ART & SCULPTURE

MATHEMATICAL INQUIRY IN THE LIBERAL ARTS

Julian F. Fleron, Volker Ecke and Christine von Renesse,
with Philip K. Hotchkiss

Discovering the Art of Mathematics

Art and Sculpture

by Julian F. Fleron, Volker Ecke,
and Christine von Renesse

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Acknowledgements

Preface: Notes to the Explorer

Yes, that's you - you're the explorer.

"Explorer?"

Yes, explorer. And these notes are for you.

We could have addressed you as "reader," but this is not a traditional book. Indeed, this book cannot be read in the traditional sense. For this book is really a guide. It is a map. It is a route of trail markers along a path through part of the world of mathematics. This book provides you, our explorer, our heroine or hero, with a unique opportunity to explore this path - to take a surprising, exciting, and beautiful journey along a meandering path through a mathematical continent named the infinite. And this is a vast continent, not just one fixed, singular locale.

"Surprising?" Yes, surprising. You will be surprised to be doing real mathematics. You will not be following rules or algorithms, nor will you be parroting what you have been dutifully shown in class or by the text. Unlike most mathematics textbooks, this book is not a transcribed lecture followed by dozens of exercises that closely mimic illustrative examples. Rather, after a brief introduction to the chapter, the majority of each chapter is made up of Investigations. These investigations are interwoven with brief surveys, narratives, or introductions for context. But the Investigations form the heart of this book, your journey. In the form of a Socratic dialogue, the Investigations ask you to explore. They ask you to discover the mathematics that is behind art and sculpture. This is not a sightseeing tour, you will be the active one here. You will see mathematics the only way it can be seen, with the eyes of the mind - your mind. You are the mathematician on this voyage.

"Exciting?" Yes, exciting. Mathematics is captivating, curious, and intellectually compelling if you are not forced to approach it in a mindless, stress-invoking, mechanical manner. In this journey you will find the mathematical world to be quite different from the static barren landscape most textbooks paint it to be. Mathematics is in the midst of a golden age - more mathematics is discovered each day than in any time in its long history. Each year there are 50,000 mathematical papers and books that are reviewed for *Mathematical Reviews*! *Fermat's Last Theorem*, which is considered in detail in Discovering that Art of Mathematics - Number Theory, was solved in 1993 after 350 years of intense struggle. The 1$ Million Poincaŕe conjecture, unanswered for over 100 years, was solved by **Grigori Perelman** (Russian mathematician; 1966 -). In the time period between when these words were written and when you read them it is quite likely that important new discoveries adjacent to the path laid out here have been made.

"Beautiful?" Yes, beautiful. Mathematics is beautiful. It is a shame, but most people finish high school after 10 - 12 years of mathematics *instruction* and have no idea that mathematics is beautiful. How can this happen? Well, they were busy learning mathematical skills, mathematical reasoning, and mathematical applications. Arithmetical and statistical skills are useful skills ev-

erybody should possess. Who could argue with learning to reason? And we are all aware, to some degree or another, how mathematics shapes our technological society. But there is something more to mathematics than its usefulness and utility. There is its beauty. And the beauty of mathematics is one of its driving forces. As the famous **Henri Poincaŕe** (French mathematician; 1854 - 1912) said:

> The mathematician does not study pure mathematics because it is useful; [s]he studies it because [s]he delights in it and [s]he delights in it because it is beautiful.

Mathematics plays a dual role as both a liberal art and as a science. As a powerful science, mathematics shapes our technological society and serves as an indispensable tool and language in many fields. But it is not our purpose to explore these roles of mathematics here. This has been done in many other fine, accessible books (e.g. [COM] and [TaAr]). Instead, our purpose here is to journey down a path that values mathematics from its long tradition as a cornerstone of the liberal arts.

Mathematics was the organizing principle of the *Pythagorean society* (ca. 500 B.C.). It was a central concern of the great Greek philosophers like   **Plato** (Greek philosopher; 427 - 347 B.C.). During the Dark Ages, classical knowledge was rescued and preserved in monasteries. Knowledge was categorized into the classical liberal arts and mathematics made up several of the seven categories.[1] During the Renaissance and the Scientific Revolution the importance of mathematics as a science increased dramatically. Nonetheless, it also remained a central component of the liberal arts during these periods. Indeed, mathematics has never lost its place within the liberal arts - except in the contemporary classrooms and textbooks where the focus of attention has shifted solely to the training of qualified mathematical scientists. If you are a student of the liberal arts or if you simply want to study mathematics for its own sake, you should feel more at home on this exploration than in other mathematics classes.

"Surprise, excitement, and beauty? Liberal arts? In a mathematics textbook?" Yes. And more. In your exploration here you will see that mathematics is a human endeavor with its own rich history of human struggle and accomplishment. You will see many of the other arts in non-trivial roles: dance and music to name two. There is also a fair share of philosophy and history. Students in the humanities and social sciences, you should feel at home here too.

Mathematics is broad, dynamic, and connected to every area of study in one way or another. There are places in mathematics for those in all areas of interest.

The great **Betrand Russell** (English mathematician and philosopher; 1872 - 1970) eloquently observed:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.

It is our hope that your discoveries and explorations along this path through the infinite will help you glimpse some of this beauty. And we hope they will help you appreciate Russell's claim that:

> ...The true spirit of delight, the exaltation, the sense of being more than [hu]man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

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[1]These were divided into two components: the *quadrivium* (arithmetic, music, geometry, and astronomy) and the *trivium* (grammar, logic, and rhetoric); which were united into all of knowledge by philosophy.

Finally, it is our hope that these discoveries and explorations enable you to make mathematics a real part of your lifelong educational journey. For, in Russell's words once again:

> ...What is best in mathematics deserves not merely to be learned as a task but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement.

Bon voyage. May your journey be as fulfilling and enlightening as those that have served as beacons to people who have explored the continents of mathematics throughout history.

Navigating This Book

Before you begin, it will be helpful for us to briefly describe the set-up and conventions that are used throughout this book.

As noted in the Preface, the fundamental part of this book is the Investigations. They are the sequence of problems that will help guide you on your active exploration of mathematics. In each chapter the investigations are numbered sequentially. You may work on these investigation cooperatively in groups, they may often be part of homework, selected investigations may be solved by your teacher for the purposes of illustration, or any of these and other combinations depending on how your teacher decides to structure your learning experiences.

If you are stuck on an investigation remember what **Frederick Douglass** (American slave, abolitionist, and writer; 1818 - 1895) told us: "If there is no struggle, there is no progress." Keep thinking about it, talk to peers, or ask your teacher for help. If you want you can temporarily put it aside and move on to the next section of the chapter. The sections are often somewhat independent.

Investigation numbers are bolded to help you identify the relationship between them.

Independent investigations are so-called to point out that the task is more significant than the typical investigations. They may require more involved mathematical investigation, additional research outside of class, or a significant writing component. They may also signify an opportunity for class discussion or group reporting once work has reached a certain stage of completion.

The Connections sections are meant to provide illustrations of the important connections between mathematics and other fields - especially the liberal arts. Whether you complete a few of the connections of your choice, all of the connections in each section, or are asked to find your own connections is up to your teacher. But we hope that these connections will help you see how rich mathematics' connections are to the liberal arts, the fine arts, culture, and the human experience.

Further investigations, when included are meant to continue the investigations of the area in question to a higher level. Often the level of sophistication of these investigations will be higher. Additionally, our guidance will be more cursory.

Within each book in this series the chapters are chosen sequentially so there is a dominant theme and direction to the book. However, it is often the case that chapters can be used independently of one another - both within a given book and among books in the series. So you may find your teacher choosing chapters from a number of different books - and even including "chapters" of their own that they have created to craft a coherent course for you. More information on chapter dependence within single books is available online.

Certain conventions are quite important to note. Because of the central role of proof in mathematics, definitions are essential. But different contexts suggest different degrees of formality. In our text we use the following conventions regarding definitions:

- An *undefined term* is italicized the first time it is used. This signifies that the term is: a standard technical term which will not be defined and may be new to the reader; a term that will be defined a bit later; or an important non-technical term that may be new to the reader, suggesting a dictionary consultation may be helpful.

- An ***informal definition*** is italicized and bold faced the first time it is used. This signifies that an implicit, non-technical, and/or intuitive definition should be clear from context. Often this means that a formal definition at this point would take the discussion too far afield or be overly pedantic.

- A **formal definition** is bolded the first time it is used. This is a formal definition that suitably precise for logical, rigorous proofs to be developed from the definition.

In each chapter the first time a biographical name appears it is bolded and basic biographical information is included parenthetically to provide some historical, cultural, and human connections.

Contents

Chapter 1

The Magical Möbius Band: Cultural Icon and Engineering Marvel

Symbols make us think.
Symbols can change the world.
And sometimes, symbols are all we have,
To help us maintain our resolve.
Even in our darkest and our most tragic days.
Peter W. Schroeder (German Journalist; -)[1]

1.1 Object of This Chapter

In this chapter we will study one of topology's most well-known objects - the Mbius band. The Mbius band is a magical mathematical object which is simple enough that it can be made and enjoyed by any kindergartener equipped with scissors, tape and a strip of paper. But it illustrates many deep mathematical principles.

It is also a beautiful mathematical object which occurs repeatedly in art and sculpture.

1.2 Topology

Topology is, in some sense, a special kind of geometry. It is a very basic kind of geometry. **Jean Piaget** (Swiss Psychologist; 1896 - 1980) tells us, "A child[s]... first geometrical discoveries are topological... If you ask him to copy a square or a triangle, he draws a closed circle." Topology is fundamental to contemporary mathematics, yet very few people know anything about it.

One nice way to describe topology is to compare it to geometry - something we know. To do this we think about geometry in a different, more modern way: in terms of its *transformation*

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[1]From the documentary <u>Paper Clips</u>.

groups and *invariants*.

In standard **Euclidean geometry** - the geometry we learned in school and that we use when we measure distances, areas, angles and the like - our transformations include *translations* and *rotations*. The two triangles in Figure 1.1 are **congruent** because you can simply translate the one on the left so it is *concurrent* with the one on the right. Carpenters use a **sliding T bevel** like that pictured in Figure 1.2 to measure/transfer angles because once the bevel is firmly set, no matter where they move it through space it will find/create congruent angles.

Figure 1.1: Congruent triangles.

Figure 1.2: Carpenter's sliding T bevel.

Each of the quantities length, area, and angle measure are geometrically invariant - they do not change under the transformations of Euclidean geometry.

While this seems both obvious and necessary, there are many other settings in which this is not what is desired. Consider, for example, **cartography**, the science of map making. We are all familiar with the **Mercator projection** map of the world, an example of which is shown in Figure 1.3.

Of course angles are invariant in Mercator projection maps - they would not be much use to sailors, pilots, or hikers if they could not use a compass for their headings and bearings. To signify that the angles are preserved in our map, our transformation from the spherical earth to the flat map is called **conformal**. Unfortunately, areas are certainly not preserved. Greenland is

Figure 1.3: Mercator projection map of the world.

not nearly as big as it appears on a world map, Africa not nearly so small. Something must be distorted to form a flat map from a round earth.

Depending on our needs we can choose what quantities remain invariant under our mapping. For example, we can sacrifice angles to keep areas invariant if we want to focus on sizes of land masses. One such map projection is called the **Gall-Peters projection** of the world which is pictured in Figure 1.4.

We may also choose to deform our map to highlight non-physical aspects of our world's composition. For example, the world map in Figure below is a **population cartogram** - the relative size of each country is based on its population.

Notice how huge India and Japan are relative to their geographic size. And, the tremendous population between them, where parts of the former Soviet Union are on a Mercator projection map, is China!

Of course, you can represent many different political, sociological, humanitarian, or economic issues via **cartograms** of this type. Software to create cartograms is available online from major *geographic information system* (GIS) organizations such as the Environmental Systems Research Institute (ESRI). There are also wonderful books, including The Atlas of the Real World: Mapping the Way We Live and Portraits of the Earth: A Mathematician Looks at Maps.

Similar distortions generate the *homunculus* which is used in psychology show deformations of the human body that represent the prevalance of different physiological mechanisms. (See the Connections section later for details.)

We've emphasized cartograms because they illustrate perfectly key principles of topology. Cartograms distort the typical shapes of countries. Yet all countries remain and no new countries have been added. Two countries that share geographic borders share borders on the cartogram, those that do not share borders geographically do not share borders on the cartogram.

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Figure 1.4: Gall Peters map of the world.

This is the same with is topology which is often called "rubber sheet geometry".

Topological transformations are those transformations which involve shrinking, stretching, bending, or twisting (as if the objects were made out of rubber or clay or string) yet do not involve cutting, separation, gluing, or joining objects or parts of objects together, nor do they involve collapsing an object down into a point.

If you can transform one object into another following these rules, the objects are called **topologically equivalent**. Among the **topological invariants** of an object are the number of *holes*, the number of *boundary components*, and whether the object is *connected*.

Examples:

- Imagine the letters L and N being formed out of a flexible, malleable material like elastic string. It is an easy matter to topologically deform the L into the N, perhaps by first straightening out the L. Hence, L and N are topologically equivalent. Each is simply a line with two endpoints as boundary components.

- Similarly, Y and E are topologically equivalent. The Y can be topologically deformed into the shape ⊢ simply by rotating the upper "arms" clockwise. Then it is simple matter to stretch the ends out to form the E. Each of these letters has three endpoints as boundary components.

- On the other hand, the letters F and P are not topologically equivalent. Gluing is not allowed in topology so there is no way to form the loop of the P from the F. Another way to see that they are not topologically equivalent is that F has three endpoints as boundary components, P just one. Alternatively, P has a loop while F does not.

In Investigations Investigation **2** - Investigation **5** you are encouraged to determine how to **partition** (i.e. group) the alphabet shown in Figure 1.6 into topological *equivalence classes*.
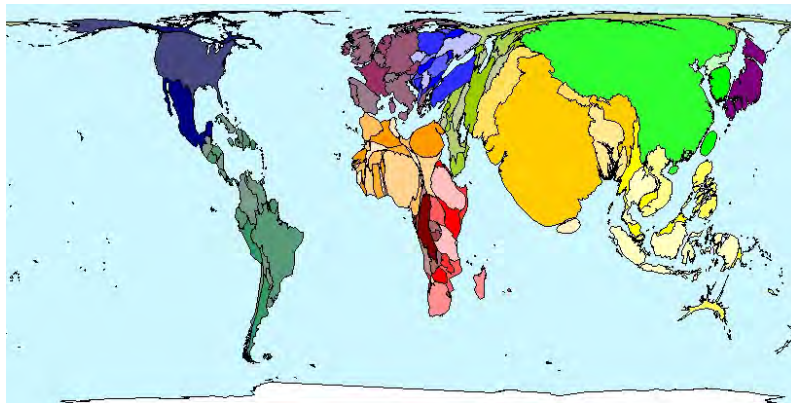
6

Figure 1.5: Population cartogram of the world.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z.

Figure 1.6: The sans serif font Avant Garde.

1.3   A Bit of Basic Topology

We start with a joke as a warm-up:

Question: What do you call somebody who cannot tell their coffee cup from a doughnut?
Answer: A topologist.

1. Explain this joke by describing the topological equivalence of a coffee cup to a doughnut.

2. Determine which letters in Figure 1.6 are topologically equivalent to the letter L.

3. Determine which letters in Figure 1.6 are topologically equivalent to the letter E.

4. Determine which letters in Figure 1.6 are topologically equivalent to the letter A.

5. The collections in Investigation **2** - Investigation **4** are called the ***topological equivalence classes*** of L, E, and A respectively. Partition the remainder of the alphabet Figure 1.6 until each of the letters has been accounted for by an equivalence class.

We now turn to the number of *sides* and *edges* - two topological invariants - of different objects.

A typical sheet of paper is rectangular, so it has four sides, right? This is certainly the case if you are speaking geometrically. But here we working topologically and we need to think of the other meaning of the term "sides" - the number of "faces" a surface has. So, as illustrated by the ***face down icon*** in Figure 1.7, we will use the word ***side*** in the sense that a sheet of paper has two sides.

What about *edges*?

6. Explain why all rectangles, pentagons, hexagons, octagons, and circles are topologically equivalent.
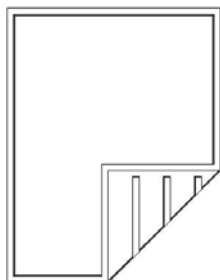
Figure 1.7: Page down icon.

7. If the term edge is to define a topological invariant of objects, explain why rectangles, pentagons, octagons, and circles must have the same number of edges.

Because a circle has a boundary which is composed of one continuous component we say that it has one edge.

8. How many edges does a typical funnel have? Explain.

9. Name, describe, or draw a surface/object that has three edges.

1.4 The Place of Topology in Modern Mathematics

The distinguished **Raymond Wilder** (American Mathematician; 1896 - 1982), who was President of both the Mathematical Association of America and American Mathematical Society as well as a member of the National Academy of Sciences, said of topology:

> Probably no branch of mathematics has experienced a more surprising growth than has... topology... Considered as a most specialized and abstract subject in the early 1920's, it is today an indispensable equipment for the investigation of modern mathematical theories.

Hermann Weyl (German Mathematician; 1885 - 1955), the great German mathematician who escaped Germany with his Jewish wife and took refuge with Albert Einstein and many other prominent scientists at the Institute for Advanced Study at Princeton University during World War II, remarked more comically, "In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics."

Like all areas of mathematics, topology has its own wealth of open problems. One of its most central problems, the *Poincare conjecture*, was recently solved by the **Grigori Perelman** (Russian Mathematician; 1966 -). He did little to publicize this landmark achievement in the early part of this decade. Once his work was discovered he won many awards, including the 2006 Fields Medal (the highest honor in mathematics) and a $1 Million Millennium Prize. He deferred both, began to question the ethical standards of several mathematicians who had worked in the same area, went into seclusion, and is reported to have quit mathematics. Perleman's discovery and the resulting

controversy surrounding the latter developments received quite a bit of press, including a page one Science Times section story in the *New York Times*, a lengthy *The New Yorker* article and an uproarious *Cobert Report*.

For more on topology in general, see Discovering the Art of Knot Theory in this series or the book Knots and Surfaces: A Guide to Discovering Mathematics which is quite similar to the books in this series and which also deals with topology.

1.5 Basic Möbius Topology

> Mathematics is something that one does.
>
> **E.E. Moise** (American mathematician; 1918 - 1998)

Required Supplies: You will need the following physical materials for the investigations

- Several dozen strips of paper approximately $1 - \frac{3}{4}$ of an inch wide. Generally, strips of 11 inches in length work well, but you may also find longer ones helpful for the independent investigation.

- Transparent tape

- Scissors

Let's make some objects and consider their topological properties.

Take one of your strips of paper, bring the ends together, and then tape the two ends together. Your object, called a cylinder or cylindrical band, should appear as pictured on the left in Figure 1.8.
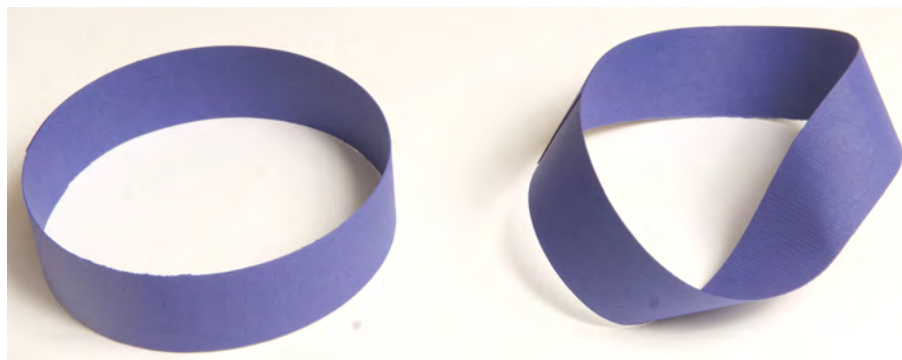

Figure 1.8: Cylindrical band and Möbius band.

10. How many sides does your cylindrical band have? Explain.

11. How many edges does your cylindrical band have? Explain.

Take another strip of paper and bring the ends together, as you did in making a cylinder. This time, before taping the ends together, flip one end over by rotating it through 180 degrees. Now tape the ends together.

You have created a Mbius band, like the one pictured on the right in Figure 1.8.

Before considering the topological properties of the Mbius band we have two things to take care of. First, some people call the twist we made in constructing the Mbius band a half-twist for obvious reasons. However, because we have no reason to consider any other type of twists, we will simply call it a **_twist_** consistently throughout. Also, we twisting we have a caution:

> **Caution:** When creating Mbius bands it is critical that you first bring the ends together to form a cylinder and then add the twists. If you twist before bringing the ends together then your hands are often in a position that results in an extra twist being added or deleted as you bring the ends together.

Returning now to the Möbius bands topological properties...

12. How many sides does the Mbius band have? Explain.

13. How many edges does the Mbius band have? Explain.

14. Are the results above surprising? Explain why or why not.

Here and below determining the number of twists, the number of sides, and the number of edges bands have is central to the next two sections. So, while you work on Investigations 15 - 20, pay particular attention to these issues.

15. Carefully describe the method you use to determine the number of twists in a band, the number of sides a band has, and the number of edges a band has.

Of course, there is no reason to stop after making a band with a single twist. We might as well see what happens as we add more twists. We will call a Möbius band created from n twists an n-**Möbius band**.

16. Create a 2-Möbius band. Can you see the two twists clearly? How many sides does it have? How many edges? Explain.

17. Create a 3-Möbius band. Can you see the three twists clearly? How many sides does it have? How many edges? Explain.

18. Create a 4-Möbius band. Can you see the four twists clearly? How many sides does it have? How many edges? Explain.

19. Create a 5-Möbius band. Can you see the 5 twists clearly? How many sides does it have? How many edges? Explain.

20. Make a conjecture about the numbers of sides and edges a general n-Möbius band will have. How certain are you of the validity of this conjecture? Explain.

While it is unlikely that you have heard of the Möbius band, you will quickly see that it plays a central role in many contemporary symbols that we see around us every day. Moreover, it has important applications in diverse areas. We provide several examples here to give you some sense

of how pervasive the Möbius band is as well as to give you some idea of the topics you'll be investigating here.

Figure 1.9 is a "Möbius bench" at an outdoor mall in Pasadena, California. There is an indie rock/electronic pop band from Shutesebury, MA called, you guessed it, "The Möbius Band." While not on a major commercial label, they are well-known enough to have an allmusic.com listing among other things. Figure 1.10 shows one of the many "Jewish love pendants" that can be purchased from catalogs and online shops. We have been unable to determine whether there is any historical connection between the Möbius band and Jewish culture. Please let us know if you find one. Figure 1.11 is a napkin ring that was found advertised in a newspaper flyer for a chain of houseware/linen stores.

