shape, and Space by Kinsey and Moore; etc. In other cases there is not as much, yet.

But there is a big unanswered question given all of this: what is geometry? If so many people are unaware of these areas of geometry which we claim are fundamental parts of geometry then what is going on? Why is this geometry so different than what we saw through elementary, middle, and high school? We think this is a fundamental question to address.

This then is the main topic of this final chapter.

To be able to answer this question there are some areas that are critical to mention - areas that we have yet to mention yet. Additionally, because most experiences with school geometry are so limited, some historical perspective is important as well.

So, in here we will guide you through an exploration of some key areas of geometry and some revolutions in the subject that will help us answer the question, "So, what is geometry anyway?"

2. Ancient Geometry

The etymology of the word geometry is from the combination of two Greek words: *ge* for "earth" and *metréō* for "to measure." Ancient geometry, as far as we can tell from the historical and archeological record, began as a practical art. **Herodotus** (Greek Historian; -) reports what he learned on a visit to the Nile River valley circa 460 B.C.:

> They said also that this king [Sesostris] divided the land among all Egyptians so as to give each one a quadrangle of equal size and to draw from each his revenues, by imposing a tax to be levide yearly. But every one from whose part the river tore

> away anything, had to go to him and notify what had happened. He then sent the overseers, who had to measure out by how much the land had become smaller, in order that the owner might pay on what was left, in proportion to the entire tax proposed. In this way, it appears to me, geometry originated.

As the surveyers often used ropes as their measuring tools, they are known as ***rope stretchers***.

The geometric abilities that developed from surveying, architecture, and navigation were quite sophisticated, as we shall see.

As a starting point we would like you to experience geometric questions, problems, and strategies that mimic those of the rope stretchers. I.e. we want you to think about how we measure things in our world.

In Investigation **11** of the opening chapter you built a scale model of a significant three-dimensional structure. As you worked on your model you certainly found many, many different quantities to be measured to find the dimensions of components of your building. Below you are asked to (re-)discover methods for making both direct and indirect measurements. If you are like most of our students, we expect that you will rediscover methods that were known to the ancient geometers and are known as *shadow reckoning.*

The principle idea in shadow reckoning is one of proportion or similarity. The power of this idea appears to have been universally recognized by ancient cutures. As **Frank Swetz** (Mathematical Historian; -) tells us:

> Many ancient societies relied on shadow observations for agricultural and religious purposes. Shadow lengths helped determine Summer and Winter solstices thus fixing a year in time upon which planting seasons could be scheduled. Egyptian and Hindu priests fixed religious rituals according to the sun's position in the sky as determined by shadow lengths. In later Islamic societies, three of the five prescribed times for daily prayer are based on shadow lengths. Existing evidence indicates that the Babylonians, Egyptians, and Chinese developed rather precise celestial observation techniques using merely a vertical staff or pole and noting shadow positions.[1]

1. Figure 6.1 shows a schematic diagram of a light source located 21' 5 15/16" above the ground. The light source is located 17' 3" from the furthest point on the base of a vertical pole. The length of the shadow cast by the pole is 26' 4 7/8" long. Describe a context in which such a situation might arise. How tall is the pole?

2. A schematic for two crossed rulers held up against the backdrop of a building is pictured in Figure 6.2. The width of the door, which can be easily found to be 12 feet, measures 2 3/8 inches on the ruler. The circular stained glass window measures 4 1/4 inches on the ruler. What is the actual diameter of this circular window?

3. How are the approaches in Investigation **2** and Investigation **??** related? I.e. it seems nice to have ready access to digital cameras, but is their use that much different than what was available to the ancients?

4. The images in Figure 6.3 are of the Carillon Tower at Stanley Park in Westfield, MA. The figure in Figure 6.4 is an illustration of the way that the tower can be sighted in a mirror via its reflection. An actual mirror sighting of this type is pictured in Figure 6.5. When sighted in this way, the distance from the camera lens to the tape on the mirror was 64 inches

[1]From "Trigonometry comes out of the shadows" by Frank J. Swetz in Learning From the Masters, Mathematical Association of America, 1995, p. 57