Figure 1.9: Möbius bench by Vito Acconci in Pasadena, CA.

Figure 1.10: "Jewish infinity love" pendant.

Fiber artists also seem to be particularly fond of Möbius bands. You can find dozens of different sets of instructions on the Internet for knitting a Mbius scarf. They can be used as capelets, neck warmers, or a combination scarf/ear warmer where the twist provides a subtle diversion at the neckline of each of these objects. In the beautiful book Making Mathematics with Needlework Amy F. Szczepański contributes a whole chapter to "Quilted Möbius bands."

Figure 1.11: Möbius napkin ring.

Here's an art project we made up. You have probably had the experience of making paper chains as a child. Many teachers use them in novel ways as teaching tools, even in mathematics classes. We give this practice a mathematical twist, excuse the pun, requiring our future teacher candidates to make paper chains out of Möbius bands! It's a simple, wonderful way to introduce compelling mathematics to children of any age. If you try it you will see that regardless of the medium, Möbius chains are more aesthetically pleasing than their cylindrical counterparts. (See Figure 1.12.)

Figure 1.12: Möbius chains as garland.

The Möbius band has even inspired a Möbius Recycling Education Program complete with a Möbius mascot! Designed by Susan Schaefer for Browning Ferris Industries, this program has been used in 30,000 classrooms in 87 countries and has won numerous awards. This program and its materials remain available online. [Mob]

The latter half of this chapter was inspired by a somewhat random Möbius sighting. As the first author was leaving a theatre with his family after having watched the moving Holocaust documentary Paper Clips, described below, they were struck by the beautiful image emblazoned on the movie poster which is shown in Figure 1.14.

They quickly conjectured that the two triangles which make up this symbol must be some sort of Möbius bands. This began a wonderful exploration of paperclip topology. It was a challenge with great mathematical and artistic rewards - we think you will enjoy rediscovering it.

21. Find and describe some of your own "Möbius sightings" that are different than those de-
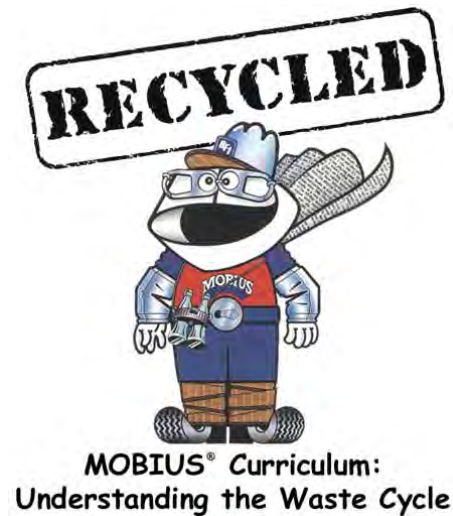
Figure 1.13: The Möbius mascot.

scribed in this chapter.

1.6 Möbius Magic

In Section 1.3 we made many different Möbius bands. If we begin dissecting these Möbius bands we will find many more surprises.

Preparation: You will need to divide strips into several equal pieces by carefully drawing equally spaced lines along the lengths of the strips. You will need to make two of each of the following:

- strips divided equally into half by one line,

- strips divided equally into thirds by two lines,

- strips divided equally into fourths by three lines, and,

- strips divided equally into fifths by four lines.

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22. Make a cylinder out of one of the strips that has been divided in half. What do you think will be created if you cut along the entire length of the line that divides it in half? Explain.

23. Cut along the line and precisely describe the result. Do you have any idea why this happened? Explain.

24. Now make cylinders out of one of each of the strips that have been divided in thirds, fourths, and fifths. What do you think will be created if you cut along the entire length of each of the lines that divide them into equal sections? Explain.
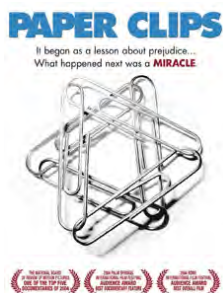
Figure 1.14: Poster for the documentary Paper Clips.

25. Cut along the lines and precisely describe the results. Do you have any idea why this happened?

26. Make a Möbius band out of the other strip that has been divided in half. What do you think will be created if you cut along the entire length of the line that divides it in half? Explain.

27. Cut along the line and precisely describe the result. Do you have any idea why this happened? Explain.

> **Caution**: Here and below you will need to use enough tape to sufficiently secure the ends of the strips when making the Mbius bands. If you don't, as you dissect them they will fall apart.

28. Make a Möbius band out of the other strip that has been divided into thirds. What do you think will be created if you cut along one of these lines and continue cutting until you have returned to your starting point? Explain.

29. Cut along the line until you return to your starting point and precisely describe the result. Do you have any idea why this happened? Explain.

30. Make a Möbius band out of the other strip that has been divided into fourths. What do you think will be created if you cut along one of the lines closest to an edge and continue cutting until you have returned to your starting point? Explain.

31. Cut along the line until you return to your starting point and precisely describe the result. Do you have any idea why this happened? Explain.

32. Now cut along the entire length of any remaining lines and precisely describe the result.

Comment: You should see that things are much more interesting than in the cylindrical case. To precisely describe the result you should include the number of bands created, the number of twists in each of these bands, the relative lengths and widths of these bands, whether they are linked or knotted together, and any other salient features you might find.

33. Make a Möbius band out of the other strip that has been divided into fifths. What do you think will be created if you cut along one of the lines closest to an edge and continue cutting until you have returned to your starting point? Explain.

34. Cut along the line until you return to your starting point and precisely describe the result. Do you have any idea why this happened? Explain.

35. Cut along the entire length of any remaining lines and precisely describe the result.

36. You should see patterns forming. State a conjecture which describes the result of cutting along one of the lines closest to an edge and continuing to cut until you have returned to your starting point on a Möbius band created from a strip which has been divided into m equal sections.

37. Similarly, state a conjecture which precisely describes the result of cutting along the entire length of all of the lines on a Möbius band created from a strip which has been divided into m equal sections.

38. In the investigations above we dissected the Möbius bands starting our cuts on lines closest to the edge(s) and later cutting along other lines. If we started with other lines would the final results be any different? Explain.

1.7 History of the Möbius Band

We hope that this is beginning to feel like a puzzle of sorts, something worthy of more investigation. Before we continue, you might be interested to know some history of this curious mathematical object.

The Möbius band was invented in 1858. The time was apparently ripe for its invention as it was discovered independently by both **Johann Benedict Listing** (Czech Mathematician; 1808 - 1882) and **August Ferdinand Möbius** (German Mathematician and Astronomer; 1790 - 1868). Obviously named after the latter, the Mbius band was first described in respective papers published in 1861 and 1865.

Surprising Möbius band properties, like those we discovering here, have long been popularized. While the book Möbius and His Band [Fau] might seem to be a book about the Möbius band itself, it is really, as the subtitle tells us, a book about the important impact of Möbius and his German contemporaries on nineteenth century mathematics and astronomy. Perhaps surprisingly, the first full-length book on the Möbius band just appeared only in 2006: Clifford Pickover's wonderful The Möbius Strip: Dr. August Möbius's Marvelous Band in Mathematics, Games, Literature, Art, Technology, and Cosmology [Pic]. If you are eager to learn more about the Möbius band's ubiquitous role in mathematics as well as its many occurrences in literature, the world of art, and technology of all sorts, you are encouraged to read this fine book.

OK, so now lets get back to unlocking the secrets of this puzzle. . .

1.8 A Calculus of Möbius Dissections

In Section 1.6 you discovered a pattern that describes what happens when you *dissect* both cylinders and Möbius bands created from strips divided into $2, 3, 4, \ldots$ equal sections. But what

happens when you dissect a 2-Möbius bands in this same way? And what happens with a 3-Möbius bands? 3- Möbius bands? And what about the general case, what happens when you dissect the general n-Möbius band in this same way?

This is your challenge for this section.

Before you continue your investigation we provide a few notes which might help you.

Educational Notes:

1. Completing this investigation will take quite a while - as long as a few hours. You will have to experiment dissecting many different Möbius bands.

2. It will be helpful to compare your work with others to help catch mistakes that might disrupt your search for useful patterns.

3. Think critically about what values of n and m might provide useful information.

4. Think about effective ways to organize your work.

5. As the number of twists and the number of equal sections increase the results of your dissections will become more complex. It is often helpful to use longer bands to help ease your construction and analysis. It also helps to have peers to help hold the bands and/or count the twists.

So now it's time for you to get to work. Here is the main focus of this section:

Independent Investigation: Determining a Calculus of Möbius Dissections

Completely characterize the result of dissecting an n-Möbius band created from a strip divided into m equal sections.

Get started and then come back if you need some help...

You're back. Most of our students make quite good progress for a while, but because this is a fairly big task they often need some help. So we have a few other notes that might be helpful now that you have worked on this for a bit.

Mathematical Notes:

1. Mathematicians are picky about their use of notation, but their consistency and organization are often quite helpful. Here we have used the variable n to stand for the number of twists in our Möbius bands and the variable m to stand for the number of equal sections the strip used to make our Möbius band has been divided into. The investigations we did above correspond to $n = 0$ and $m = 2$ (24 and 25), $n = 1$ and $m = 2$ (26 and 27), $n = 1$ and $m = 3$ (28 and 29), etc.

2. Remember as you work that the number of bands created, the number of twists in each of these bands, the relative lengths and widths of the bands, and whether the bands are linked or knotted together are all important.

3. The relative lengths and widths of the bands that are created in the dissection is what's important, not their actual dimensions. For ease of reference, it makes sense to refer to the length and width of the original Möbius band by L and W respectively.

4. If your bands become knotted it is important to note that they are knotted. However, you are not expected to identify the type of knot. Identifying and classifying knots is mathematically and historically a very interesting part of topology which is at the heart of Discovering the Art of Knot Theory for those of you that are interested in learning more.

5. As the resulting dissections "grow" knots or even become knotted together it becomes difficult to count the number of twists directly. While topologists do not generally allow cutting or pasting, they do have forms of *surgery* they practice. Carefully holding an untwisted section of a band between your left and right hand, a peer may cut the band, spread it apart to allow knotted sections of this or other bands to fall between the cut and then tape the cut sections together again. This surgery will help you gradually unknot things without introducing any new twists - if you are careful - and thereby allowing you to analyze configurations that otherwise would be inaccessible.

6. Similarly, if you have precisely described one band in a looped or knotted structure, you may simply excise it via surgery without affecting the properties of the remaining bands.

You're back again. Are you done? Here's one way to check.

Am I Done?
You are likely done if you can quickly answer each of the following questions:

1. What is the result of dissecting a 27- Möbius band created from a strip divided into 139 equal sections?

2. What is the result of dissecting a 59- Möbius band created from a strip divided into 18 equal sections?

3. What is the result of dissecting a 156- Möbius band created from a strip divided into 93 equal sections?

4. What is the result of dissecting a 234- Möbius band created from a strip divided into 5,322 equal sections?

Having completed this investigation you might wonder about the title we've given it. The term

calculus is often used to refer to the mathematical fields of *differential calculus* and *integral calculus* that were independently invented by **Isaac Newton** (English Mathematician and Physicist; 1642 - 1727) and **Gottfried Wilhelm Leibniz** (German Mathematician and Philosopher; 1646 - 1716) and which remain one of the most fundamental descriptive tools used by human beings to describe change over time. When you hear about AP Calculus in high school or Calculus in college, this is what is being referred to. If you are interested in learning more about these subjects please see Discovering the Art of Calculus in this series.

But why'd we use the term calculus here?

The word calculus actually has a broader meaning - "a method of computation or calculation within a symbolic system." This meaning comes from the Latin root calculare which means to calculate. One uses the phrase the calculus to differentiate the field made up of differential and integral calculus from other calculi such as *propositional calculus* which is a branch of mathematical logic.

39. Using this broader meaning, explain why it is appropriate for our Independent Investigation to be called a "Calculus of Möbius Dissections."

40. Now that you have a calculus of Möbius dissections, use it to explain why the order of the cuts (considered in Investigation 38) doesn't matter.

1.9 Möbius Band Applications

One application of the Möbius band is as a machine belt connecting a motor to a component to be rotated. Used in many machine shops, this application of Möbius bands was patented by B.F. Goodrich in 1957. [Gar2, p. 129]

41. Why would a Möbius belt be better than a cylindrical belt for use in machine shops?

Numerous other applications of the Möbius band are can be found and many are included in [Pic]. We include one more application here as it leads directly to the next model we will create.

Take two of your paper strips. Make two Möbius bands, each with a single twist. However, give one band a right-hand twist while you give the other one a left-hand twist. Don't worry about which is the right-twist and which is the left-twist, just make sure that once they are completed they appear to be fundamentally different Möbius bands. (See e.g. Fig. 15.) We'd like to see how these two Möbius bands are related.

42. Can you transform one of your Möbius bands into the other using translations, rotations, or the malleability of the paper?

43. Now consider the reflection of one of these Möbius bands by viewing it in a mirror. What do you notice?

Mathematicians have terms that describe the properties you have just seen:
A geometric object is called **chiral** if it cannot be transformed into its mirror image using rotations and/or translations in the *ambient space* in which it is considered. A chiral object and its mirror image are called **enantiomorphs** or **enantiomers**.

Since chirality is a geometric principle, involving only rotations and translations, we will stop thinking about topology for the remainder of this section.

Here's another example. As a rigid geometric object in two dimensional space (e.g. written on a piece of paper), the symbol ↻ is chiral. ↺ and ↻ are enantiomorphs. Notice that if we considered ↺ as a ridig geometric object in three dimensional space it would no longer be considered chiral.

44. For each of the letters given below, exactly as they appear in the given typeface, draw a mirror image:
A B C D E F G H I J K L M N O P Q
R S T U V W X Y Z.

45. Use Investigation 44 to determine which of the letters shown there, in the given typeface, are chiral in two dimensions. (Hint: There are 12.)

46. Show how you could make very minor modifications to the characters L and Q so they were no longer chiral.

47. Find some everyday objects that are chiral in two or three dimensions. Describe them and how you know they are chiral.

By now you may be wondering what the big deal about chirality is. It turns out to be important in many areas, including at the molecular level. It was first observed by **Louis Pasteur** (French Chemist and Biologist; 1822 - 1895). Topological issues such as chirality are studied throughout chemistry and molecular biology - particularly in the area of DNA. [Fla, p. ??]

A specific example arose in the late 1950's when women were given the drug Thalidomide to combat morning sickness. Horrifically, this caused thousands of devastating birth defects in babies. [Raj] The reason - Thalidomide of a specific chirality was tested, but when the drug was produced it was produced without regard to chirality. The mixed chiral drug that was sold had a different, devastating effect.

Sherri Finkbine (American Actress; 1932 -) was one woman whose pregnancy was horribly impacted by Thalidomide. Her choice regarding her pregnancy played an important role in the early political debate about abortion in the United States. Her story was told in a made for TV movie <u>A Private Matter</u> starring **Sissy Spacek** (American Actress; 1949 -).

Remarkably, more than 50 of the "world's top one hundred drugs are chiral, including such well-known medicines as Lipitor, Paxil, Zoloft, and Nexium." [Pick, p. 52] Thankfully, we have learned from this terrible example of Thalidomide and are aware of differing effects of topologically different molecules.

You're going to make something pleasant from your enantiomorphic Möbius bands, a creation of **Theoni Pappas** (American Mathematics Educator; circa 1940 -) [Pap, 2004].

Take your left- and right-handed Möbius bands and join them together at right angles, as shown in Fig. 16. Be sure to tape them together firmly on "both" sides.

48. How many sides does this new object have? Explain.

49. How many edges does this new object have? Explain.

50. Cut each of the Möbius bands making up this object longitudinally down their centers, cutting right through their common intersection. You should get two "lovely" linked objects. Describe them.

Figure 1.15: Enantiomorphic Möbius bands.

Figure 1.16: Joined enantiomorphic Möbius bands.

1.10   Möbius Band Icons

Awareness ribbons, like the AIDS and breast cancer awareness ribbon in Figure 1.17, are commonplace and are used to symbolize many different causes. What are the mathematical properties of such a ribbon?

To find out you simply have to make an awareness ribbon using one of your strips of paper. In constructing your ribbon, be careful to construct it so that:

- Its twist is the same as in Figure 1.17 below - the same face of the ribbon should be showing on both of the flat components.

- You trim the ends of the ribbon in the appropriate directions to make the points.

51. Release the ribbon and try to form it into a cylinder by joining parts of the edge together without overlapping. Can you? Explain.

52. Can you form the ribbon into a Mbius band by joining parts of the edge together without overlapping? If so, specify the number of twists that the ribbon must have.

An even more prominent contemporary icon is the **recycling symbol**. Shown in Figure 1.18, this symbol was created by a 23 year-old college student named **Gary Anderson** (American Graphic Designer and Architect; 1947 -) to help commemorate our nation's first Earth Day celebration on 22 April, 1970. One of the most universally recognized symbols of modern times, the symbol's history has only become more widely known recently. (Jones 1999 and Peterson 2003)

This prominent symbol is also a Möbius band! Try it. Make a Möbius band and, as illustrated in Figure 1.19 flatten it to make the recycling symbol.

53. What are some reasons the Möbius band might be an appropriate symbol for recycling?

54. What is it about the Möbius mascot pictured in Figure 1.13 that makes the name Möbius appropriate for this character?

Figure 1.17: AIDS and breast cancer awareness ribbons.

55. Make several Möbius bands with different numbers of twists. Which can be flattened into a triangular shape resembling Figure 1.18? State what you observe as a conjecture, specifying clearly what evidence you have for your conjecture.

With no licensing restrictions, there is not a single, universal recycling symbol. All sorts of artists have adapted the symbol. Pictured in Figure 1.20 below are three alternative recycling symbols.

56. Explain why the recycling symbols in Figure 1.20 are, topologically, fundamentally different than the "official" recycling symbol in Figure 1.18.

57. Using Investigation 56, determine what type of Möbius band one needs to form the "official" recycling symbol?

58. Using Investigation 56, determine what type of Möbius band one needs to form the "alternative" recycling symbols in Figure 1.20?

59. The "alternative" recycling symbols with the topological properties of Figure 1.20 are more commonplace than the original. Why do you think this might be so?

The final symbol we will consider was briefly described in the introductory section "Möbius Sightings." As noted, when the first author first saw this symbol, he immediately thought that the paper clip star in Figure 1.14 was constructed using Möbius bands. This symbol was created for the movie poster of the Holocaust documentary Paper Clips. This documentary [Paperclips, 2004] is an award-winning 2004 documentary about Whitwell (TN) Middle School's Holocaust study.
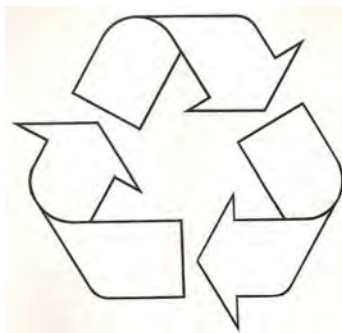
Figure 1.18: The recycling symbol.

Figure 1.19: Möbius band and flattened Möbius band.

To try to understand the numerical magnitude of the Holocaust genocide, middle school students in this small town attempted to collect 6 million paper clips. They chose paper clips because they were used by Norwegians as symbols of silent protest against the Holocaust. It is a profoundly moving documentary with deep lessons about teaching and learning.

Of course, the symbol on the movie poster represents the **Star of David** - the universal symbol of Judaism. Constructed from two superimposed equilateral triangles, this symbol has many other symbolic interpretations: the Seal of Solomon, a hexagram, a snowflake, and the Star of Bethlehem.

So what exactly does the paper clip star have to do with Möbius bands? Shown on the left in Figure 1.21 is a Möbius strip with three thin rectangles removed from its interior. This strip represents three paperclips chained together and then twisted into a Möbius band. Flattened into a recycling-type symbol, as in the center of Figure 1.21, one sees an exact analogue of the triangles, like those pictured in Figure 1.22, used to make up the paper clip star. This figure is also strikingly similar to the wonderful "Möbius Strip I" woodcarving of M.C. Escher, pictured on the right in Figure 1.21.

Figure 1.20: Alternative recycling symbols.

Figure 1.21: *Genus* 3 and flattened genus 3 Möbius bands; M.C. Escher's "Möbius Strip I".

1.11   The Paper Clip Möbius Microworld

Required Supplies: You will need the following physical materials for the investigations

- 2" uncoated paperclips

- Thin wire (e.g. crafting wire or unbraided picture hanging wire)

- Needle-nose pliers

So how can we make such a paperclip star?

Get some paperclips and chain them together three at a time. Have them face the same direction, for obvious aesthetic reasons, then chain them front to back to form paper clip Möbius bands in the shapes of triangles, as shown in Figures 1.22. Can you do it? We could, and did. However, it didn't help us build pleasing paper clip stars. These triangles are structurally unstable and there is nothing to hold the two triangles together without leaving an ugly, unstable mess.

What would a mathematician do? Well, when faced with a puzzle, they play. Eventually, somehow we made a single paper clip Möbius band that was stable: you could move it, but you could also set it down in a way that it formed a strong, structurally solid base. Now we were onto something. Alas, we couldn't seem to make another one! It was time to really analyze things once and for all: right or left chirality; one, or three, or five,... twists. What kind of paper clip Möbius band is best?

In Knot theory, the important branch of topology which makes up Discovering the Art of Knot Theory, *crossings* play a critical role. This is a natural way to analyze the paper clip Möbius bands, hereafter called **pc Möbi**. Three particular pc Möbi are shown in Figure 1.22.

Figure 1.22: The pc Mobi U-O-O, left, and two others.

Consider the pc Möbi in Figure 1.22. You should see that they are different. How might we name them? For consistency, we need some conventions. We will always position the pc Möbius so a paperclip runs horizontally along the bottom of the figure. Notice that where each pair of paper clips comes together there are two crossings. As illustrated in Figure 1.23, let's concentrate on the crossings the interior, travel in a clockwise direction, and let's always begin labeling our crossings at the top vertex. The interior length of the paper clip headed NE goes under the interior length of the paper clip headed SE. We'll call this an **under crossing** and have labeled it with a "U". Continuing around clockwise we see that at the next crossing the interior length of the SE paper clip goes over the interior length of the W paper clip, forming what we will call an **over crossing**. We label it with a "O". The third vertex has the W paper clip crossing over the NE paper clip; an over crossing which we label with a "O". So, we name this pc Möbius U - O - O.

Notice that we will not allow paperclips to be "folded over" one another. The pc Möbius in Figure 1.24 has an illegal configuration at the bottom left vertex - there are four crossings instead of two. The horizontal paperclip must be reflected, as shown in the figure.

60. Label each of the crossings of the pc Möbi shown in the center of Figure 1.22 and then determine the appropriate name for this pc Möbi.