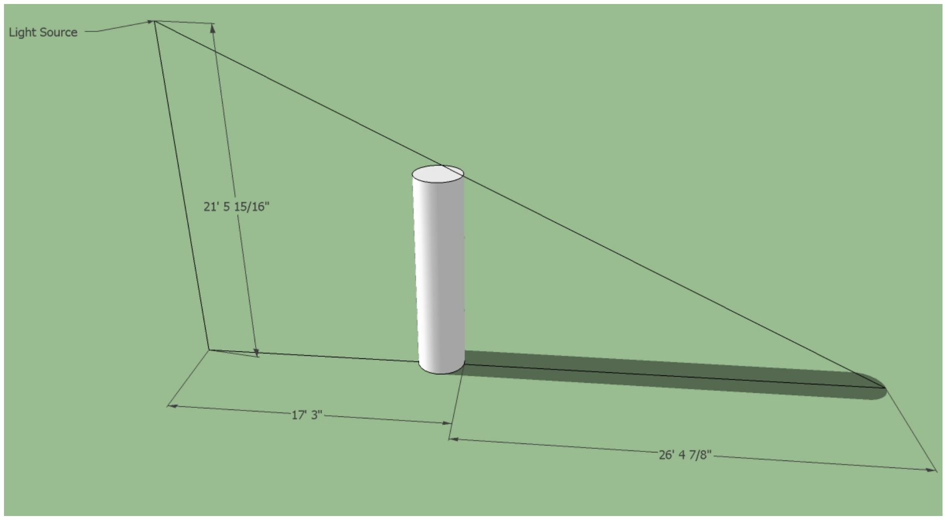

FIGURE 6.1. Diagram for Investigation **1**.

FIGURE 6.2. Diagram for Investigation **2**.

horizontally and 40 inches vertically. The horizontal distance from the tape on the mirror to the center of the tower's dome was 118 feet. How tall is the Carillon Tower?[2]

5. Figure 6.6 show's Google SketchUp's character "Bruce" standing next to a vertical pole of unknown height. Sunlight casts their shadows on the ground. The lengths of the shadows are 9' 6" and 30' 8". If Bruce is 5' 10", how tall is the pole?

6. **Plutarch** (Greek Historian; -) relates the reverence with which the King of Egypt held the mathematical work of **Thales of Miletus** (Greek Philosopher and Mathematician; circa 625 B.C. - 547 B.C.) using shadow reckoning:

> Although he admired you for other things, yet he particularly liked the manner by which you measured the height of the pyramid without any trouble or instrument.[3]

[2]It is interesting to note that the Park's WWWebsite lists the height of the tower as 98 feet. This is badly inaccurate, as you will see from your results.

[3]From Burton, The History of Mathematics, p. 94.

Figure 6.3. The Carillon Tower at Stanley Park in Westfield, MA; front view and view from base.

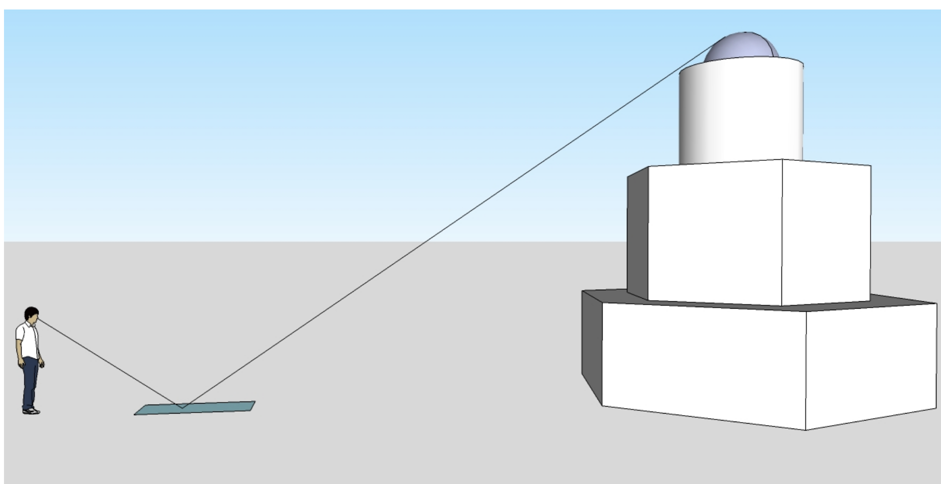

Figure 6.4. The experimental setup for *mirror reckoning*.

Use one of the approaches above to explain how Plutarch could have determined the height of one of the pyramids with no other instrument other than a pole and ruler.

Perhaps a more impressive indirect measurement is the measurement of the circumference of the earth that was completed by **Eratosthenes of Cyrene** (Greek Mathemtaician and Geographer; circa 276 B.C. - 195 B.C.) some 2,250 years ago.

One of the greatest urban legends of all time is the myth that **Christopher Columbus** (Italian Explorer; 1451 - 1506) discovered the earth was round by attempting to reach the East by sailing West. In fact, all extant evidence suggests that by several millennia *before* the birth of Christ the world's major cultures *all* believed the earth was spherical. The Christopher Columbus legend stems largely

Figure 6.5. Citing the Carillon Tower using mirror reckoning. Note the piece of tape at the top the image of the dome.