61. Repeat Investigation 60 to label and name the pc Möbi on the right of Figure 1.22.

62. Show that each of the pc Möbi in Figure 1.22 can be transformed into any of the other pc Möbi in this figure via an appropriate rotation.
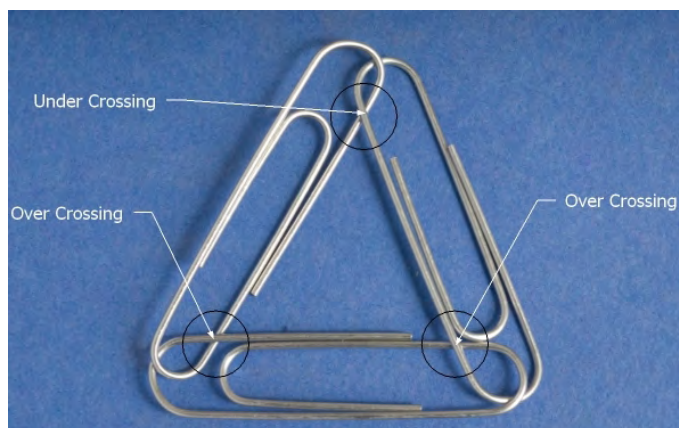
Figure 1.23: The crossings on a PC Möbi labelled; this is the U-O-O PC Möbi.

63. If we begin with a pc Möbi from Figure 1.22 and rotate it so the resulting pc Möbi has a paperclip running horizontally along the bottom of the figure, can the result be a pc Möbi any different than one of those in this same figure? Explain.

The properties you have described in Investigations 62 and 63 show that the three pc Möbi in Figure 1.22 form an ***equivalence class*** under rotations by 0, 120, and 240 degrees.

64. Using the naming scheme above, name all of the possible pc Möbi that can be made from three paperclips.

65. Divide all of the different pc Möbi into equivalence classes under rotation. How many different equivalence classes are needed account for all of the pc Möbi described in Investigation 64? Explain.

Of course, the symmetries of an equilateral triangle include reflections too, as shown in Figure 1.25.

Take 3 paperclips and make a pc Möbi. Don't worry too much about which one you made, just quickly make one. Odds are that it is an o-o-u, a u-u-o, or a rotated version of one of these as the other pc Möbi are harder to make.

66. By drawing and labeling appropriate diagrams, show what happens to the crossings of your pc Möbius as you reflect it via $R_{vertical}$.

67. Repeat 62) for the reflection R_{right}.

68. Repeat 62) for the reflection R_{left}.

69. Does your pc Möbius remain unchanged under all of these reflections?

70. Does your pc Möbius stay in the same equivalence class under these reflections?
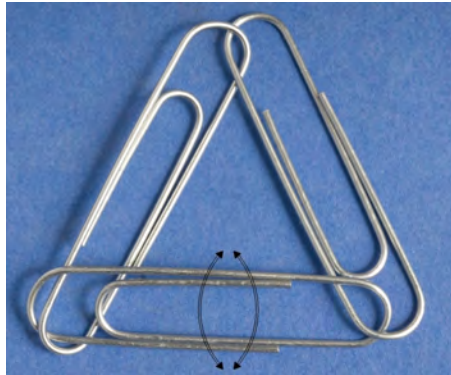
Figure 1.24: An illegal pc Möbi; the horizontal clip can be flipped down.
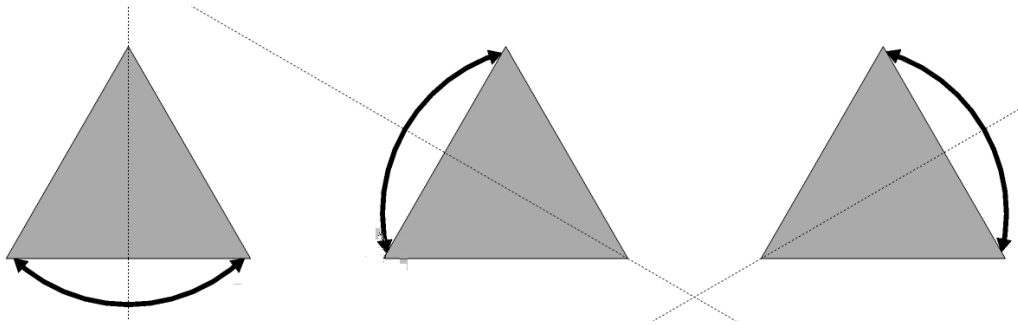
Figure 1.25: Reflections of an Equilateral Triangle; $R_{vertical}$, R_{right}, and R_{left} respectively.

Later, after you have learned how to make representatives from each of the equivalence classes, you will see that your answer to Investigation 70 holds for all pc Möbi.

As you hold your pc Möbius you should notice that you can often reflect a single paperclip about its long axis, as shown in Figure 1.26.

71. Reflect each of the paperclips in your pc Möbi which have opposite labels at the opposite ends. Describe what happens.

72. If there are any paperclips in your pc Möbi that have the same labels on opposite ends, show that reflecting this paperclip forms an illegal configuration.

The investigations above illustrate the utility of equivalence classes. They have partitioned the pc Möbi into disjoint sets that are invariant under rotations, reflection, and "legal paperclip reflections". In more concrete terms, what this shows is that once you have made a pc Möbius you cannot fundamentally change it without unclipping it.

There is also a very nice connection with Möbius bands here. One can check by experimenting that the only Möbius bands that can be flattened into triangles as in Figure 1.19 are those with

Figure 1.26: Reflecting a single clip in a pc Möbius.

1 or 3 twists. And we have the enantiomorphic pairs of left and right versions of each of these. I.e. our equivalence classes of pc Möbi correspond exactly to the equivalence classes of Möbius recycling symbols that can be made - those that you considered in Investigations 64 - 70.

1.12   Building PC Möbi

Using the results of our analysis in the previous section, we can no build all of the topologically distinct pc Möbi that you classified above. This is, in fact, the key to building the paperclip stars.

With a bit of practice, the steps below should allow you to make any pc Möbi that you want in under a minute.

1. Decide exactly which pc Möbi you want to create.

2. Chain three paper clips together end to end in the same direction.

3. Form the chained paperclips into the shape of a triangle with the unlinked vertex at the lower right.

4. Adjust the lower left vertex so it has the appropriate crossing you desire. If the crossing is not correct, simply reflect the horizontal paperclip along its long axis and this will change the crossing from over to under or under to over.

5. Adjust the top vertex so it has the appropriate crossing you desire. As in Step 4, if the crossing is not correct, just reflect the paperclip running southeast over its long axis and this will change the crossing from over to under or under to over.

6. Chain on at the lower right with the desired crossing. This can be accomplished as illustrated in Figure 1.27 which illustrates all possible situations and desired outcomes.

Slide open arm up through bottom for **Inner open arm → over crossing.**	Slide open arm down through top for **Inner open arm → under crossing.**
Slide open arm down through top for **Outer open arm → over crossing.**	Slide entire clip up through bottom then link open arm for **Outer open arm → under crossing.**
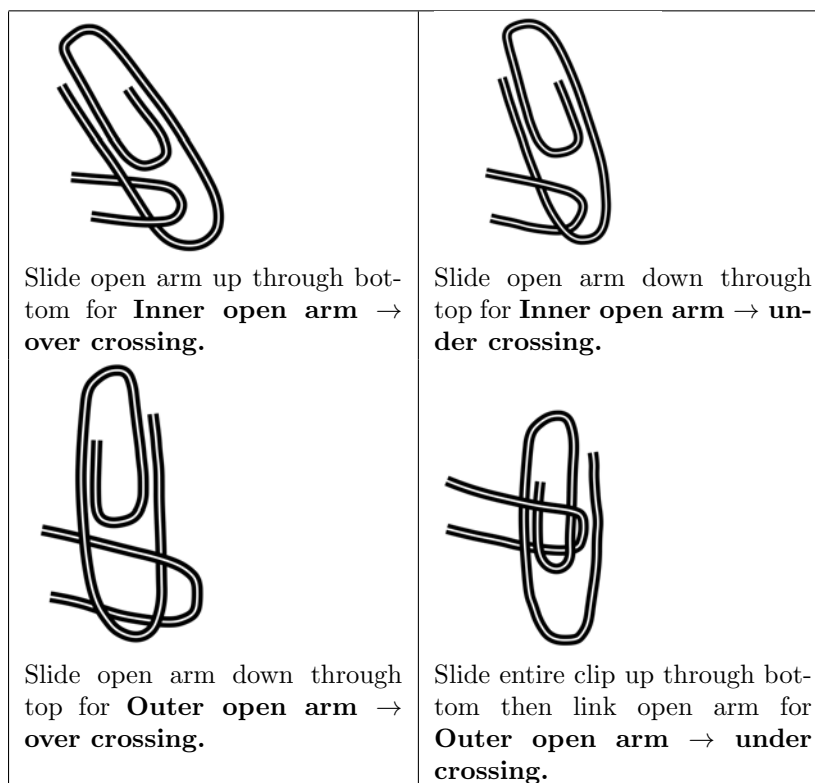
Figure 1.27: Creating desired final crossings in a PC Mobi.

> **Building Task 1** - Make an O-O-U pc Möbi, an U-U-O pc Möbi, an O-O-O pc Möbi, and an U-U-U pc Möbi.

1.13   Paper Clip Stars

Now you're almost ready to make your star. If you manipulate the O-O-O or U-U-U pc Möbi you will see that they have a structural stability that all of the others lack. Indeed, this is why it was harder to make the former than the latter.

Here's what you do to make a paper clip star: Make two pc Möbi, both O-O-O, both U-U-U, or one of each. Notice that each is stable in both *concave up* or *concave down* orientations. Orient both with the same concavity. Then weave one pc Möbi through the center openings of the paper clips that make up the other. The latter will momentarily loose its concavity. When you are done, a little pressure on the latter returns its concavity and provides the needed structural stability! To make this stability permanent simply use 3 small pieces of wire to connect the two pc Möbi.

> **Building Task 2** - Build a paper clip star.

You now have your own piece of mathematical art, a paper clip star made from Möbius bands

that resembles those in Figure 1.28.
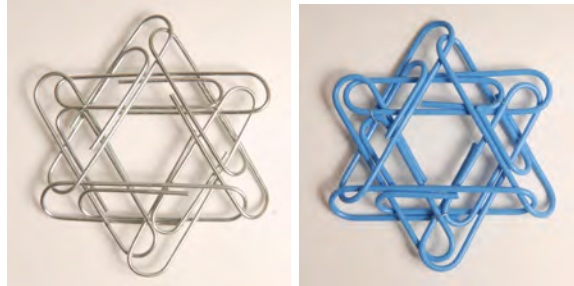
Figure 1.28: Paper clip stars from Möbi.

You get to decide what your star symbolizes. With the use of different color paperclips, which are a little harder to work with because of their added thickness, we can come full circle back to the documentary "Paper Clips" by creating paperclip models of the inmate markings worn by prisoners in the Concentration Camps during the Holocaust. Several are shown in Figure 1.29.

Figure 1.29: Paper clip stars from pc Möbi which model those worn in the Concentration Camps by homosexual male Jews, all Jews, and Jewish political enemies respectively.

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[Ric] Lloyd Richardson, jr., "The Möbius strip: An elementary exercise providing hypotheses formation and perceptual proof," Arithmetic Teacher 23: 2 (1976): 127-9.

1.15   Connections

1.15.1   Geography

The map in Figure 1.30 is called a **Mercator projection** map of the world.

73. Why is the Mercator projection so widely used? I.e. what properties does it have that makes it a particularly important map?

74. Provide a rough, qualitative description of how the Mercator projection is created.

75. Despite its importance, there are major distortions when one uses such a map to represent the surface of the earth. What are some of these distortions.

76. Find a different kind of map projection. For this projection:

   - Show how this projection represents the world in a map.

   - Describe how this projection is created.

   - Describe the important physical characteristics that this projection correctly represents.

   - Describe important physical characteristics that this projection distorts.

77. Figure 1.31 shows an "upside" down map of the United States. Does this alternative map bother you? Why?

78. Find another alternative map that represents the area in which it maps in a way that challenges your perceptions that the typical map of this area has engrained in you.

79. Repeat Investigation **78** for a different kind of alternative map.

The maps in Figure 1.32 - Figure 1.35 are **cartograms** where the area of each country is distorted to illustrate a certain property of that country. For example, a **population cartogram** is a map such that each country's area is proportional to its population. Countries like Japan with enormous populations are shown to be large in the map despite their small geographical size.

80. Find five different cartograms that you think are valuable in illustrating certain data. For each, identify the type of data used to construct the cartogram and then, in several sentences, describe why you think this is a useful cartogram. Be sure to choose diverse cartograms, representing different social, economic, physical, political, judicial, religous, etc., data.

81. The cartograms below are based on HIV/AIDS rates, greenhouse gas emissions, child mortality rate and gross domestic product. For each, identify which data you believe generates which cartogram. Give precise reasons for your answers.
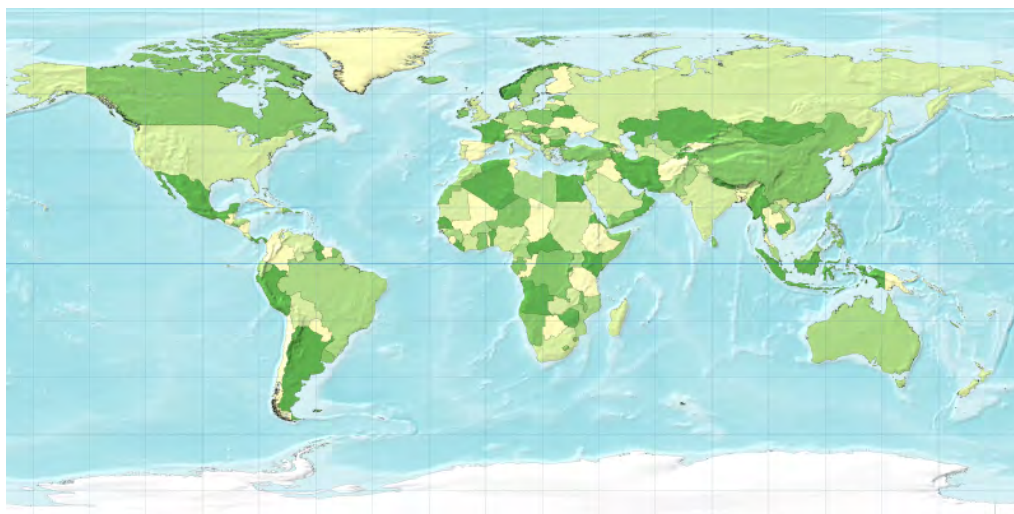
Figure 1.30: Mercator projection of the world.

1.15.2 Physiology

Figure 1.36 below is a representation of a human body that has been topologically deformed to illustrate a certain *physiological* mechanism. In other words, it is a analogue of the cartogram but its subject is human function.

82. What physiological mechanism do you think is represented by this model? Explain in detail.

83. For the mechanism that you have chosen, describe important differences that would appear in a female model.

84. Choose another physiological mechanism and describe the analogous model that it would generate.

Portraits of the Earth: A Mathematician Looks at Maps, Timothy G. Feeman, American Mathematical Society, 2002.]

Figure 1.31: Mercator projection of part of North America.
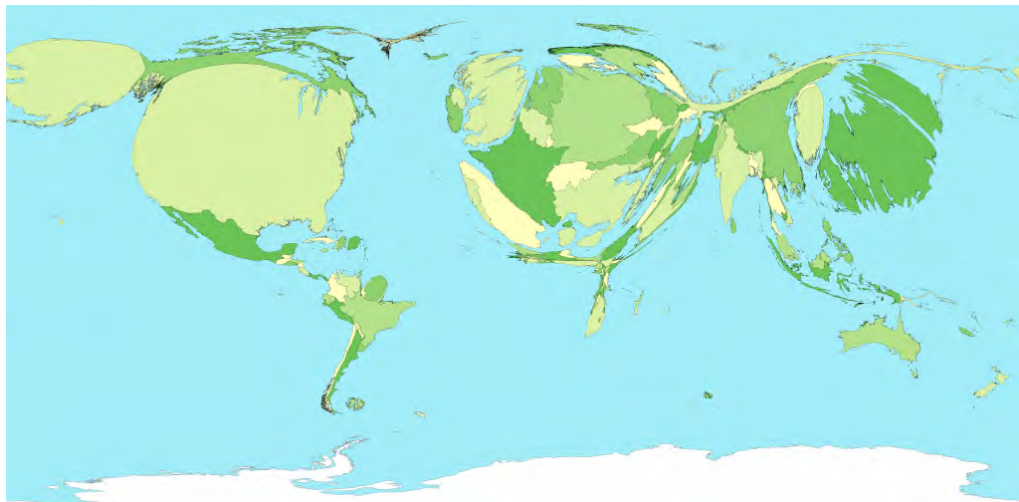
Figure 1.32: World cartogram of ...
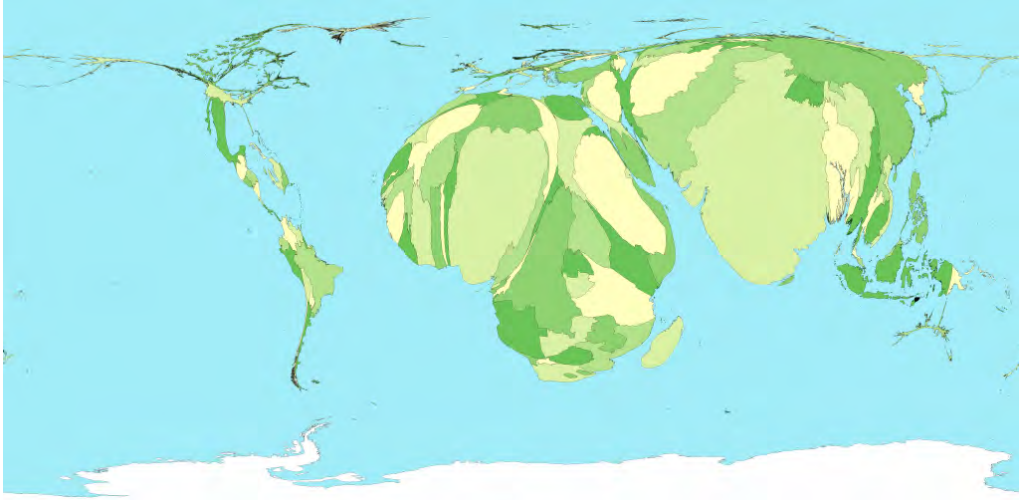
Figure 1.33: World cartogram of ...
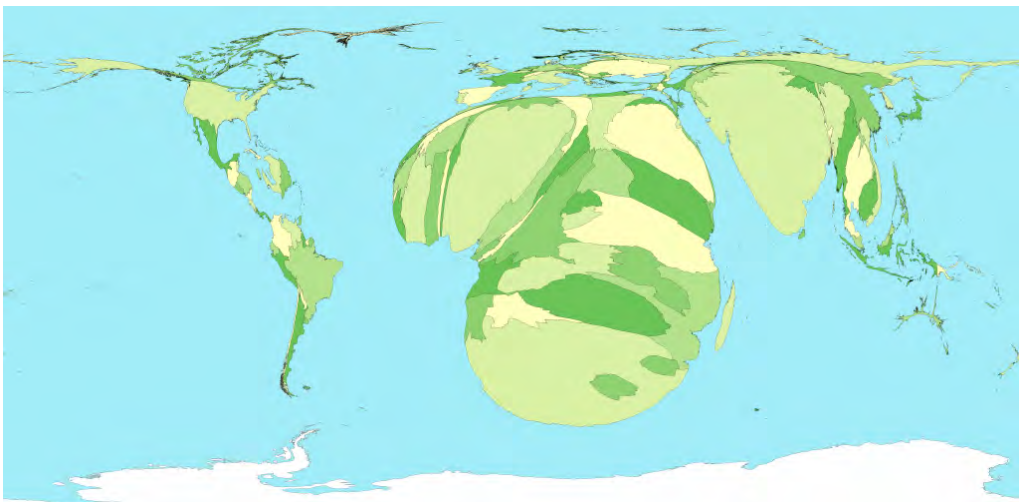
Figure 1.34: World cartogram of ...
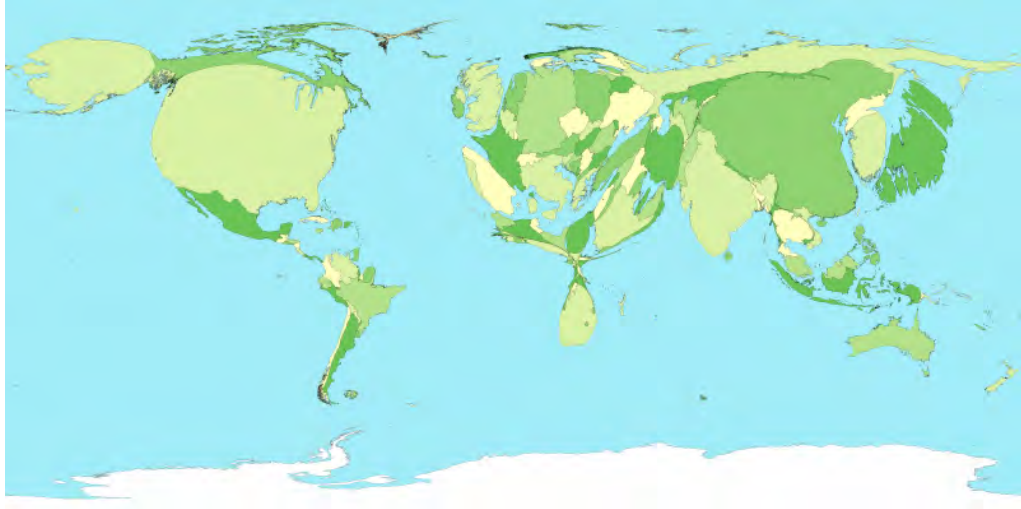
Figure 1.35: World cartogram of ...

Figure 1.36: A ... (Hint: 10 letters, starts with h.)

Chapter 2

Flexagons, Magic, Genius, and the Nobel Prize

> I don't know anything, but I do know that everything is interesting if you go into it deeply enough.
>
> **Richard Feynman** (American Phsyicist; -)

> I learned from her [my mother] that the highest forms of understanding we can achieve are laughter and human compassion.
>
> **Richard Feynman** (; -)

2.1 Richard Feynman

> In questions of science the authority of a thousand is not worth the humble reasoning of a single individual.
>
> **Galilei Galileo** (Italian Physicist, Astronomer, Mathematician, and Philsopher; 1564 - 1642)

Richard Feynman was an American Physicist born on 11 May, 1918 in Far Rockaway, Queens, New York. As a child he did not speak until he reached the age of three. His parents were particularly nurturing, his mother Lucille's support taking more social and human forms while his father Melville's support was more intellectual and scientific. Always encouraging his son to question authority, Feynman did fulfill his father's dream of having his son become a scientist. Indeed, he was one of the preeminent twentieth century physicists.