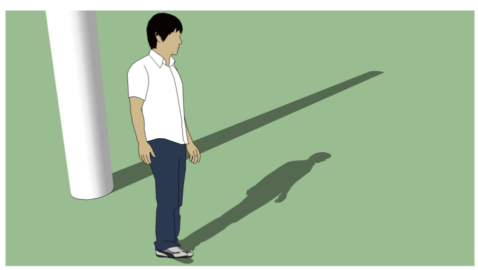

Figure 6.6. Set-up for Investigation **5**; Google SketchUp's Bruce and a pole of unknown height.

from the widely popular work The Life and Voyages of Christopher Columbus which was written by **Washington Irving** (American Author; 1783 - 1859) in 1828.

7. Classroom Discussion: Arguments attributed to the ancients' view that the earth was spherical were, not surprisingly, geometric. Describe how sailors' views of ships on a distant horizon could help them conclude that the earth is round. Alternatively, describe how views seen during a lunar eclipse could help astronomers conclude that the earth is round. Can you think of other evidence that the ancients may have used to conclude that the earth is round?

Eratosthenes described his result in his book On the Measurement of the Earth which was lost but is described by many subsequent authors, leaving little question of his approach. The approach is shown schematically in Figure 6.7. Eratosthenes noticed that at noon on the Summer Solstice in Syene the sun was directly overhead; presumably something he found by looking down a well and seeing no shadow. At exactly the same time a vertical pole at Alexandria cast a shadow measuring $7\frac{1}{5}$ degrees.

8. Explain why each of the angles labelled $7\frac{1}{5}$ must be congruent.

9. Eratosthenes measured the distance between the well and the staff to be about 490 miles. [4] Using this measurement and the angle from the previous investigation, determine the circumference of the earth that Eratosthenes would have obtained.

10. How close is Eratosthenes value to the actual circumference of the earth? How remarkable is it that the circumference was known this exactly so long ago?

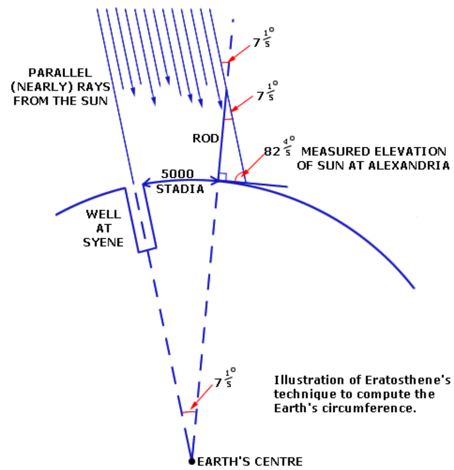

Figure 6.7. Schematic of the set-up for Eratosthenes measure of the Earth's circumference.

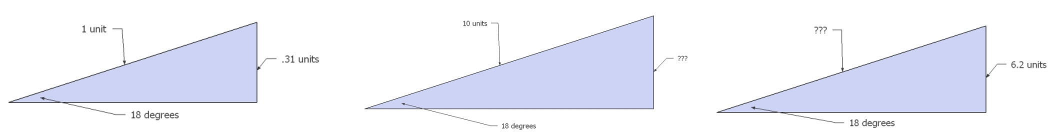

Figure 6.8. Three triangles with base angles of 18 and 90 degrees.

11. Figure Figure **??** show three right triangles with base angles of 18 degrees. The height of the first triangle, correct to two decimal places, is 0.31. Determine the missing lengths of the other two triangles.

12. Given the height of *any* right triangle with base angle 18 degrees, is it possible to determine the length of the hypotenuse? Explain.

13. Conversely, given the hypotenuse of *any* right triangle with base angle 18 degrees, is it possible to determine the height? Explain.

In fact, the positive observations in Investigations **12-13** are what help form the basis of trigonometry. In this case, it allows us to define the *sine* of an angle. All that is required to define the sine

[4] Of course, this measurement is difficult and there remains debate about how precisely this distance was measured. See, e.g. Burton's History of Mathematics, pp. 205-6.

for other angles is knowledge of the ratio of the opposite side to the hypotenuse. Rough estimates for these values can be found experimentally, as was done in ancient cultures. They can be found theoretically, as was done later, and compiled into tables. With the advent of computers and calculators, numerical algorithms can compute these values to a very high degree of accuracy.

The knowledge of these ancient rope stretchers brought us here:

> Shadow reckoning dominated calendrical computation and time keeping for over two thousand years. They shaped the sciences of land survey, cartography, and navigation. Concern with shadow ratios led to the eventual derivation of the basic trigonometric functions that we know today... Certainly, shadow reckoning constituted an important phase of early mathematical activity.[5]

14. Classroom Discussion: Are there mathematical principles that each of the investigations above share? What are they and how does their application differ in the problems?

3. Euclid and the Platonic Ideal

Having developed the practical side sufficiently to perform the tasks they needed, geometry continued to be widely practiced. Great structures were build (Great Pyramids, give many other examples ???) A significant division opened that remains a fundamental part of the organization of mathematics and many of the sciences. In addition to the practitioners who applied mathematical ideas to the world around them, there developed a significant group which concerned themselves with the more theoretical aspects of mathematics.

Careful. Is this really a good way to characterize this? Certainly this is a good place for a plug for A People's History of Science. Euclid and the Pythagoreans - who we look back at with reverence because it makes a nice, tidy story, borrowed much of what they learned from practitioners whose names are long lost.

> Mathematics owes its existence and a great deal of its development to surveyors, merchants, clerk-accountants, and mechanics of many millennia.[6]
>
> **Clifford D. Conner** (; -)

Nowhere is this illustrated better than the Platonic ideal in geometry.

Pythagoreans

Ruler and compass constructions. This is an Art form. What are the limits to this art form? Links to the reasoning book where the Three Greek problems are considered.

(The equations of lines and circles are y=mx+b and $(x-a)^2 + (y-b)^2 = r^2$. I.e. we have only linear and quadratic equations. We can get square roots (have them do them!), but where cube roots? Fourth roots from square roots of square roots. But fifth roots? Then state the theorem about constructible numbers. Send them to George Martin and Hadlock and new symmetry book if they want more details.) While a dichotomy between pure and applied remains to this day, each is informed by the otherWhat are the activities for this section?

Discussion about the rigor of Euclid.

The Platonic ideal where lines, triangles, and circles rule. We live in 3D and draw in 2D. There is nothing else.

This is not true. Lines are not everything. This is coming in many different ways.

What Euclid did. How Euclid is portrayed in contemporary mathematics:

Separate Euclid's Elements as a single book at a single point in time from his inspiration to follow logic?

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[5]From "Trigonometry comes out of the shadows" by Frank J. Swetz in Learning From the Masters, Mathematical Association of America, 1995, p. 67

[6]P. 3 of A People's History of Science: Miners, Midwives, and "Low Mechanicks"

The early study of Euclid made me a hater of geometry.

Arthur Cayley (; 1821 - 1895)

Euclid's work ought to have been any educationist's nightmare. The work presumes to begin from a beginning; that is, it presupposes a certain level of readiness, but it makes no other prerequisites. Yet it never offers any "motivations," it has no illuminating "asides," it does not attempt to make anything "intuitive," and it avoids applications to a fault. It is so "humorless" in its mathematical purism that, although it is a book about "Elements," it nevertheless does not unbend long enough in its single-mindedness to make the remark, however incidentally, that if a rectangle has a base of 3 inches and a height of 4 inches then it has an area of 12 square inches. Euclid's work never mentions the name of a person; it never makes a statement about, or even an (intended) allusion to, genetic developments of mathematics...In short, it is almost impossible to refute an assertion that the Elements is the work of an insufferable pedant and martinet.

S Bochner (; -)

Only impractical dreamers spent two thousand years wondering about proving Euclid's parallel postulate, and if they hadn't done so, there would be no spaceships exploring the galaxy today.

Marvin Jay Greenberg (; -)

It has been customary when Euclid, considered as a textbook, is attacked for his verbosity or his obscurity or his pedantry, to defend him on the ground that his logical excellence is transcendent, and affords an invaluable training to the youthful powers of reasoning. This claim, however, vanishes on a close inspection. His definitions do not always define, his axioms are not always indemonstrable, his demonstrations require many axioms of which he is quite unconscious. A valid proof retains its demonstrative force when no figure is drawn, but very many of Euclids earlier proofs fail before this test...The value of his work as a masterpiece of logic has been very grossly exaggerated.

Bertrand Russell (; -)

What are the problems that we are going to have students work on? What is the interactive part?

Need a few axiomatics. 180 degrees in a triangle is good. Can get them doing several different proofs and then show they Euclid's.

In 306 the stuff that I did at the outset was good. What can be constructed? Should have them do some of these things with a compass and a straightedge. Really get them doing hands on things. Can also do some of the construction things that relate to the Three Greek Problems. These are nice things to work with.