He was also a wonderful character who loved exploring all facets of the world - natural, scientific, cultural, artistic, and historic. Later in life he wrote two widely read popular books Surely You're Joking Mr. Feynman and What Do **You** Care What Other People Think?. As you might tell from the titles, they are spirited reads which recount adventures of all sorts. The celebrated **Freeman Dyson** (; -) has called the former book "the best antitode against academic snobbery."[1] They are highly recommended.

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[1]From From Eros to Gaia, Pantheon Books, 1992, p. 311.

In addition to his role as an important physicist and teacher of physics, Feynman was fairly well-known for his role in the inquiry into the space shuttle Challenger disaster. The space shuttle Challenger departed on its ninth mission at 11:38 a.m. E.S.T on 28 January, 1986. It had a crew of seven, including **Sharon Christa Corrigan McAuliffe** (American Teacher and Astronaut; 1948 - 1986) who was to become the first teacher in space as part of the Teacher in Space Program. Because of her role, children all over the country watched the liftoff live in schools. 73 seconds after liftoff Challenger broke apart in a ball of smoke. All seven astonauts were killed. Shortly after this disaster a Presidential Commission on the Space Shuttle Challenger Accident, also known as the Rogers Commission, was formed. Feynman was one of twelve members of this commission.

Described in detail by Feynman in the second part of What Do **You** Care What Other People Think? the commission found itself nearly overwhelmed by political posturing, people more concerned with public relations than truth, lawyers protecting many of the contractors, and NASA officials worrying more about losing funding than determining the cause of the disaster. As a scientist, Feynman made great headway serving as "gumshoe" who was perfectly at home investigating in the labs, in the factories, and with the engineers and technicians that were actually involved in the day to day shuttle development, design, and operation. Feynman, with the help of others like **Roger M. Boisjoly** (American Engineer; 1938 -) who had warned not to launch Challenger and was later awarded the Prize for Scientific Freedom and Responsibility by the American Association for the Advancement of Science for his efforts, focussed on rubber *O-rings* that helped seal joints in the fuel tanks under the great stress of liftoff. It was 36 degrees Fahrenheit when the launch took place, 15 degrees lower than any previous launch.

In a televised moment that eloquently illustrated the great power of the scientific reasoning of a single individual, Feynman interrupted the Senate hearings on the disaster to dip one of the O-rings in a glass of icewater. The O-ring had been clamped with a C-clamp that he bought at a hardware store on his way to the hearings. He then announced the following to the distinguished audience:

> I took this stuff that I got out of your seal and I put it in ice water, and I discovered that when you put some pressure on it for a while and then undo it it doesn't stretch back. It stays the same dimension. In other words, for a few seconds at least and more seconds than that, there is no resilience in this particular material when it is at a temperature of 32 degrees. I believe that has some significance to our problem.
>
> (; -
>
>)

The generally outspoken Feynman underspoke on this day. The O-ring and the temperature at launch were precidely the problems that doomed the Challenger. Short videos of this event can be found online.

Freeman Dyson recalls this event as follows:

> His finest hour as a communicator came when, already mortally ill, he performed an expiement at a televisted sessions of the shuttle inquiry... The public saw with their own eyes how science is done, how a great scientist thinks with his hands, how nature gives a clear answer when a scientists asks her a clear question.[2]

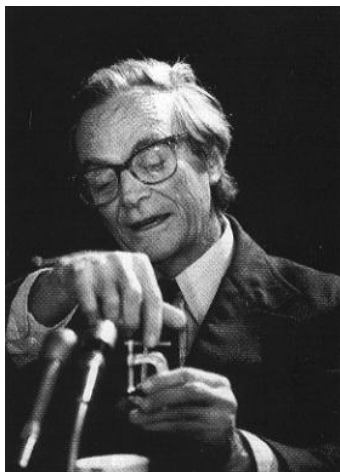[2]From From Eros to Gaia, Pantheon Books, 1992, p. 312.

Figure 2.1: Richard Feynman and the cold O-rings in the Senate hearings.

(; -

)

2.2 Learning as Play

> I don't know what's the matter with people: they don't learn by understanding, they learn by some other way - by rote or something. Their knowledge is so fragile!
>
> **Richard Feynman** (; -)

Certainly Feynman's expertise as a scientist was important in his role on the Rogers Commission. However, if you look at final result - an O-ring clamped with a $2.00 C-clamp and a glass of ice-water - it becomes clear that it was a more practical form of knowledge that prompted the penultimate conclusion. In Surely You're Joking Mr. Feynman Feynman describes in elloquent detail the way in which he learned by playing, exploring, discovering, inventing, and understanding - always with his father nearby nurturing but not telling. It is hands-on type of learning, of understanding the world around you, that lead to the o-rings.

This type of learning is compatible with the discussion of what's worth knowing in the opening chapter of Discovering the Art of Mathematics: A Student Tool. It is unclear what lessons at the hand of the world with his father at his side may have lead to the understanding he needed to focus on the o-ring. Yet it is the same sort of serendipity as Barbara Shipman's discovery of the relationship between the waggle dance and 6-dimensional flag manifolds that are described in the Student Toolbox. And, it is certainly exactly the type of discovery that Feynman's father would be so very proud of.

Many view science and mathematics as rote tasks - but to the scientist and the mathematician their understanding of their subject is more akin to a puzzle or a game. It is a mystery to be solved. Scientists and mathematicians are quite fond of puzzles and games, and it would not surprise them

to see a whole volume. Discovering the Art of Mathematical Puzzles and Games in a series such as this. Rubik's Cube, Sudoku, the Fifteen Puzzle, and Peg Solitaire are mathematical - they are miniature "microworlds" for mathematical exploration.

As children many of you probably made "Cootie-Catchers", an origami-like object that is manipulated with your fingers to tell fortunes:

Figure 2.2: Cootie Catcher

Cootie Catchers have been around almost as long as paper has been widely available and are common throughout most North American, European, and Far Eastern countries.

In this chapter we will investigate a very similar object, the *Flexagon* one with deep mathematical properties that was developed through a similar spirit of play that probably lead to the development of the Cootie Catcher.

2.3 Flexagons

In 1939, **Arthur H. Stone** (English Mathematician; 1916 -) was a graduate student in Mathematics at Princeton University, have arrived from England not long before. The American paper he got for his English three-ring binder was about an inch too wide. He was in the habit of tearing off this extra paper and folding it. One of the objects he built became known as the hexaflexagon, which you will build and analyze below. He was so fascinated with this object he set to work showing his peers. They became quite a hit in the Department of Mathematics - Princeton University's own version of a Cootie Catcher rage! Soon a "Flexagon Committee" was formed, including Stone, **John W. Tukey** (American Mathematician; 1915 - 2000) - then a lecturer in mathematics, **Bryant Tuckerman** (American Mathematician; 1915 - 2002) - then a graduate student in mathematics, and Feynman - then a graduate student in physics. The Committee invented all sorts of flexagons and worked out all of their mathematical properties before the Second World War broke up the group.

In the 1950's Tuckerman demonstrated flexagons for winners of a science talent contest each year and they were subsequently resurrected by the popular mathematical author **Martin Gardner** (; -). They have been in wide circulation since then and there have been several books written about them, including: Flexagons Inside Out by Les Pook.

2.4   Investigations

1. Find out a few interesting things about each of the members of the flexagon committee. In particular, determine whether each of these people lead productive, successful lives.

2.5   Making and Flexing a Flexagon

We will begin by making a **Hexahexaflexagon**. You will need a perfectly rectangular strip of paper that is about 12 times as long as it is tall. Adding machine tape of width between 2 and 3 inches works perfectly. Alternatively, you can carefully cut a strip of paper that is $1\frac{3}{8}$ inches wide and 17 inches long from a sheet of 11 inch by 17 inch paper.

You must fold your strip into a series of 19 *congruent equilateral triangles* that are exactly adjacent to one another, as shown in the top image in Figure 2.3. To make these folds perfectly, each fold will need to be exactly 60 degrees. The **Folding tool for Flexagons** in Figure 2.8 should be quite helpful.

2. Describe what geometric principles you used to help insure that the folds in your strip were correct.
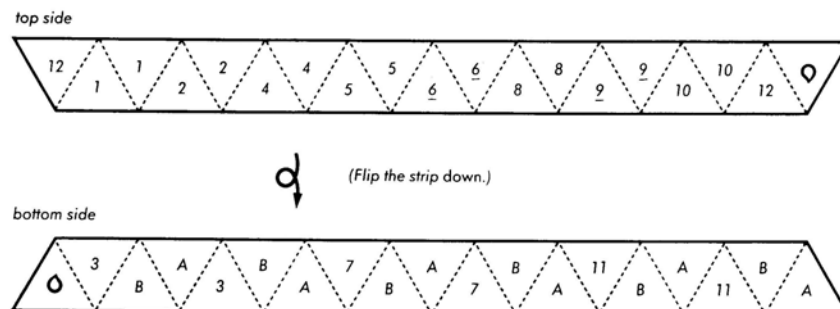
Figure 2.3: Pattern for making a Hexahexaflexagon

Once you have your 19 triangle folds, clip off the extra ends and fold each of the folds in the opposite direction so the folds can operate in either direction. Orient your strip exactly as in the upper image in Figure 2.3. In pencil, copy the labels onto your strips as indicated. Then flip your strip over toward you, so it is oriented exactly as in the lower image in Figure 2.3, and copy the appropriate labels onto the other side.

You are now ready to make your Hexahexaflexagon. Fold your strip once so that the two faces labelled "1" are touching. Then fold your strip once so that the two faces labelled "2" are touching. Continue folding in this way until all of the faces labelled "1" - "11" have been folded onto each other. At this point the only labelled faces you should see are those labelled "A", "B", "1", and the two droplet symbols for glue. Your strip should be folded into a shape that is almost hexagonal. Arrange the shape so the two faces labelled "12" are touching and the two remaining

43

faces with the droplet symbols can be folded on to each other. Spread a small amount of glue onto the faces with the droplet symbol and glue these faces together. Your Hexahexaflexagon should be in the shape of a hexagon with all faces on one side of the Hexahexaflexagon labelled "A", the opposite side labelled "B".

Now you have to learn how to flex your flexagon. Once you see it and try it a few times, it's easy. If you don't have somebody to show you, there are numerous videos available on YouTube. In words it is a bit more cumbersome, but not too bad:

Flexing a Flexagon

- Hold your Hexahexaflexagon flat.

- Fold one creased edge up, into a **mountain fold**.

- Fold the adjacent creased edge down, into a **valley fold**.

- Repeat the two previous steps each twice more so you have mountain and valley folds alternating around the six creased edges.

- If you push the valley folds together, your flexigon will have folded up into a 3-pointed star; a shape that looks almost like a Y from above.

- With your thumbs grab the point closest to you where the three valleys have come together. You should see that if you pull this point away from the center that the edges fold back down into a flattened hexagon.

This move is called a **flex** of the Hexahexaflexagon.

2.6   Flexagon Faces

3. As you flex your hexa-hexaflexagon (henceforward simply referred to as a flexagon), what happens to some of the faces?

4. What is happening to the flexagon that allows the faces to disappear and reappear like this?

5. How many different faces do you seem to find as you first start flexing your flexagon?

6. Do you think that there might be more faces that can be found simply by flexing? Try to find some and then list those that you have found by listing the markings on the faces.

7. You should see several patterns in the markings on the faces that you have found. Describe these patterns.

8. Look back at the original strip that you used to make your flexagon. Based on the markings on this original strip, can you guess what other faces should be possible to make by flexing your flexagon?

9. Return again to the original strip you used to create your flexagon. Count how many faces make up this strip. Compare this number with the number of triangular faces that appears

on each face of the flexagon. These quantities should suggest how many different faces can appear on your flexagon. Explain.

10. Compare your answers to Investigations 8 and 9.

11. Can you explain why the proper name for this flexagon is a hexa-hexaflexagon based on your investigations so far? Explain.

There are two kinds of geniuses, the "ordinary" and the "magicians." An ordinary genius is a fellw that you and I would be just as good as, if we were only many times better. There is no mystery as to how his [her] mind works. Once we understand what they have done, we feel certain that we, too, could have done it. It is different with the magicians Even after we understand what they have done, the process by which they have done it is completely dark Richard Feynman is a magician of the highest order.[3]

Mark Kac (Polish Mathematician; 1914 - 1984)

Continue to experiment for a while with your flexagon. See what patterns you can find, what you can discover that might help you figure out the secret of the symmetries of the hexa-hexaflexagon. Along the way you should be able to answer some of the following questions:

12. In a given state, how many different ways are there that you can flex your flexagon?

13. In a given state, how many layers thick, including the top layer, are the sections under each triangular face in your flexagon? (Count the two glued layers as a single layer.)

14. How is the answer to Investigation 13 number related to the proper name your flexagon?

15. How are Investigations 12 and 13 related? Explain.

16. If you end up with a given collection of symbols showing on the face of your flexagon (e.g. six As), will the state of your flexagon be the same, with the symbols on the face appearing in exactly the same way, each time these symbols form a face on your flexagon? Explain.

When you want some help, the next section has a few moves that might help you find out more about the patterns.

> **Warning:** You should only flex your flexagon in the standard way described above. You cannot "pull" or "weave" sections through each other to form different face combinations. It may be an interesting mathematical problem to analyze this new situation, but it is beyond our reach and will make the subsequent guided discovery mute. So we will not allow this right now. You are invited to try it on your own if you want - just be ready to end up with a mangled flexagon that you can no longer amaze your friends with.

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[3]Quoted in <u>Genius</u> by James Gleick.

2.7 3-Cycles

Hold your flexagon at a specific vertex and flex it. Keeping track of the vertex you flexed at, hold your flexagon on an adjacent vertex and flex it at this next vertex in the same way. Continue to do this, always moving to an adjacent vertex (in the same order, either always clockwise or always counter-clockwise) and flexing.

17. Write down the sequence of faces that you see when you flex your vertex in this way.

18. After a few steps, something surprising happens. What is this? Does it continue to happen indefinitely?

19. How might you draw a diagram that represents the flexagon states that you have just described in the pattern in Investigation 17?

2.8 Getting Stuck

Having flexed the flexagon while moving to adjacent vertices at each stage, you might wonder what happens when you flex your flexagon while staying at the same vertex. Do this.

20. Write down the sequence of faces that you see when you flex the vertex in this way, staying at the same vertex.

21. After several steps something surprising happens. What is this?

22. Since you can't flex, what might you do next?

23. Start flexing again. What do you notice?

24. You should now be able to find all of the faces of the flexagon. Explain how you do this.

2.9 The Hidden Patterns of the Hexa-hexaflexagon

We would like to determine how we can move from face to face to find any face that we want in a predictable way. It is a bit harder to find patterns when the faces have such convoluted markings on them. So let's color them. Pick several different colors (you should know how many you might need from above) and color each face of your flexagon with a different color as they appear in your flexing.

   If we could find a map that tells us how to travel around all of the possible faces by flexing then we will have unlocked the secret.

> **Project:** Use the investigations you have completed so far to help you try to unlock the secret of the hexa-hexaflexagon by collecting data. You should do this by trying to look for patterns, sequences of faces, and/or strategies while you flip the flexagon. You may want to keep track of your data using trees, lists, networks, spreadsheets, or other means. You may want to keep track of top and bottom colors, the number of layers, symbol orientations, or whether a given state forces a specific move or whether you have choices. This should take quite some time to do - an hour or more. This is a critical part of the process. If you just jump ahead you will be cheated from the real magic of discovery.

In deductivist style, all propositions are true and all inferences valid. Mathematics is presented as an ever-increasing set of eternal, immutable truths. Counterexamples, refutations, criticism cannot possibly enter. An authoritarian air is secured for the subject Deductivist style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem are doomed to oblivion while the end result is exalted into sacred infallibility.[4]

<div align="right">

Imre Lakatos (Polish Mathematician and Philosopher; 1922 - 1974)

</div>

2.10   Other Flexagons

There are many other flexagons. There is the tri-hexaflexagon, the tetraflexagon, dodeca-hexaflexagon, and a whole zoo of others. My favorite is the Pirate Tri-hexaflexagon invented by Peter Hilton, Jean Pedersen, and Hans Walser. This flexagon is created by folding the following strip:
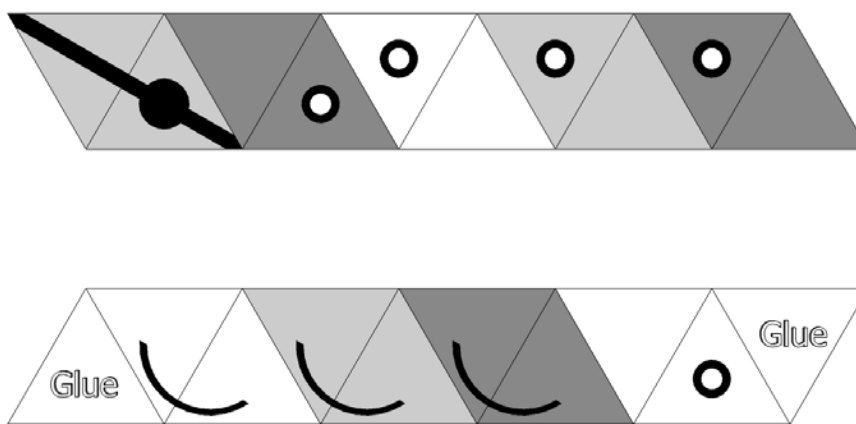
Figure 2.4: Pirate Trihexaflexagon template.

---

[4]From <u>Proofs and Refutations</u>

Not only is the Pirate Tri-hexaflexagon quite cute, it is a simpler version of the hexa-hexaflexagon. As the famous mathematician George Polya, who wrote so eloquently about problem solving, reminded us:

> If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problems? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem?[5]

(; -

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So maybe by analyzing this simpler example we can help nurture our investigation of the hexa-hexaflexagon.

Create the Pirate Tri-hexaflexagon.

25. Flex the Pirate tri-hexaflexagon repeatedly. How many different faces do you see?

26. What do you notice about the order in which the faces occur?

27. Now turn the flexagon over and begin flexing it. What do you see now?

28. What pattern do you see?

Notice that there are happy Pirates, with a patch over the left eye at times, and sad Pirates, with a patch over the right eye at times. These are different orientations of the tri-hexaflexagon. This is important to remember when returning to the hexa-hexaflexagon:

> **Warning:** The flexagon, like many other mathematical objects, has an orientation. When you begin writing down patterns, sequences of flexes, and diagrams which represent the relationship between flexagon states all of these things will be specific to the orientation of your flexagon. The orientation is reversed simply by flipping the flexagon over!! If something that you previously understood seems not to be working it could be that you inadvertently flipped your flexagon over. Unfortunately, there is no easy visual way to tell which orientation you are in at a given time without devising an analogue of the happy/sad Pirate for our hexa-hexaflexagon. So be careful.

29. How many layers are there on adjacent triangular faces on the tri-hexaflexagon?

Suppose that you wanted to build a dodeca-hexaflexagon. Use what you have observed about the tri-hexaflexagon and hexa-hexaflexagon to answer the following questions:

30. How many different triangular faces would there be on each side of the strip that is used in constructing the dodeca-hexaflexagon?

31. How many layers would there be on adjacent triangular faces on the dodeca-hexaflexagon?

---

[5]From How to Solve It.

2.11   Flexagons' Hidden Structure - Möbius Bands

The physical structure of flexagons that enable them to have their magical properties lie with another amazing mathematical object, the Möbius band. We briefly not this connection here.

If you have already completed the chapter on Möbius bands skip to Investigation 35.

You will need a few sheets of standard copy paper, scissors, and tape. Cut a strip of paper about $\frac{3}{4}$ inch wide and 11 inches long. Draw a single line down the center of this strip. Bring the two ends of this strip together to form a loop, matching up the two ends of your line and insuring that there are no twists in your strip. Now place three half-twists in your loop by turning one end of the strip over three times in succession in the same direction. Then firmly tape the ends together. The resulting object should look like figure 1a) below:

Fig. 1 - Three-twist Möbius bands

By carefully positioning each of the three half-twists equally around this three-twist Möbius band you can flatten your band down into a figure that is identical to figure 1b) or figure 1c). The only difference between these objects is their orientations - Möbius bands are chiral, any Möbius band and together with its mirror image make up enantiomorphs, a pair of distinct objects. Many pharmacological molecules are chiral, an issue of profound importance in medicine.

Möbius bands have many remarkable properties that are investigated in Discovering the Art of Patterns in this series. But just to give a bit of context we note that the recycling symbol is based on the Möbius band:

Fig. 5 - Recycling Symbol

32. Structurally the two recycling symbols in Fig. 5 are different. Explain how they are different.

33. Which of the symbols above has the same structure, up to enantiomorphy, as the flattened Möbius band that you made? Explain.

34. As a graphic designer, which recycling symbol would be easier to create? Explain.

Actually, it was the harder of the two symbols in Fig. 5 that is the "real" recycling symbol that was created by a 23 year-old college student named Gary Anderson to help commemorate our nation's first Earth Day celebration on 22 April, 1970. Many who design their own recycling symbols miss this distinction and create the simpler one.

35. Make a few more flattened, three-twist Möbius bands. Use paper strips that are 1" wide and 11" long, 1 1/4" wide and 11" long, and 1 1/2" wide and 11" long. How do these objects compare to the original that you made?

36. Keeping a length of 11", is there any limit to how wide you can make your strip and still create a flattened, three-twist Möbius? Explain.

37. Make a flattened, three-twist Möbius out of a strip of paper that is exactly 2" wide and 10 1/2" long. (It is easiest just to overlap " of the length of an 11" strip.) Describe precisely how this object is related to your hexaflexagon.
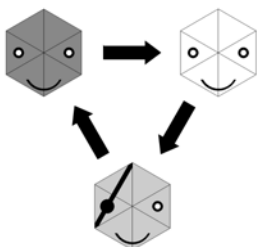
Figure 2.5: The *Tuckerman traverse* for the Pirate trihexaflexagon.

2.12 The Tuckerman Traverse

A simple diagram can be used to show the sequence of faces that appear in the Happy Pirate:

38. Make an analogous diagram for the Sad Pirate.

39. Use this approach to make a diagram which illustrates the sequence of faces that appears as you traverse a specific 3-cycle on your hexahexaflexagon.

40. As you go through the 3-cycle in Investigation 39, determine which states have required moves and which can bring you to two different states. Label these appropriately on your diagram.

41. Move to a state on the 3-cycle in Investigation 39 where you have a choice of the next state. Flex the flexagon so you will move out of the 3-cycle you have been in. Expand your diagram appropriately to include the subsequent states you find yourself in, continuing around a different 3-cycle if you can.

42. Repeat Investigation 41 several more times.

43. Eventually your diagram will become complete. Draw your diagram and explain in detail how you can use it to navigate the entirety of the flexagon.

44. Explain how you know your diagram is complete.

The diagram you have found is called the **Tuckerman traverse** for the hexa-hexaflexagon.

2.13 Face Colorings and Face Configurations

The Tuckerman traverse completely unlocks the secret of the hexa-hexaflexagon in terms of the way faces arise as the flexagon is flexed in different ways. Use your diagram to consider the following questions.

45. Suppose that you wanted to flex your flexagon to show all 6 of the different face colorings. What is the minimum number of flexes you can accomplish this in?

46. Is there a requirement where you should start to make such a *minimal flexagon face coloring tour*? Explain.

47. Suppose you wanted to flex your flexagon to show all 9 of the different face configurations that are possible in a given orientation. What is the minimum number of flexes you can accomplish this in?

48. Is there are requirement where you should start to make such a *minimal flexagon face configuration tour*? Explain.

2.14   Flexagons and the Nobel Prize

[Feynman] could not, or would not, distinguish between the prestigious problems of elementary particle physics and the apparently humbler everyday questions that seemed to belong to an earlier age. No other physicist since Einstein so ecumenically accepted the challenge of all nature's riddles.[6]

James Gleick (; -)

The study of flexagons lead to the Tuckerman traverse. This diagram was a forerunner to the Richard Feynman's invention many years later of the *Feynman diagram*. Such diagrams are schematic diagrams which illustrate the states of particles in subatomic particle physics and quantum field theory. One such diagram is shown in Figure 2.6.
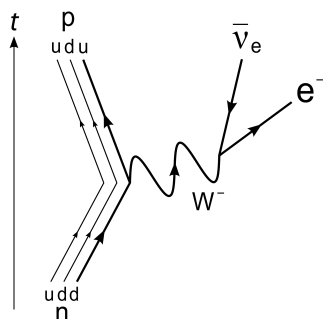
Figure 2.6: A *Feynman diagram* for beta decay.

In 1965 Richard Feynman won the Nobel Prize in physics for his work on quantum electrodynamics. Feynman diagrams played a central role and remain a prominent device in modern physics.

49. How remarkable is the connection between flexagons and the Feynman's Nobel Prize?

50. What moral might this historical vignette suggest about the nature of learning, discovery, and education? Explain.

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[6]Quoted in Genius.
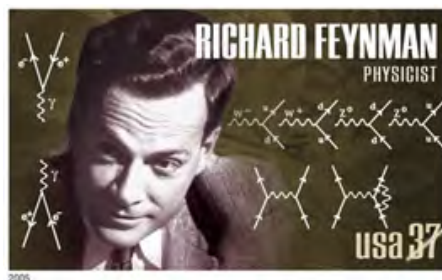
Figure 2.7: United States postage stamp honoring Feynman.

Reflective Essay: Return to the accomplishments of the people on the Flexagon committee. How remarkable is it that they all became so accomplished? What is it about the flexagon that might have impacted their success? Do you think creativity like this can be measured by standarized exams? What impact does any of this have on your view of "what's worth knowing" and "learning as play"?

2.15   Bibliography

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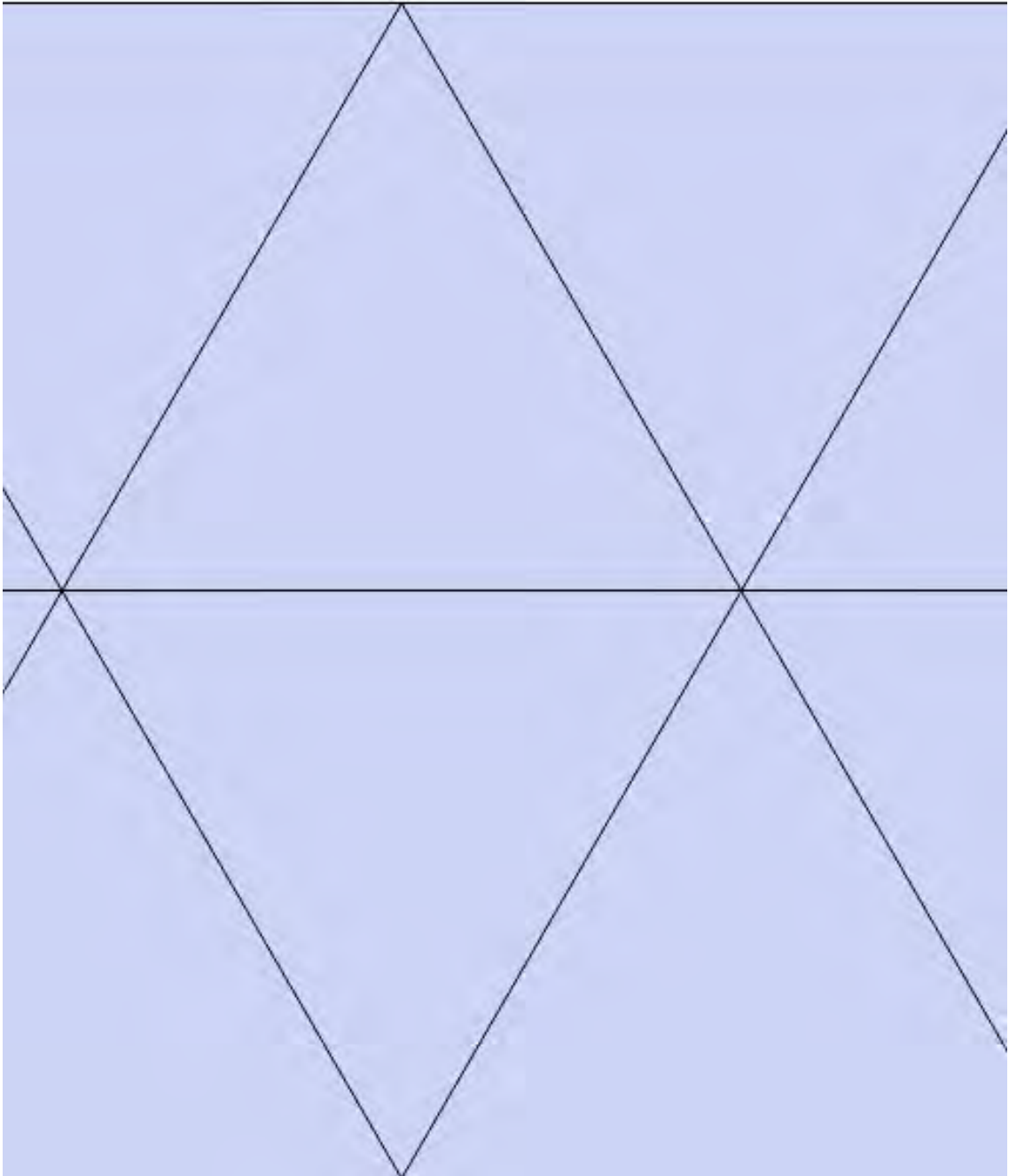
Figure 2.8: Folding tool for Flexagons

Chapter 3

Straight-cut Origami

3.1 The Fold-and-cut Problem

1. Take a piece of paper, fold it up in any way you like, as long as the end result is flat. Now make one complete straight cut (i.e., a cut along a line). Unfold the pieces, and see what you get. Try a few examples. Share with your class.

2. Describe a few characteristics that all these shapes have in common.

3. Will any and all shapes obtained by this process have these characteristics? Explain your reasoning in detail.

4. In general, what types of shapes do you think can be obtained in this way? Be as precise as you can.

5. What kinds of shapes are *not* possible? Describe a few examples. Explain why you are convinced that they are not possible.

In this chapter, we will investigate the **Fold-and-cut Problem**, sometimes also called the problem of Straight-Cut Origami: Given a shape, such as those in Figure 3.1, is there a a way to fold it up so that you can cut it out with a single straight cut? If there is such a way, how do we actually fold it? How can we describe all the shapes for which this is possible?

NOTE: For many of the figures in this chapter, the Appendix includes versions large enough for folding and cutting; see Page 73 and following. Whenever such a larger version exist, we include a reference in the caption of the small figure. Most of these figures were created using the free dynamic geometry software GEOGEBRA; see Section 3.5.

3.2 Erik Demaine

We learned about this problem from a 2005 New York Times article about Erik Demaine, a mathematician at the Massachusetts Institute of Technology who –then in his early twenties– has
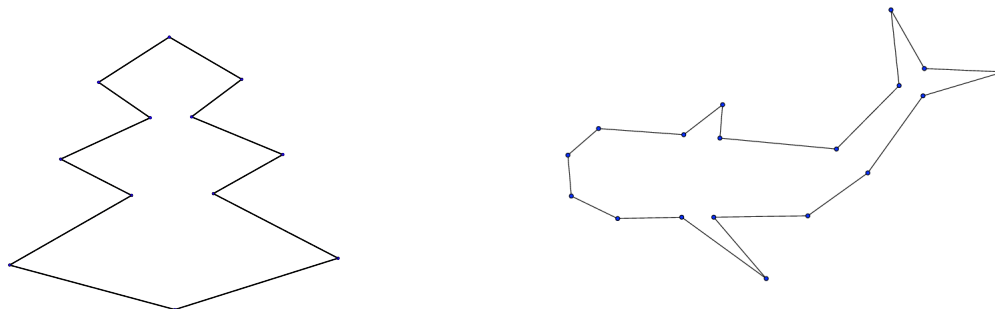
(a) Japanese *sangabisi* crest (large: Figure 3.13). (b) Whale for Straight-cut Origami (large: Figure 3.26).

Figure 3.1: Two fold-and-cut challenges.

become the leading theoretician in the emerging field of origami mathematics, the formal study of what can be done with a folded sheet of paper. Much of the material in this chapter is based on our attempts to understand the mathematics of the fold-and-cut problem as laid out in a series of papers by Erik Demaine and his collaborators. For more information about these papers and lectures, see the "Further Investigations" in Section 3.8.

Here is what Erik Demaine tells us about the history of the problem:

The first published reference to folding and cutting of which we are aware is a Japanese book, Wakoku Chiyekurabe (Mathematical Contests), by **Kan Chu Sen** (; -), published in 1721. This book contains a variety of problems for testing mathematical intelligence. One of the problems asks to fold a rectangular piece of paper flat and make one complete straight cut, so as to make a typical Japanese crest called *sangaibisi*, which translates to "three folded rhombics." [*See Figure 3.1(a) for a rendition of a* sangaibisi *crest.*]

Another early reference to folding and cutting is a July 1873 article National Standards and Emblems in Harper's New Monthly Magazine, volume 47, number 278. This article tells the story about Betsy Ross and her relation to the American flag. It claims that in 1777, George Washington and a committee of the Congress showed Betsy Ross plans for a flag with thirteen six-pointed stars, and asked whether she could make such a flag. She said that she would be willing to try, but suggested that the stars should have five points. To support her argument, she showed how easily such a star could be made, by folding a sheet of paper and making one cut with scissors. The committee decided to accept her changes, and George Washington made a new drawing, which Betsy Ross followed to make the first American flag.

Folding and cutting may have been used for a magic trick by Houdini, before he became a famous escape artist. His 1922 book Paper Magic (E. P. Dutton & Company, pages 176-177) describes a method for making a five-pointed star. According to David Lister, this book

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appears to have been ghostwritten by another magician, Walter Gibson.

Another magician, Gerald Loe, studied the fold-and-cut idea in some detail; his 1955 book Paper Capers (Magic, Inc., Chicago) describes how to cut out arrangements of various geometric objects, such as a circular chain of stars.

Martin Gardner wrote about the fold-and-cut problem in his famous series in Scientific American (Paper cutting, chapter 5 of New Mathematical Diversions (Revised Edition), Mathematical Association of America, Washington, D.C., 1995.). Gardner was particularly impressed with Loe's ability to cut out any desired letter of the alphabet. He was also the first to state cutting out complex polygons as an open problem. What are the limits of this fold-and-cut process? What polygonal shapes can be cut out?

3.3 Symmetric Shapes

If you boldly tried to fold the whale shape in order to cut it out with a single straight cut, you may have realized that this is not so easy. Often in mathematics, when we encounter a problem that seems too difficult to tackle with the tools we have at hand, we shift to a somewhat easier problem, in the hope of gaining a deeper understanding of the problem and of learning new tools. Let us therefore start with some shapes that are less complex and that exhibit some symmetry.

Figure 3.2: Flower with line of symmetry.

Consider the shape shown in Figure 3.2 and the **line of symmetry** drawn into it. The left hand side of the shape is (close to) a mirror image of the right hand side of the shape, so that if you fold the shape along this line of symmetry, the two sides will match. We say that the shape has **reflectional symmetry**; for more on geometry and symmetry, see references [**?**, **?**, **?**].
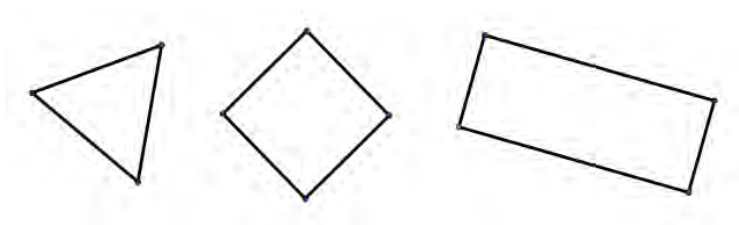
Figure 3.3: Three Geometric Shapes for Straight-cut Origami (large: Figures 3.14-3.16).

Figure 3.3 shows three simple geometric shapes: an equilateral triangle, a square, and a rectangle.

6. Find a way to fold up the equilateral triangle in Figure 3.3 so that you can cut it out with a single straight cut. Once you succeed, clearly mark on both pieces of paper (the triangle and the outside) the actual fold lines you used. Explain your strategy.

7. As before, find the straight-cut folding for the square. Explain your strategy.

8. Now, find the straight-cut folding for the rectangle. Explain your strategy.

9. Take a new printout of each shape, draw in all the lines of reflection. What do you notice when you compare these with your fold lines? Write down everything you notice.
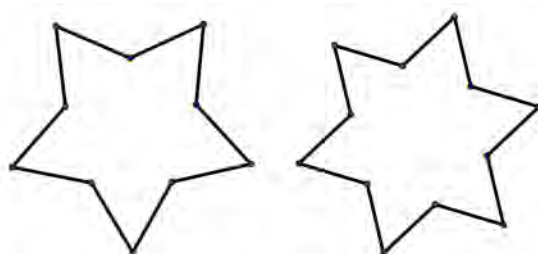
Figure 3.4: Two Star Shapes for Straight-cut Origami (large: Figures 3.17-3.18).

Figure 3.4 shows two stars similar to those that Betsy Ross may have worked with in designing the American flag: a five-pointed star and a six-pointed star.

10. Find a way to fold up the five-pointed star so that you can cut it out with a single straight cut. Once you succeed, clearly mark on both pieces of paper (the star and the outside) the actual fold lines you used. Explain your strategy.

11. Now, find the straight-cut folding for the six-pointed star. Explain your strategy.

12. Take another five-pointed star and shade the interior. As you fold it up again, make note whether shaded areas get folded onto shaded or unshaded paper (or both). Explain any patterns you see. Would the result have been the same for the six-pointed star?

13. Take a new printout of the six-pointed star, draw in all the lines of reflection and compare them with your fold lines. What do you observe? Would the result have been the same for the five-pointed star?

14. **Classroom Discussion:** Share your observations about straight-cut folds for symmetric shapes. Do you feel that you would be able to fold and cut any polygon with at least one line of symmetry? Try a few examples to check whether your conjecture is correct.

All the shapes discussed in this section are what are called **polygons**. The word "polygon" derives from the Greek πολυσ (polys) meaning "many" and γωνια (gōnia), meaning "knee" or "angle". Traditionally, a polygon is a plane figure that is bounded by a closed path or circuit, composed of a finite sequence of straight line segments (i.e., by a closed polygonal chain). These segments are called its **edges** or **sides**, and the points where two edges meet are the polygon's **vertices** or **corners**. An *n*-**gon** is a polygon with *n* sides. The interior of the polygon is sometimes called the **body of the polygon**.

   A **regular polygon** is a polygon that is **equiangular** (all angles are equal in measure) and **equilateral** (all sides have the same length).

15. When we fold a paper flat and cut along a straight line, will it always separate into two pieces, or can it result in more than two pieces?

16. When we fold a paper flat and cut along a straight line, will the shape(s) you cut out always consist of polygons? Show your thinking and explain your reasoning.

17. Will the resulting shape(s) always be *symmetric*? Explain.

18. Will the shape(s) you cut out always consists of *regular* polygon(s)? Explain.

3.4 Irregular Shapes

We noticed that lines of symmetry are very helpful in folding polygons so that we can cut them out with a single straight cut. Yet, with shapes that are less symmetric, it is no longer clear how to proceed.
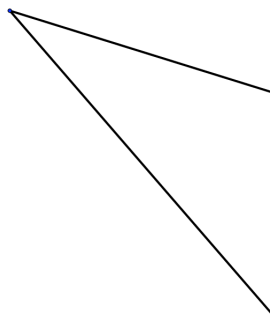
Figure 3.5: Single Angle for Straight-cut Origami (large: Figure 3.19).

19. Find a way to fold the single angle shape in Figure 3.5. How does the folding of this shape relate to the work you did with the symmetric shapes in Section 3.3? Explain.

You may find the following series of investigations more challenging than previous ones. It may take you more than just one attempt with each shape, sometimes *many* more. You may observe that using ideas gained from working with symmetric shapes may not be enough to fold these irregular shapes. Do not be discouraged. Where could you go for some new ideas? Do not discard the results of your attempts. Instead, use them as resources to analyze carefully what the results look like, and why they do not *completely* accomplish the task. Also keep an eye on what happens to interior and exterior areas when folding.

Mathematicians have a name for the line (or line segment) that divides an angle into two equal parts: they call this an **angle bisector**.

20. Find a way to fold the double-angle shape in Figure 3.6.

21. Find a way to fold the ***irregular triangle*** in Figure 3.7 (three sides all of different lengths).

22. **Classroom Discussion:** Other than angle bisectors, what other kinds of lines did you use to fold up the shapes in Figure 3.6 and Figure 3.7? How would you describe those?

23. An angle bisector is a line of symmetry for one angle. Is it also a line of symmetry of the *entire* shape? What are the consequences of folding along an angle bisector all the way through the shape?
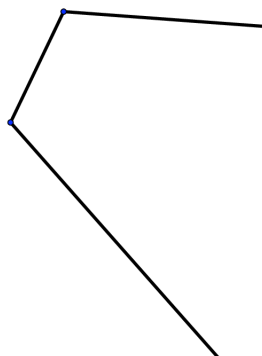
Figure 3.6: Double Angle for Straight-cut Origami (large: Figure 3.20).
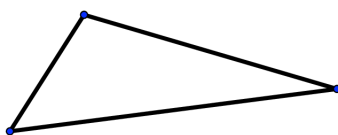
Figure 3.7: An Irregular Triangle for Straight-cut Origami (large: Figure 3.21).

24. Find a way to fold the ***irregular quadrilateral*** in Figure 3.8 (four sides all of different lengths).

25. Did your observations about folding the double-angle, or the irregular triangle, help you in folding the irregular quadrilateral? Explain.

26. Were there any new ideas that you used for this shape? Explain.

27. Classroom Discussion: Given a line ℓ and a point P not on ℓ, we can construct the **perpendicular**. This is a line through P which makes a 90° angle with ℓ. Look back at your folding patterns and find perpendiculars. What do you notice?
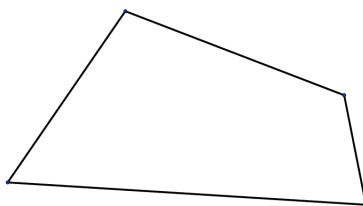
Figure 3.8: An Irregular Quadrilateral for Straight-cut Origami (large: Figure 3.22).

28. Next consider the shapes in Figure 3.9. In what ways are these shapes different from ones you have considered in this section so far? Explain.

29. Create three geometric shape of your own that belong to this new type. Explain why each belongs to this new type.
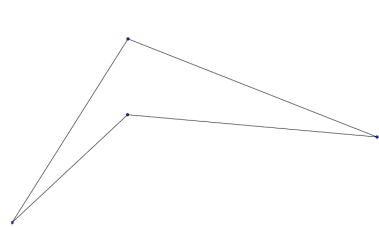
(a) Quadrilateral (large: Figure 3.23).
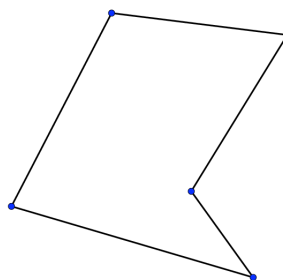
(b) Pentagon (large: Figure 3.24).
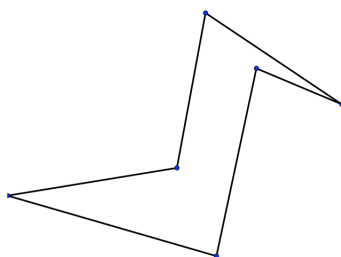
(c) Hexagon (large: Figure 3.25).

Figure 3.9: Three Shapes for Straight-Cut Origami.

30. Find a way to fold the quadrilateral in Figure 3.9(a) so that you could cut it out with a single straight cut.

31. Find a way to fold the pentagon in Figure 3.9(b) so that you could cut it out with a single straight cut.

32. Find a way to fold the *non-convex hexagon* in Figure 3.9(c) (six sides, segments between some of the vertices fall outside the shape) so that you could cut it out with a single straight cut.

33. Once you know how to fold and cut these shapes, make another folded copy of each but *do not cut it out*. Carefully unfold it, making sure to note which of the fold lines were used in

your final version, and which were not. Clearly mark all the fold lines that were needed for your final version.

34. **Writing Assignment:**   Using the marked and labeled shapes you created in Investigation **33** as resources, clearly describe a geometric way in which these folds relate to the original lines of the polygon. Write a complete and careful summary of your observations and findings.

3.5 Dynamic Explorations

We have now seen how certain geometric lines (angle bisectors and perpendiculars) relate to certain fold lines in straight-cut origami. In this section, you will use the dynamic geometry software called GEOGEBRA. The ability to dynamically change the shape may allow you to more fully explore the relationship between a shape and its straight-cut fold lines.

35. Download the free GEOGEBRA software from `http://www.geogebra.org/`. Figure 3.10 shows a screenshot of a GEOGEBRA session editing the *sangabisi* crest. The main window contains an image of the shape which you can modify by clicking and dragging. A row of square icons along the top gives access to various geometric tools; hovering over them with the mouse cursor will pop up the names of the tools. A complete list of "free" and "dependent" objects (points, lines, etc.) is shown along the left. An input bar at the bottom accepts direct typed input. Objects may be manipulated using either one of these interfaces.