Should use the term ***synthetic geometry*** in discussing Euclid. But should also consider the rise of Analytic geometry - it really is a big deal. Can do this with the links to dynamic geometry software which they should certainly see. Part of this should certainly be the triangle centers stuff. This is nice because it fist in well historically too. The Euler line is after Descartes. Is this something that was done analytically? I imagine so. It would be nice to have this historical thread in here. The GeoGebra script that we can make for the triangles - does it show all of the algebraic information as an applet? If so, this would be great seeing these parallel representations working together. It really does illustrate the connection in an important way.

15. Classroom Discussion: Having spent many years in mathematics classes, having gone through the material on logic, reasoning and proof in the Student Toolbox, and having worked again a bit with Euclidean geometry, discuss the following: Is Euclid's Elements a model for logical reasoning? Has the structure of Euclid's Elements and its influence on mathematics classes you have been in provided beneficial in your mathematics education? Should we give Euclid's work different prominence depending on whether we are considering the history and/or philsophy of mathematics and thought versus when we are considering questions of teaching, learning, and pedagogy? Do you have an opinion the role the style, structure, and approach Euclid's Elements should have in contemporary mathematics classrooms? (Note: You may want to revisit this last question once you have proceeded further through the investigations that make up this book.)

4. Analytic Geometry

> Not till Descartes, 1985 years after the death of Plato, published his analytic geometry, did geometry escape from its Platonic straightjacket.[7]
>
> **E.T. Bell** (; -)

Have this section be a continuation of the chapter on Euclid only now dressed in the guise of algebra. We can see how powerful this is in many ways.

Use the triangle centers investigations via GeoGebra scripts. This is a very powerful venue for conjecturing. This is the real nature of mathematics - how mathematics works. See how this shift empowered those like Euler to find the Euler line which was there all along, waiting to be discovered.

Links to Morgan's theorem - another perfect story to illustrate the potential of all to make contributions to mathematics. Also gives a nice link to the importance of different ways of viewing things - the computer as a tool to explore. Maybe include Papert quotes here?

5. Non-Euclidean Geometry

> Most people are unaware that around a century and a half ago a revolution took place in the field of geometry that was as scientifically profound as the Copernican revolution in astronomy and, in its impact, as philosophically important as the Darwinian theory of evolution.
>
> **Marvin Jay Greenberg** (; -)

[8]

What is a line? Lines on a cylinder with ribbon.

Maps and map projections. Our earth - where we live - is only locally Euclidean!!

6. Klein's Erlanger Program

> The universe is an enormous direct product of representations of symmetry groups.
>
> **Steven Weinberg** (American, Nobel Prize Winning Physicist; 1933 -)

Is this too much for this? Are there other big hooks that we can put in here? Poincare conjecture is really a topological thing. Are there other big open problems that typify what modern geometry is about?

Symmetry.

Links back to perspective drawing which is a subset of projective geometry.

Although this particular program was not ultimately how mathematicians view modern geometry, the flavor of its unique approach was a revolutionary new idea whose spirit lives on in our contemporary conception of geometry.

Farmer references.

Kinsey and Moore - Chapters 5 and 8.

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[7] From Men of Mathematics, p. 32.

[8] From Euclidean and Non-Euclidean Geometries: Development and History.

FIGURE 6.9. Images of "The Bean" in Chicago, Illinois' Millennium Park.

Not sure how to do this without repeating what has been done in these nice books.

Google SketchUp does some of this stuff nicely.

In the heaven of the great god Indra is said to be a vast and shimmering net, finer than a spider's web, straching to the outermost reaches of space. Strung at each intersection of its diaphanous threads is a reflecting pearl. Since the net is infinite in extent, the pears are infinite in number. in the flistening surface of each pearl are reflected all the other pearls, even those in the furthest corners of the heavens. IN each reflection, again are reflected all the infinitely many other pearls, so that by this process, reflections of reflections continue without end. Fronticpiece from Indra's Pearls: The Vision of Felix Klein.

7. The Shape of the Universe

Jeff Weeks stuff. Tic Tac Toe on a cylinder? History of this stuff? This seems a bit out of place because the Erlanger program is the overarching flavor of contemporary geometry. Maybe this can go right after Non-Euclidean geometry because it talks about our earth's geometry? Nice lead in. But then things would be a bit out of historical order. Maybe the last section could be Contemporary Views of Geometry and the Erlanger program could be one of the topics.

8. So What is Geometry?

Notice self-similar title. Have several discussion questions. Formulate this in a broad, modern way. See article by Atiyah among other things.

8.0.1. *Further Investigations.* Modern measuring tools and their relationship to geometry. GPS, parabolic reflectors, triangulation, orienteering,)

9. JF Notes

Look up stuff in A People's History of Science!