36. Play around and create a few shapes, using the following tools: NEW POINT, LINE THROUGH TWO POINTS, SEGMENT BETWEEN TWO POINTS, PERPENDICULAR LINE, ANGLE BISECTOR, INTERSECT TWO OBJECTS; you may need to click on the little triangles in the bottom right corner of each of the icons in order to see some of these options.

37. Choose the MOVE tool (first icon on the left) to move some of your lines or points. Explore how to save, export and print your workbook.

38. In order to change basic aspects of your objects, such as the line thickness or color, right-click on the object and choose OBJECT PROPERTIES. You can also switch off labels this way. Full documentation is available via HELP.

39. Create an irregular quadrilateral similar to that in Figure 3.8 and draw in the geometric lines that tell you where it needs to be folded for straight-cut origami. Change your quadrilateral using the MOVE tool and explore how the fold lines change as a result. Write down your observations.

40. Print out your quadrilateral and see whether you can fold it up. Compare whether your printout correctly predicted *all* the fold lines you find. Write down your observations.

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41. INDEPENDENT INVESTIGATION: Now create several shapes of your own. Explore how to construct as many of the fold lines as you can. Share your shapes with the class.
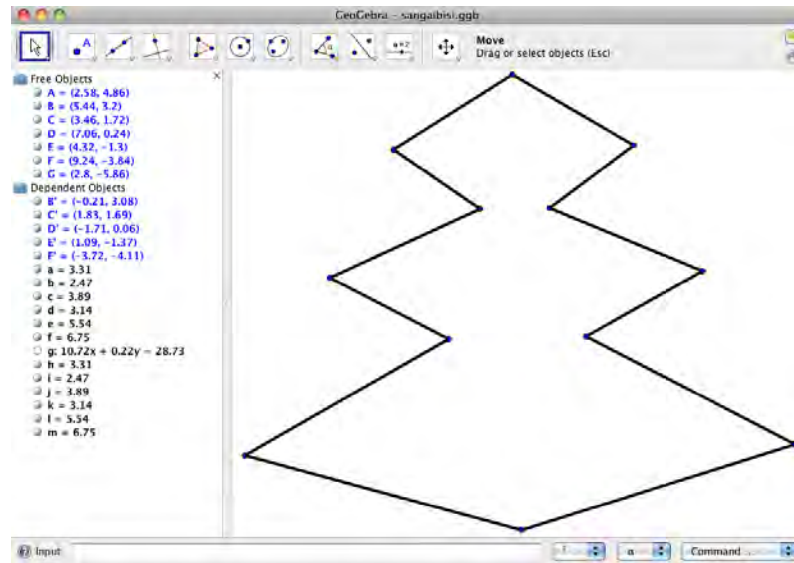
Figure 3.10: GEOGEBRA screenshot showing the *sangabisi* crest.

3.6 Further Investigation: Whale

Here is a challenge: can you fold up the whale in Figure 3.11 so that you can cut it out with a single straight-cut?
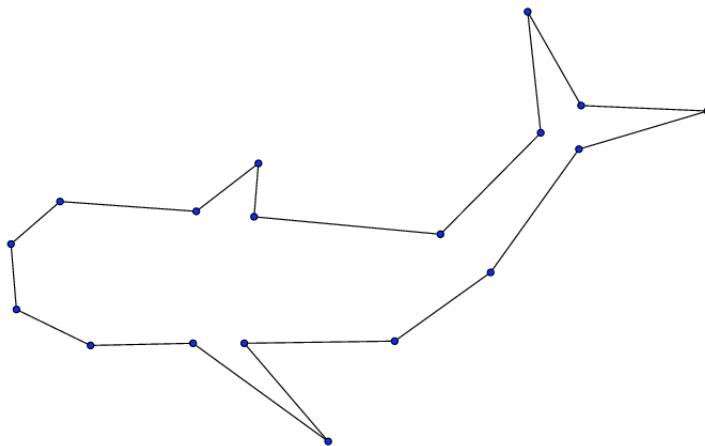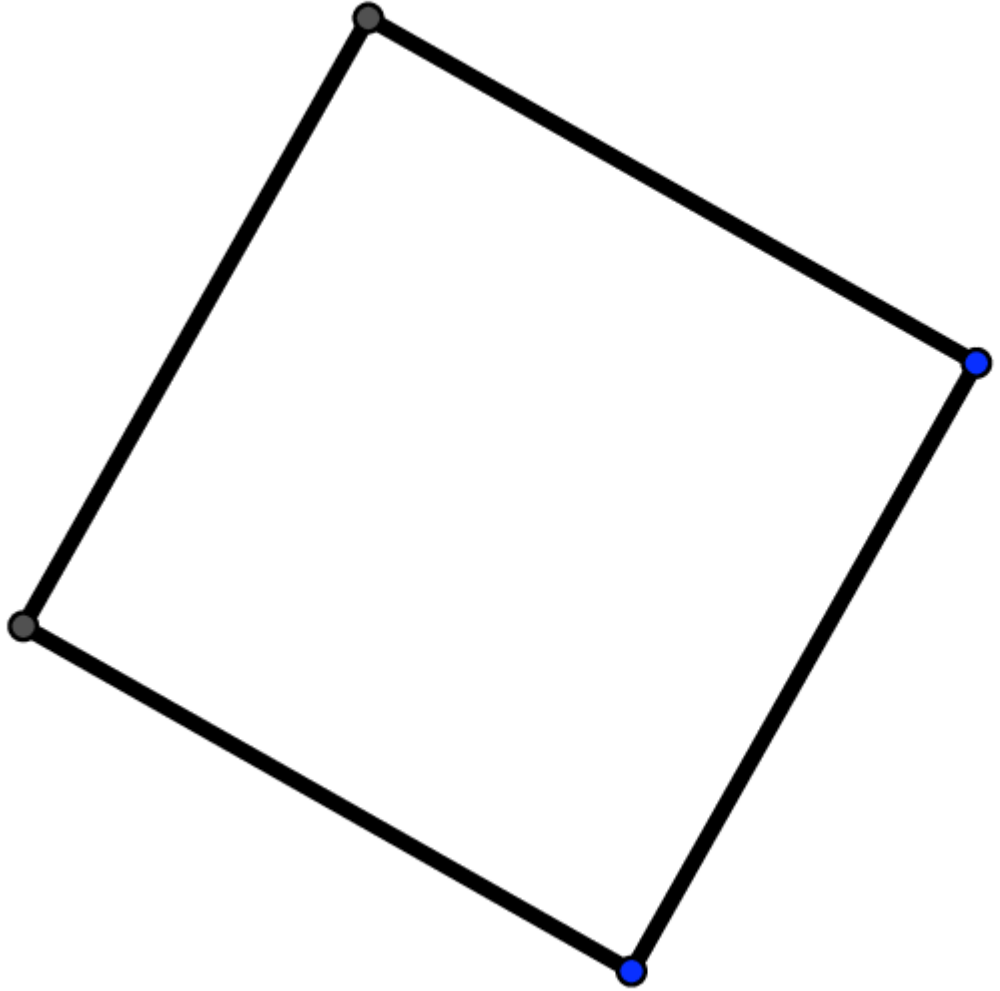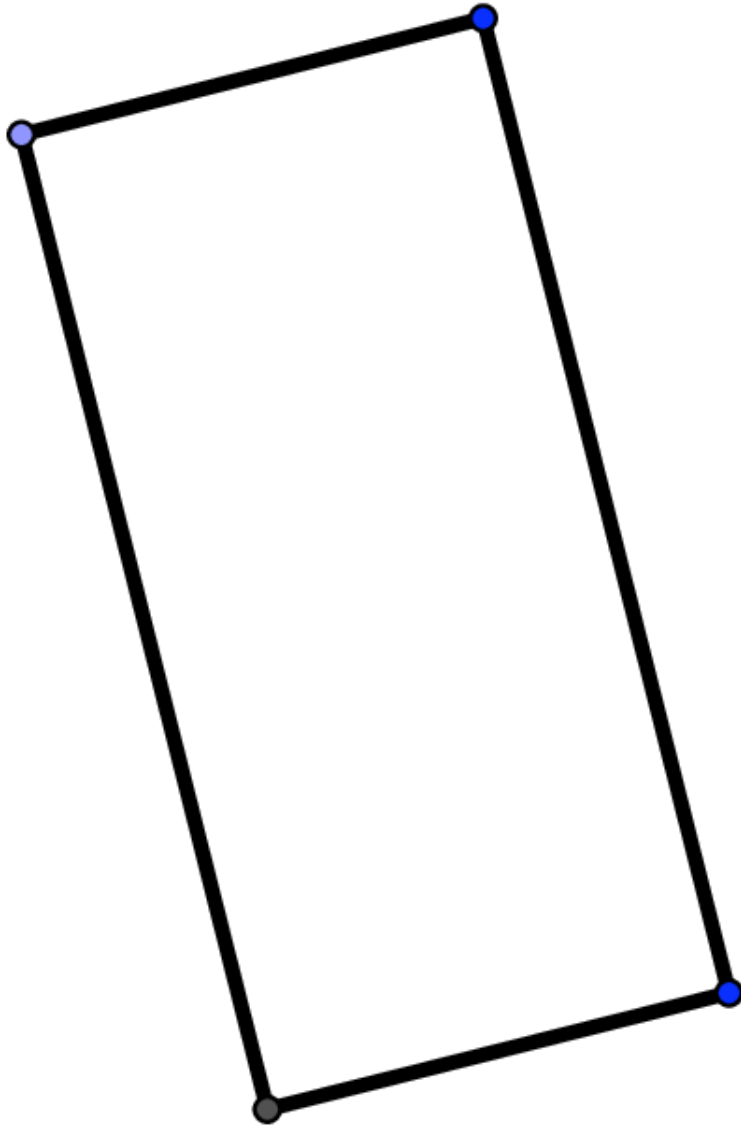
Figure 3.11: Whale for Straight-cut Origami (large: Figure 3.29).

As usual in mathematics, if a problem is really difficult, we try to break it down into smaller problems that are hopefully easier to solve. In the following investigations, we first look at three parts of the whale separately: the head, the middle including the fins, and the tail, respectively are shown in Figures 3.12(a)-3.12(c). Feel free to use GEOGEBRA to investigate potential fold lines for these shapes. If you feel bold, skip the separate parts of the whale and jump right in and fold the full whale.

F1. Figure 3.12(a) shows the head of the whale. Find a way to fold it up so you could cut it out with a single straight cut.

F2. The whale's middle section including the two fins is shown in Figure 3.12(b). Find a way to fold it up so you could cut it out with a single straight cut.

F3. Now consider the whale's tail in Figure 3.12(c). Find a way to fold it up so you could cut it out with a single straight cut.

F4. For any of these investigations, did you need to develop any new ideas beyond the ones you've seen before? Explain.
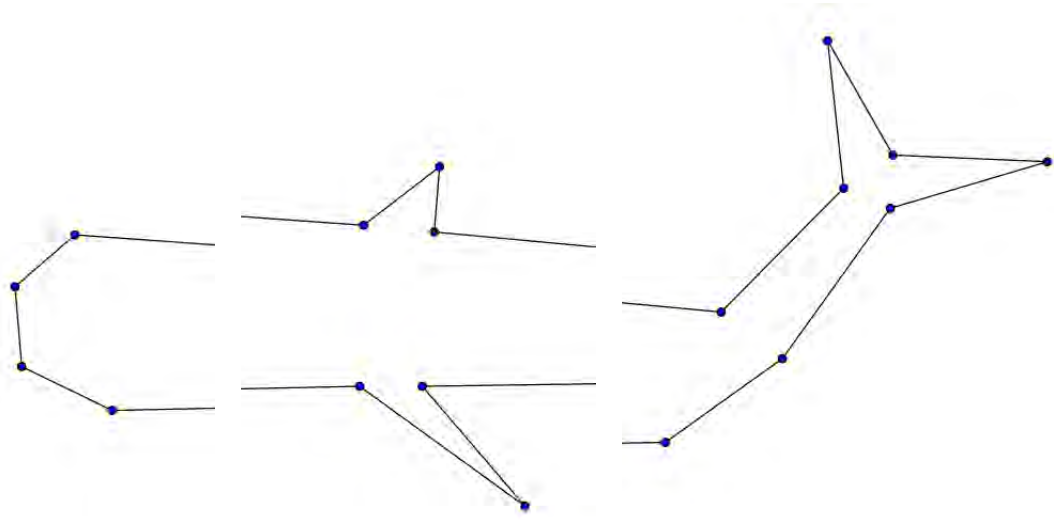
(a) Whale Head (large: Figure 3.27). (b) Whale Middle (large: Figure 3.28). (c) Whale Tail (large: Figure 3.29).

Figure 3.12: Three Parts of the Whale for Straight-Cut Origami.

F5. Based on your understanding of the three separate pieces, find a way to fold up the complete whale; see Figure 3.11.

3.7 Further Investigation: Eric Demaine's Shapes

Eric Demaine has a few shapes online at

> `http://erikdemaine.org/foldcut/examples/`.

F6. Try to fold and cut a few of these. Do as many as you like. Describe how these models relate to your investigations in this chapter.

3.8 Further Investigation: The Whole Story – Paper, Video, and Books

RESOURCES: Erik Demaine's paper "Folding and Cutting Paper"– co-authored with Martin Demaine and Anna Lubiw – explores and explains many of the mathematical ideas behind finding *all* the fold lines; see `http://erikdemaine.org/papers/JCDCG98/paper.pdf`. In addition, you can find a video of a lecture by Erik Demaine online where he talks about this material in more detail; see `http://courses.csail.mit.edu/6.849/fall10/lectures/L07.html`.

The full story is laid out in more detail in two books co-authored by Eric Demaine and Joseph O'Rourke: a more expository and easier overview can be found in Chapter 5 of "How To Fold It", [**?**]; the dense and full proof is contained in Chapter 17 of "Geometric Folding Algorithms" [**?**].

F7. Based on what you learn in the paper, the video and/or the chapters in the books, describe the following concepts and how they relate: Straight Skeleton Graph; Perpendicular Graph; Corridors. How do these tools and concepts show why (almost) any polygonal shape can be folded and cut out in a single straight cut.

3.9 Appendix: Large Shapes
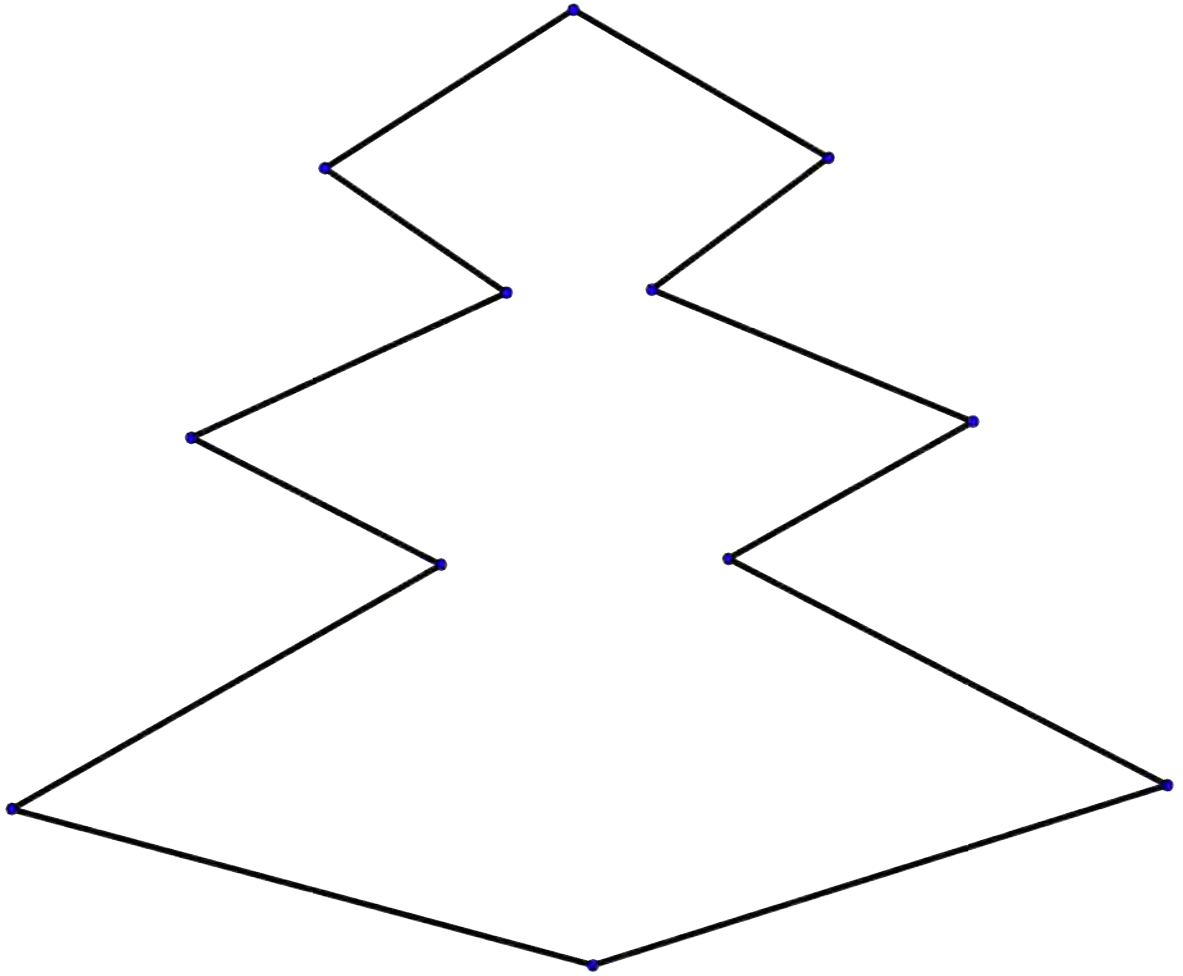
Figure 3.13: Japanese *Sangabisi* crest.
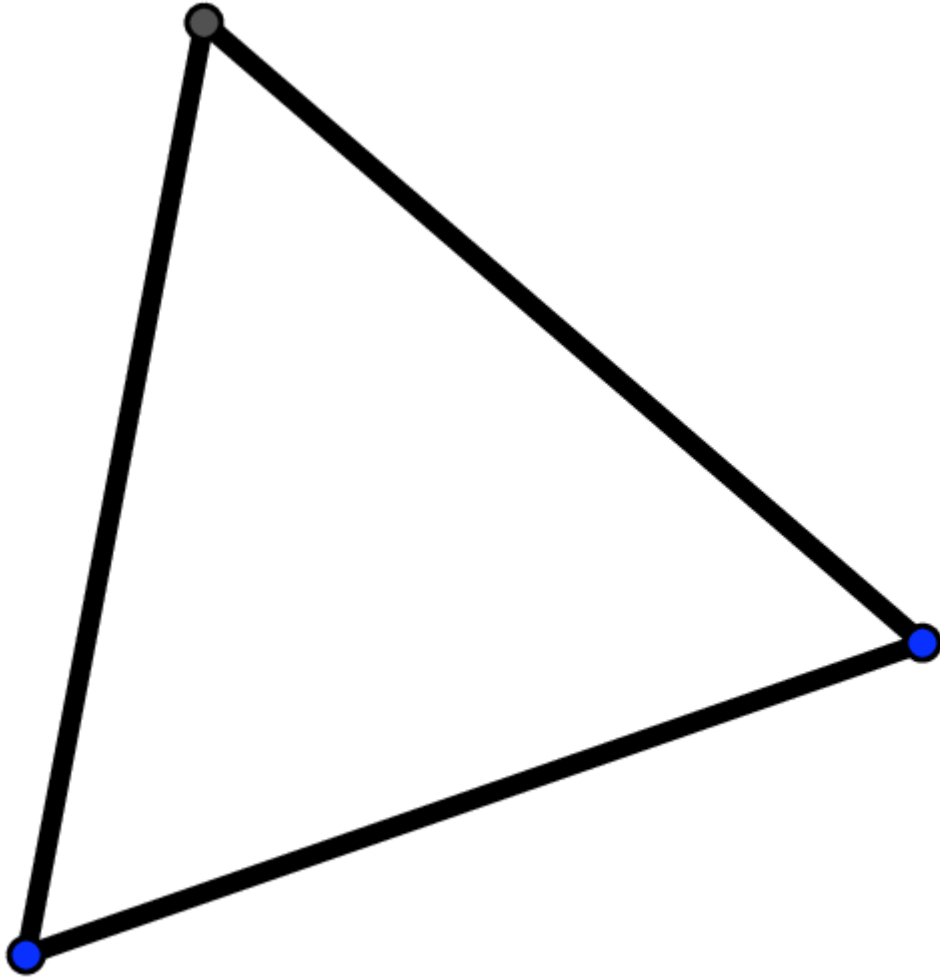
Figure 3.14: Equilateral triangle for Straight-cut Origami.

Figure 3.15: Square for Straight-cut Origami.

Figure 3.16: Rectangle for Straight-cut Origami.

Figure 3.17: Five-pointed Star for Straight-cut Origami.

Figure 3.18: Six-pointed Star for Straight-cut Origami.

Figure 3.19: Single Angle for Straight-cut Origami.
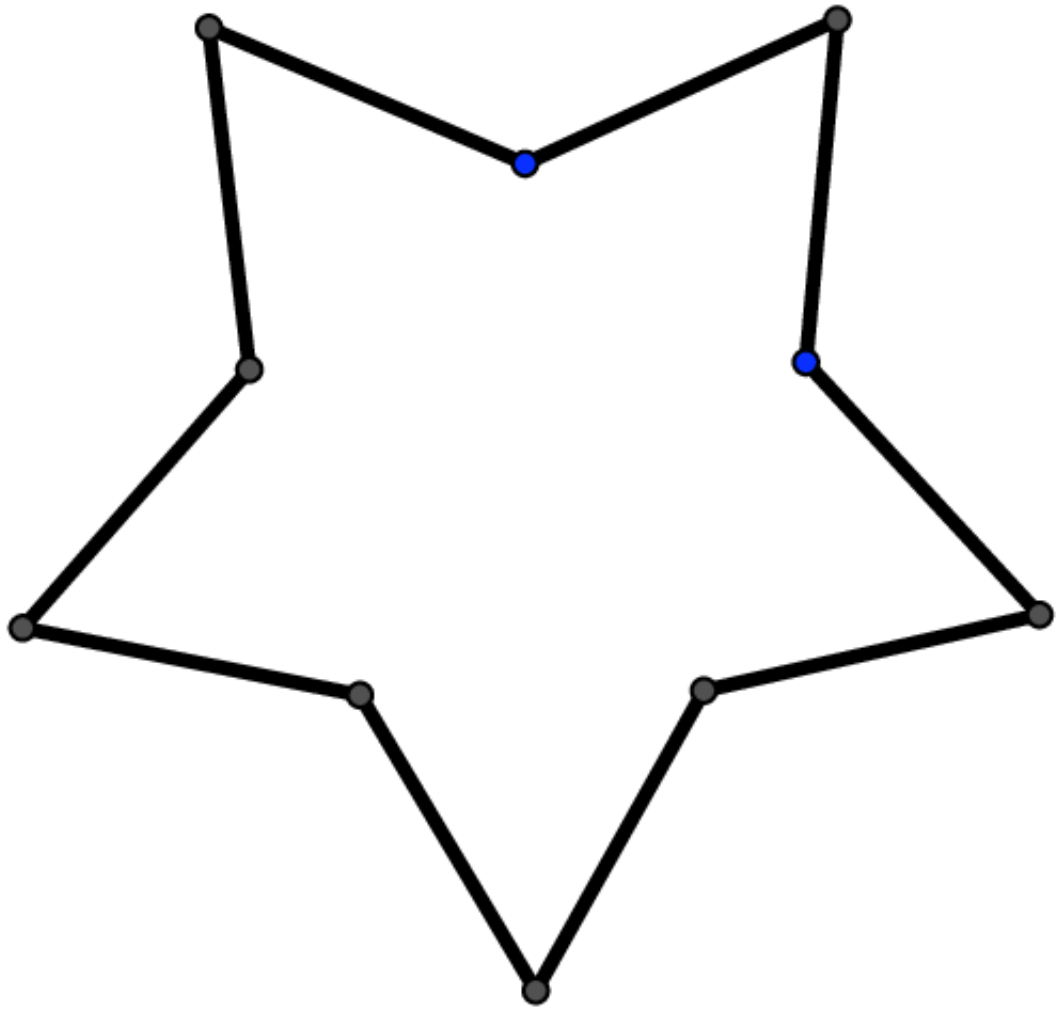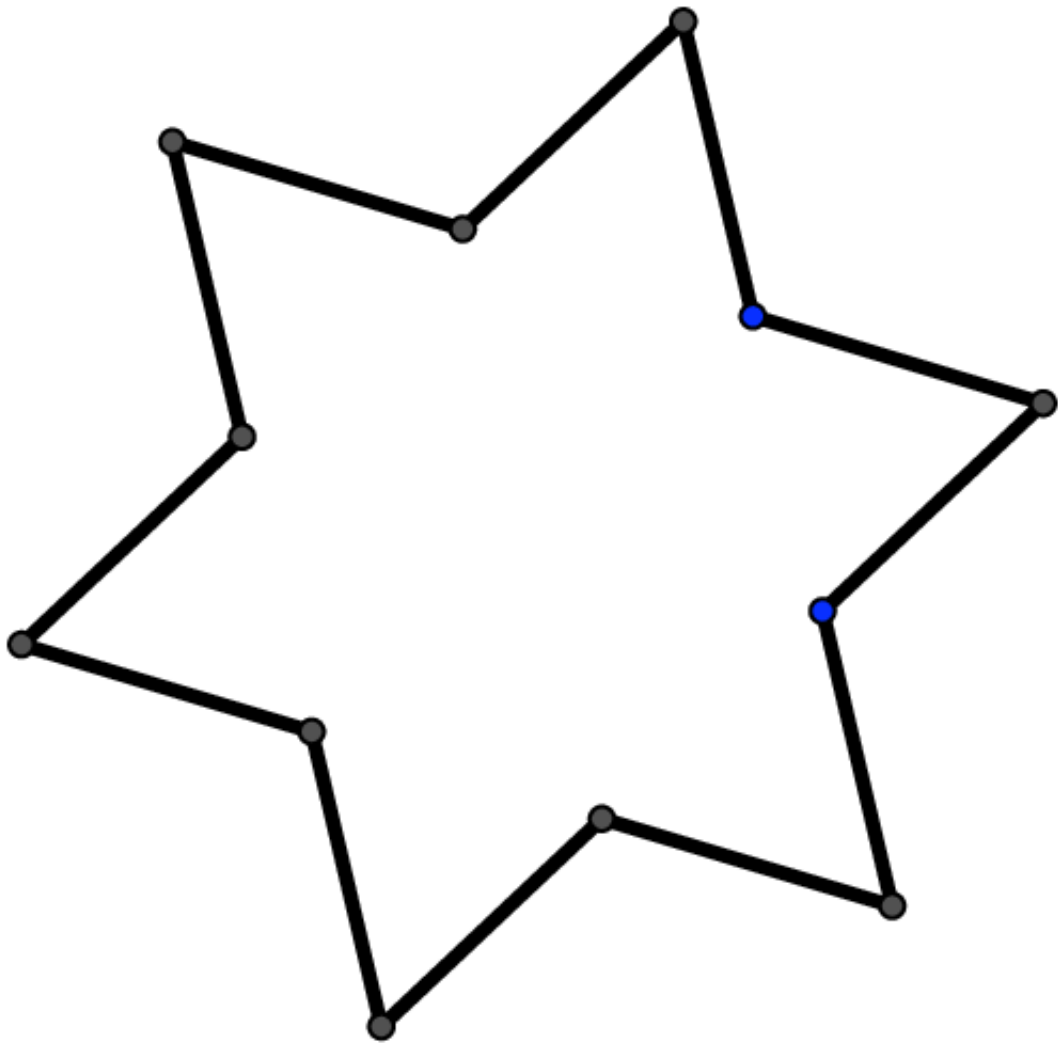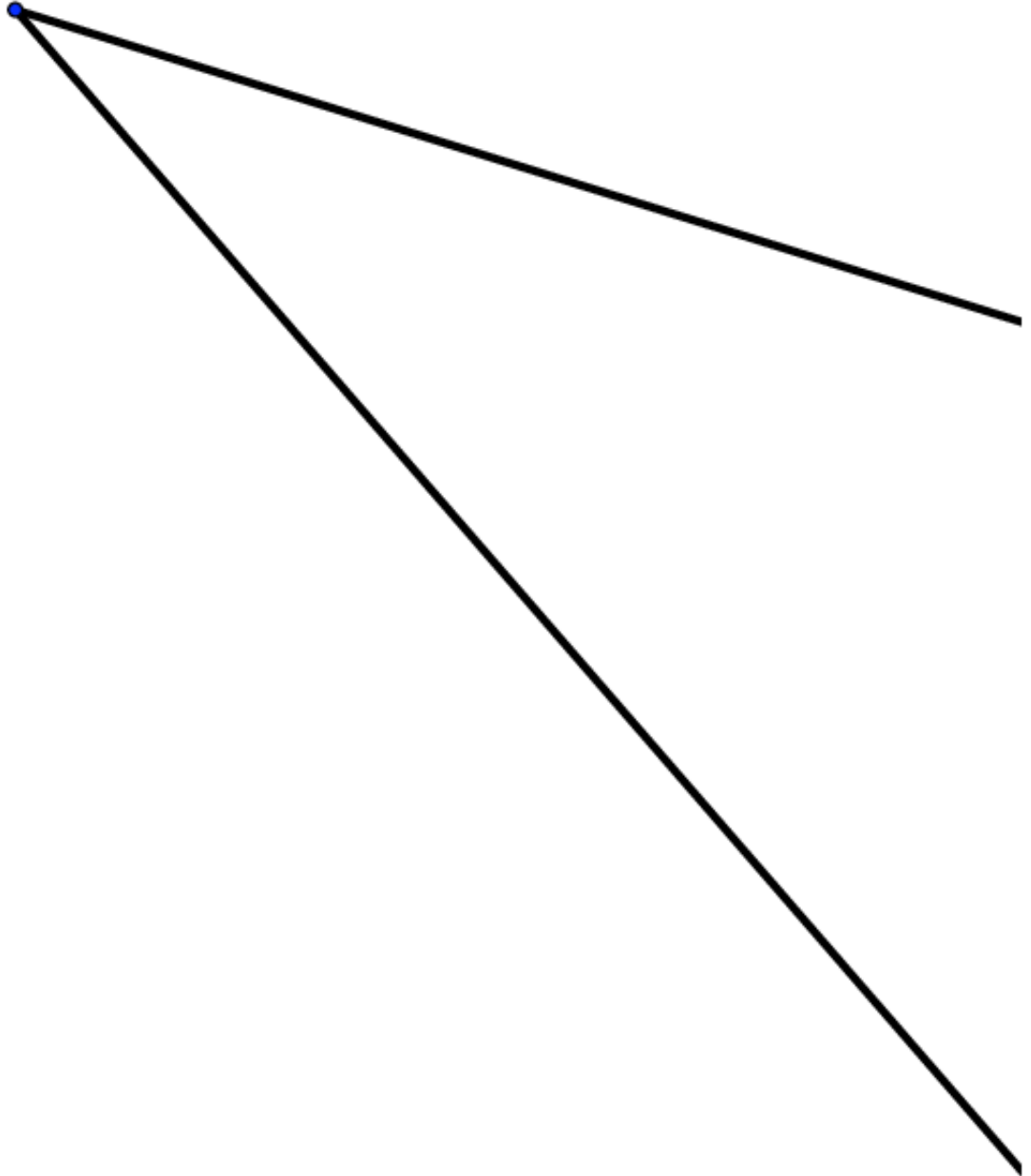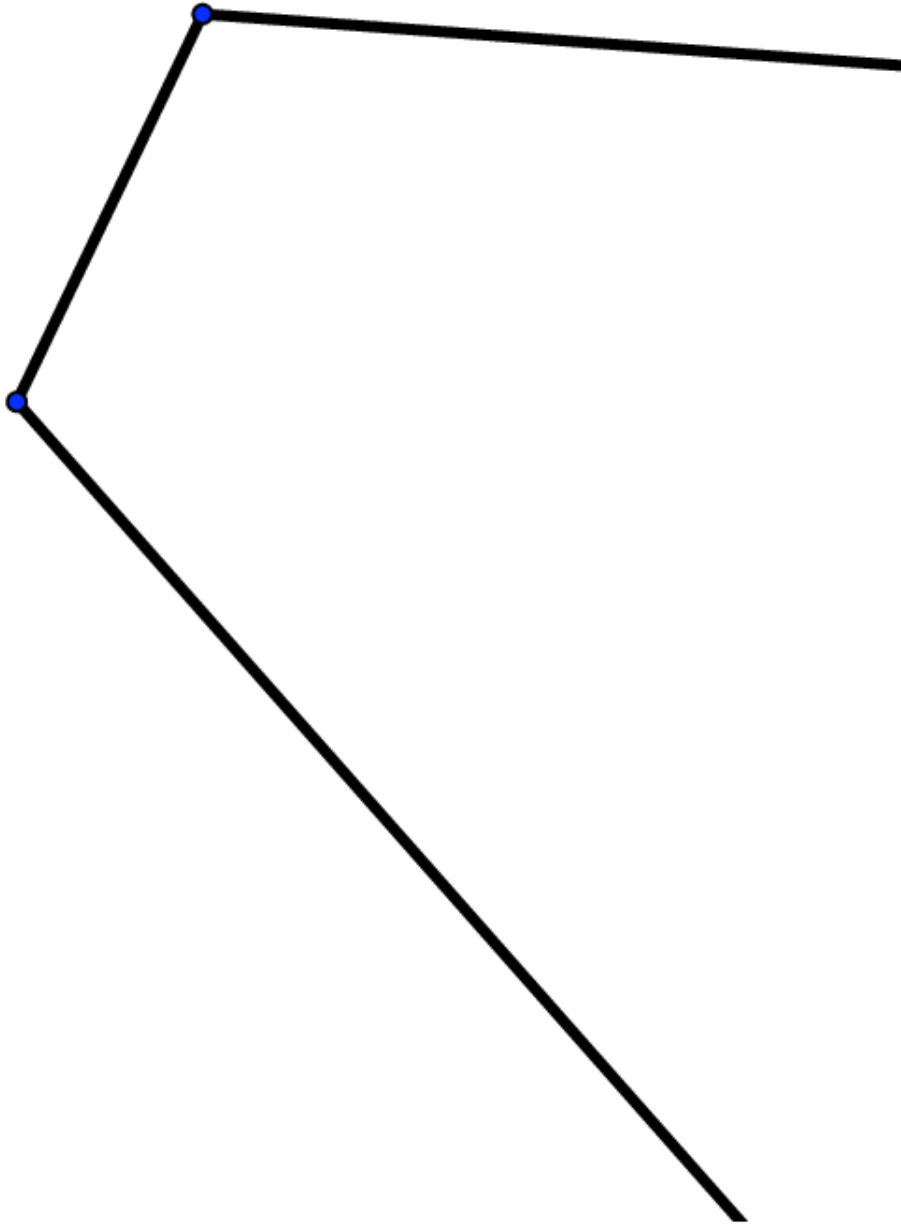
Figure 3.20: Double Angle for Straight-cut Origami.

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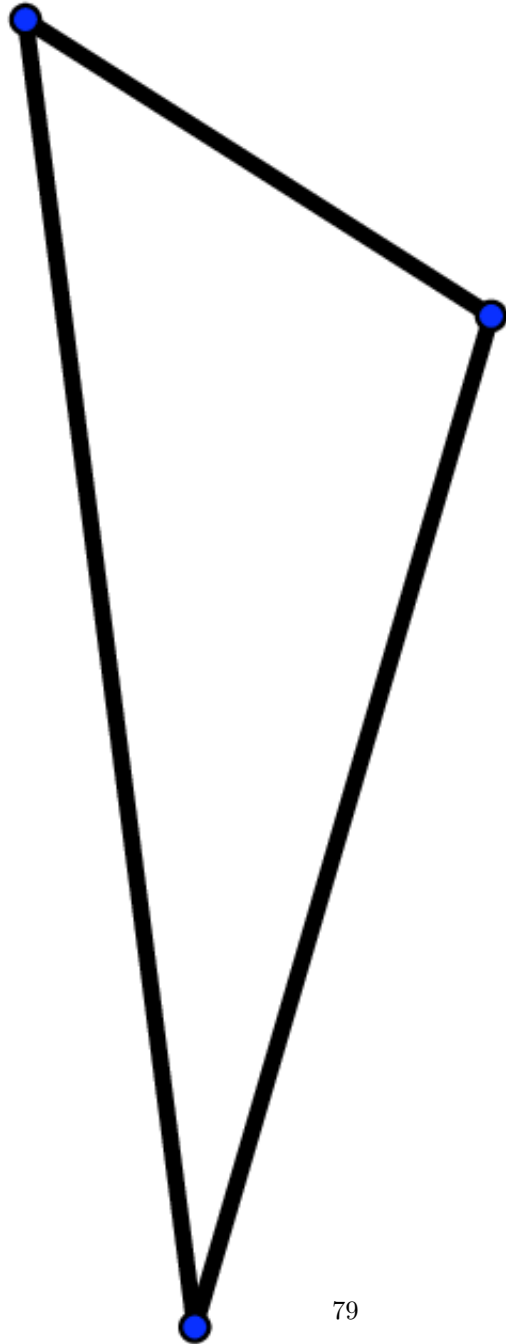
Figure 3.21: An Irregular Triangle for Straight-cut Origami.
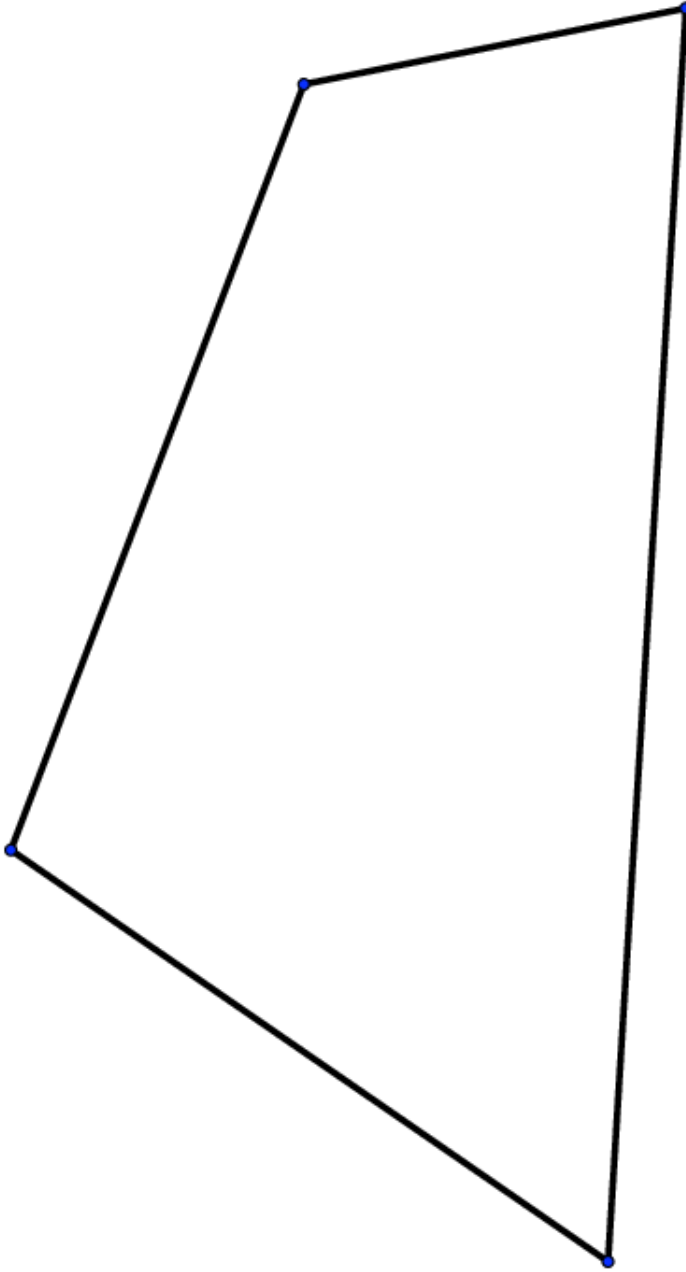
Figure 3.22: An Irregular Quadrilateral for Straight-cut Origami.
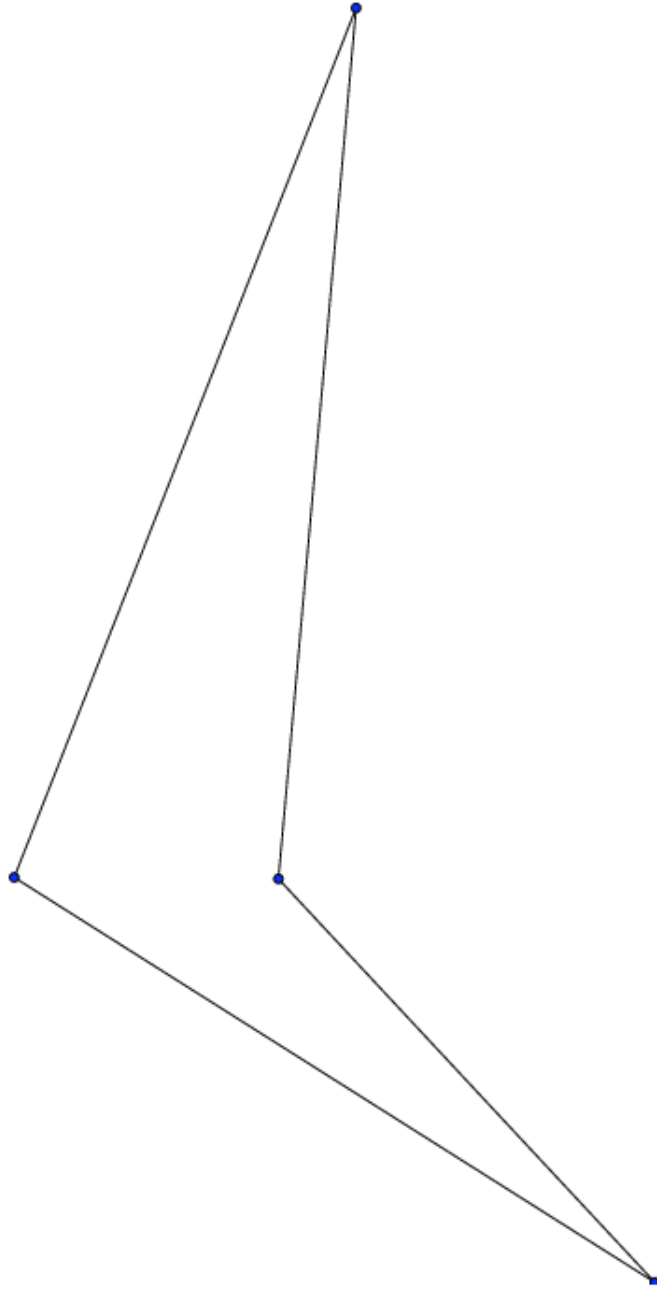
Figure 3.23: An Irregular Quadrilateral for Straight-cut Origami.
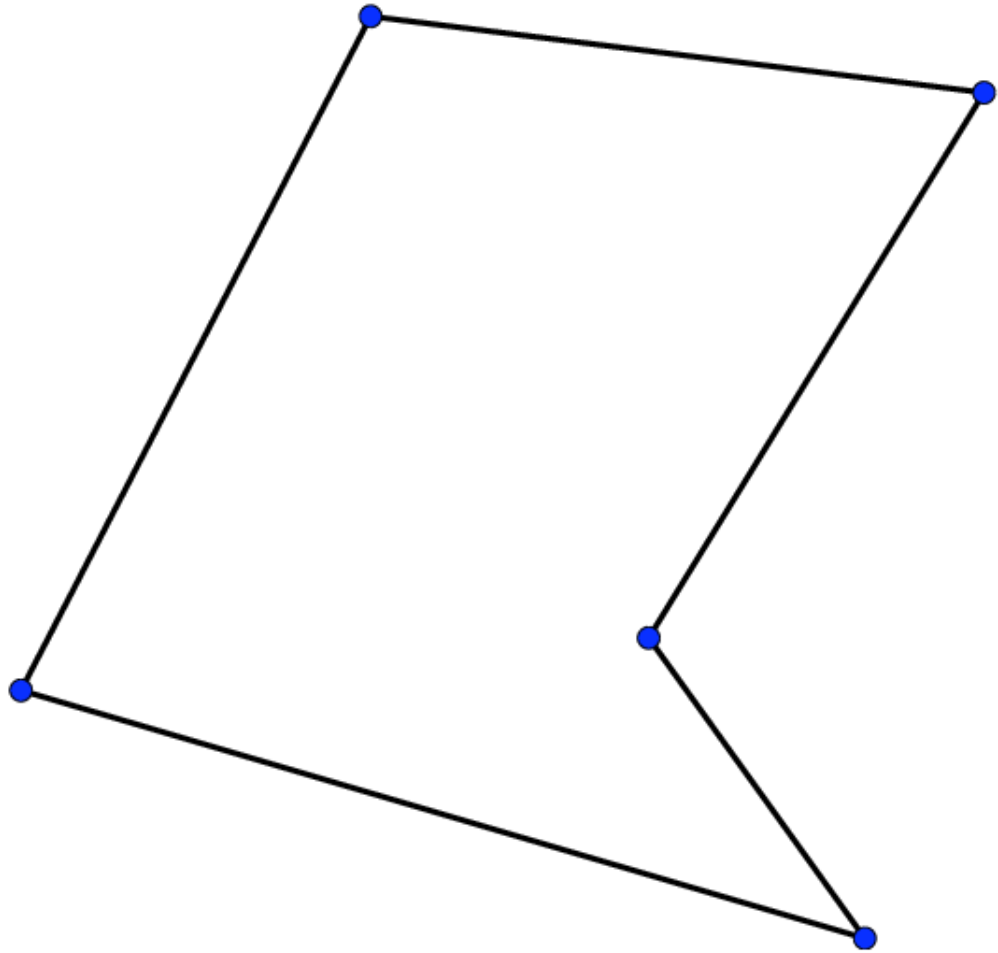
Figure 3.24: An Irregular Pentagon for Straight-cut Origami.
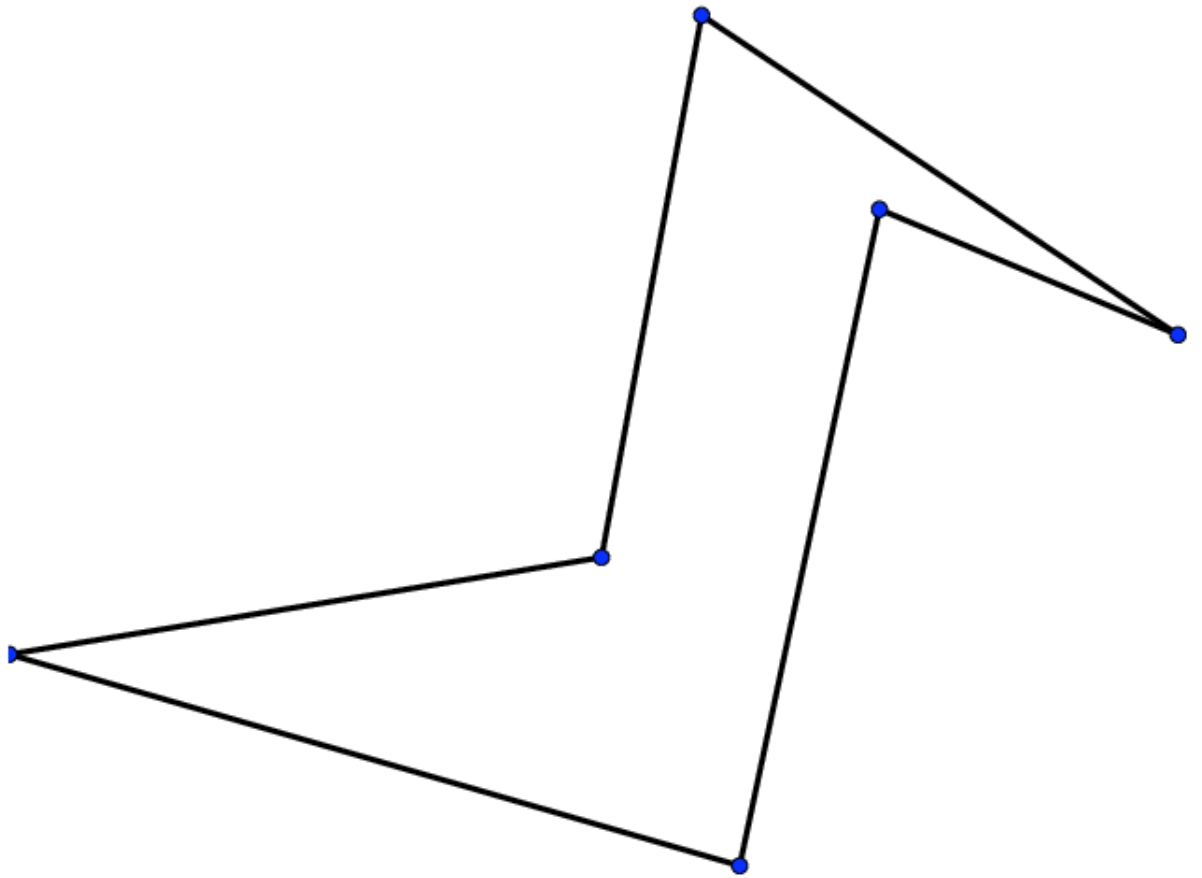
Figure 3.25: An Irregular Hexagon for Straight-cut Origami.

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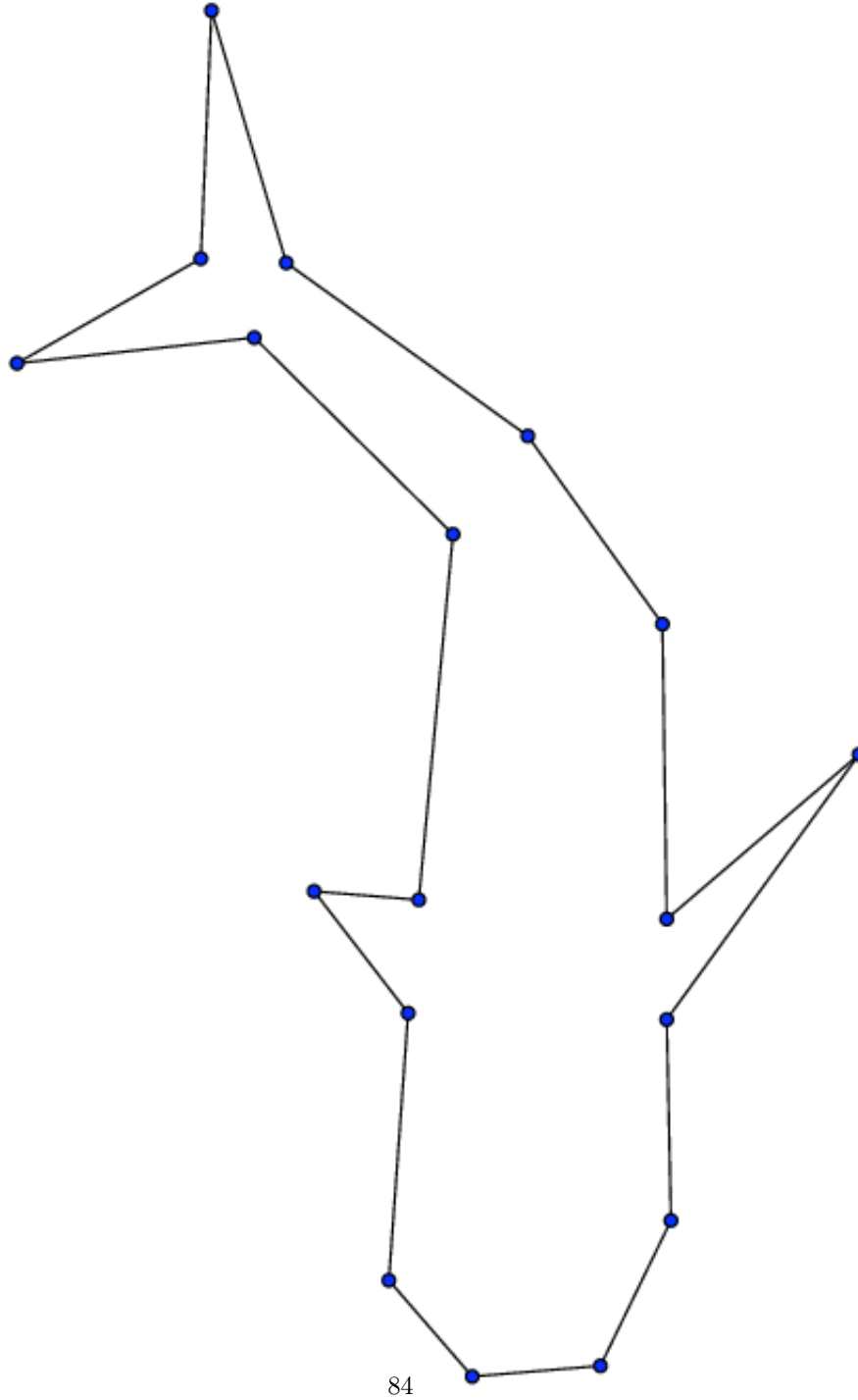
Figure 3.26: Whale for Straight-cut Origami.

Figure 3.27: Whale Head for Straight-cut Origami.
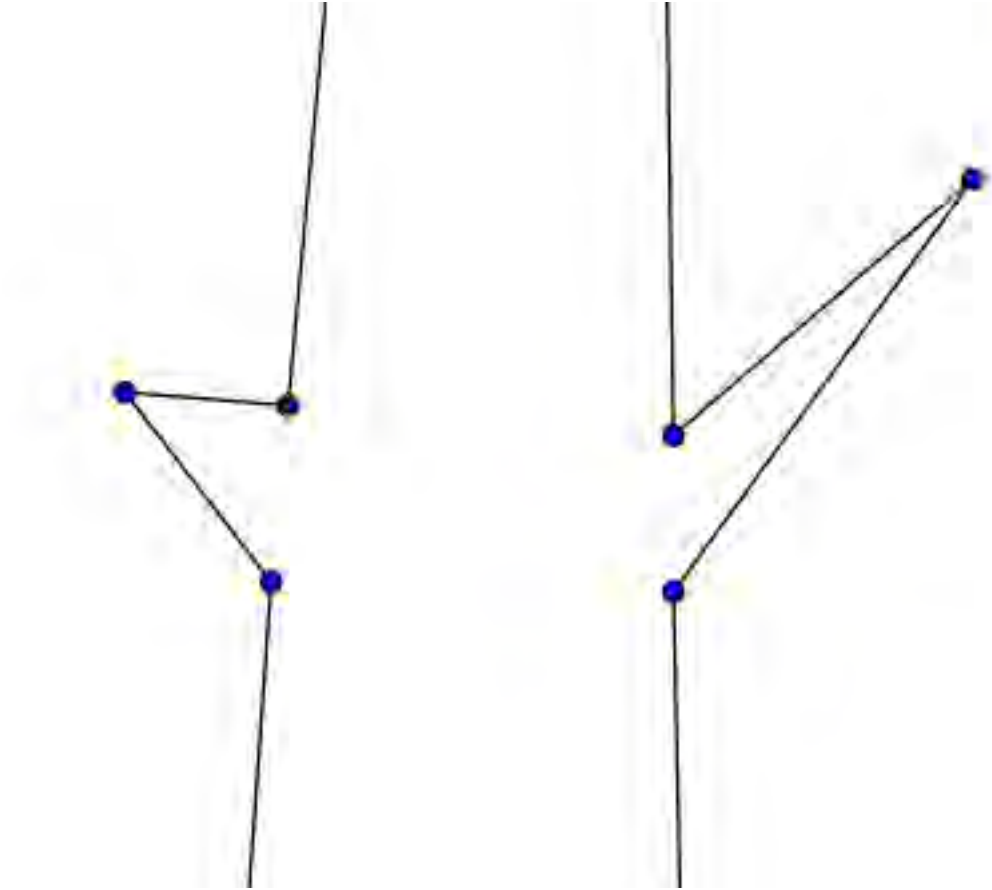
Figure 3.28: Whale Middle for Straight-cut Origami.
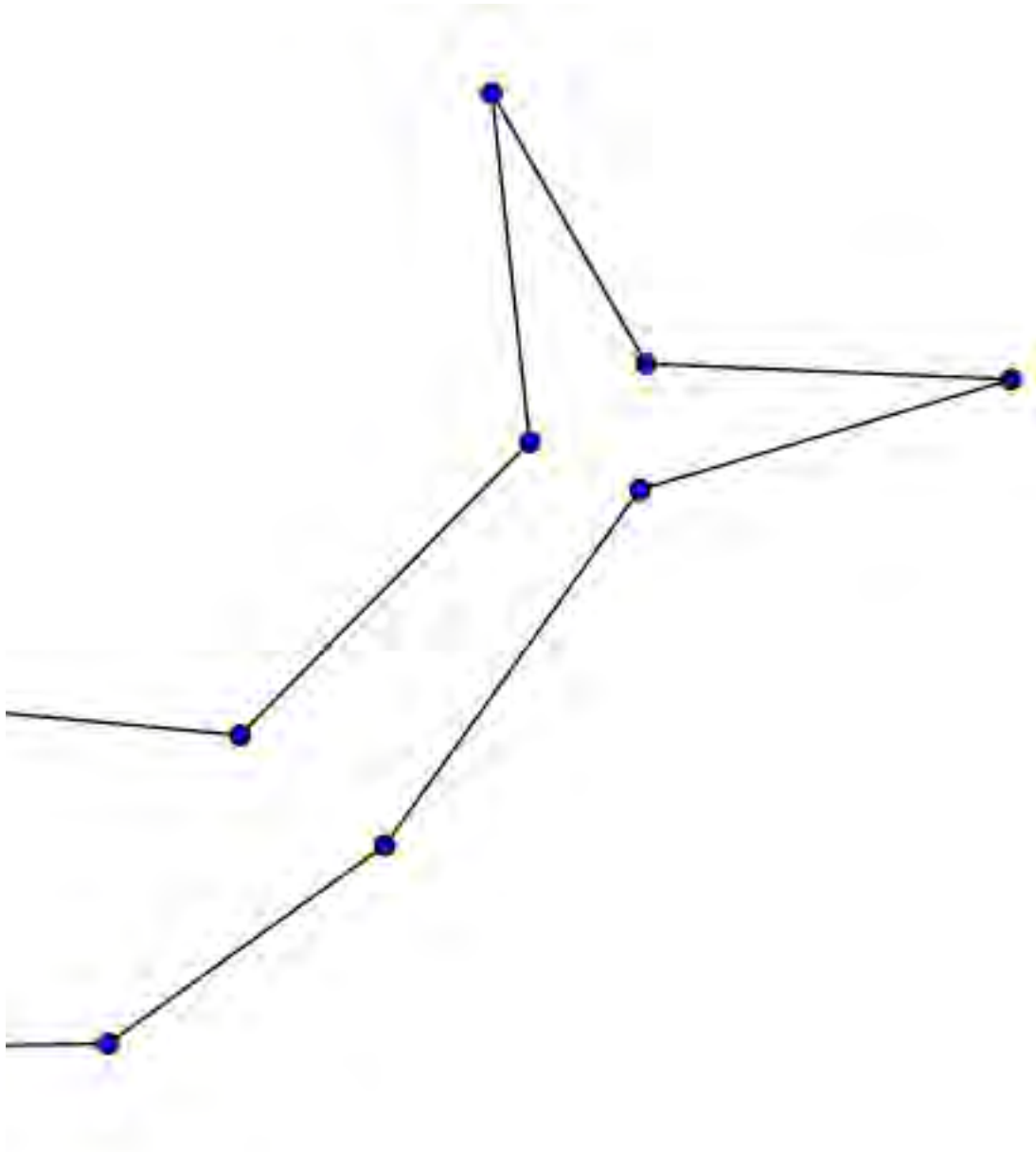
Figure 3.29: Whale Tail for Straight-cut Origami.

Chapter 4

Panorama Photography

> I don't know anything, but I do know that everything is interesting if you go into it deeply
> enough.
>
> **Richard Feynman** (American Phsyicist; 1918 - 1988)

4.1 StreetView and the Viewing Sphere

The goal of this chapter is to make a 360 degree panorama that captures everything we see around
us. How can we create a picture of all of that for somebody else to show them how interesting our
surroundings are?

Have you ever looked at Google Street View?

1. Go to `maps.google.com`, find a major street that you are familiar with. Drag the little
   yellow "Street View" person to a location for which Street View is available (indicated by
   blue lines). Take a look all around. Describe what you see.

2. Can you tell what's right above the camera? Can you tell what's right below the camera?
   What portion of the real world can you see in this Google Street View image? Explain.

We would like to find a way to record and make sense of such a Street View panorama for the
location you are in right now. [1]

As an example, consider the image in "Street View" of the beach at the "Daughters of the
American Revolution State Forest" in Goshen, MA (`http://goo.gl/tDQ5jm`)[2] or of a classroom
in Wilson Hall at Westfield State College ().[3]

Imagine that the viewing sphere is centered in the room you are in, sort of hanging in the
middle. Imagine your head is inside the sphere with one eye at the center, looking around, what
would you see? Now you want to draw the outlines of what you see on the inside of the sphere
(like a mini IMAX movie sphere). The following prompts will help you accomplish this:

---

[1] `https://www.google.com/maps/@42.1296483,-72.7953262,3a,75y,32.57h,63.19t/data=!3m4!1e1!3m2!`
`1sqi7SRQrH30-2pOiZV9mTTQ!2e0`)

[2] Full link: `https://www.google.com/maps/views/view/110195214316923802478/gphoto/6063795923312702690`)
and Classroom (link: `https://www.google.com/maps/views/view/110195214316923802478/gphoto/`
`6077800831661312866`), for skeleton)

[3] Full link: `https://www.google.com/maps/views/view/110195214316923802478/gphoto/6080253931131232050`

3. Pick two corners of the room. What does the line connecting these corners look like on the sphere? How would you describe this curve?

4. Draw a few other curves connecting the two points on the sphere. Explain what is special about the curve you drew first.

5. We would like you to investigate the skeleton of this room, including the lines where the walls meet the floor and the ceiling, and where the walls meet each other. What does this skeleton look like on the sphere? Explain your reasoning.

6. Identify one or two objects in the room and describe what they would look like on the sphere.

7. Where do parallel lines in the room map to on the sphere? Explain why.

4.2 Making Flat Images

Now we have a perfect picture of our surroundings on the sphere–its a 360 degree panorama. While we could hang these picture spheres on a wall, they do not stack nicely in a photo album. So how could you make a flat image of the full 360 degree panorama? Be ready to be creative.

8. Create a way to make a flat image. Describe where each point on the sphere would end up on your flat image (this means you have a function).

9. Using your function, draw the skeleton of the room on your flat image. What do you notice?

10. Where do parallel lines in the room end up on your flat image? Explain why.

Carl Friedrich Gauß (German mathematician, pronounced "Gauss"; 1777 - 1855) proved in his "Theorema Egregium" (Latin for "Remarkable Theorem") that it is impossible to find a "perfect" map from the sphere to the plane. Any resulting flat image will be squished or stretched in places.

11. Describe how your function stretches or squishes the image of the viewing sphere.

To prepare for the upcoming Independent Investigation, we want to understand which objects in the outside world show up as specific geometric objects on the sphere (see Figure 4.1 for illustrations of these terms).

12. Identify lines in the real world that show up on meridians on the sphere.

13. Identify lines in the real world that show up on latitudes on the sphere.

14. Identify points in the real world that show up as poles on the sphere.

15. Identify lines in the real world that show up on the equator on the sphere.

16. Identify some lines in the real world that show up on great circles on the sphere.

---

17. INDEPENDENT INVESTIGATION: For each of the functions your class developed in Investigation **8** describe in detail what happens to the following geometric objects (see Figure 4.1 for illustrations of these terms):

   a) meridians,

   b) latitudes,

   c) poles,

   d) the equator, and

   e) great circles.

The goal is for you to make sense of *all* the functions developed by your class.

Further Investigations

F1. Where do circles on the sphere go to in your flat image? Explain why.

(a) Poles and the equator on a sphere.
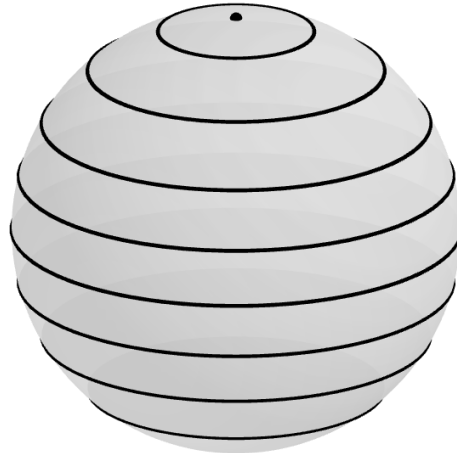
(b) Poles and latitudes (including the equator) on a sphere.
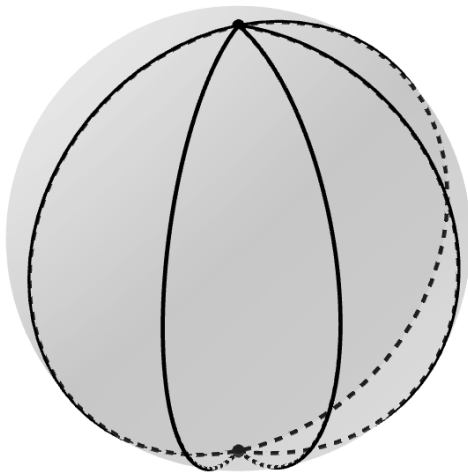
(c) Poles and meridians (or longitudes) on a sphere.

(d) A great circle, the intersection of the sphere with a plane through the center of the sphere.

Figure 4.1: Introducing various geometric objects on a sphere: **poles**, the **equator**, **latitudes**, **meridians**, and **great circles**.

4.3   Equi-rectangular Projection

You probably noticed that drawing these images is really difficult. In this section, we will finally start using cameras to create real 360 degree panoramas.

18. Download one of the following Apps (depending on whether you have an iOS or Android device) "Google PhotoSphere" (iOS) or "Google Camera" (Android)[4]. Take a 360 degree panorama in a spot where there are not many people.

19. How does this App connect to the idea of the Viewing Sphere discussed previously?

20. Notice that the App saves a flat image, as well. Mathematicians call the functions that create flat images **projections**. The App uses the so-called *equi-rectangular projection*. How does this flat image from the App compare to the projections developed in your class? What is similar, what is different? Describe everything you notice.

Figure 4.2: Equi-rectangular Projection of D.A.R. State Forest beach.

Figures 4.2-4.3 show the images of the projection created by the App for the D.A.R. State Forest beach and the Wilson Hall classroom we have seen before.

21. Describe which objects you notice as straight or curved under this new projection in Figures 4.2-4.3. Are these objects curved or straight in the real world?

22. Predict which straight objects in the real world show up as *straight* under the equi-rectangular projection.

23. Predict which straight objects in the real world show up as *curved* under the equi-rectangular projection.

---

[4]iOS: Google Photo Sphere; Android: `https://play.google.com/store/apps/details?id=com.google.android.GoogleCamera&hl=en---(Androidmaybetter,asitalsocreatestinyplanets)`

Figure 4.3: Equi-rectangular Projection of a Wilson Hall classroom at Westfield State University.

24. Which way do they curve? Explain.

25. Explain why straight objects in the real world can show up as curved or straight under the equi-rectangular projection.

4.4 Stereographic Projection

One very ancient idea for creating flat images of features on a sphere is called the ***Stereographic Projection***. Figure 4.4 shows a transparent sphere sitting on top of a plane, touching the plane at its South Pole. We want to see where the point P on the sphere ends up when projected onto the plane. Imagine a light source at the North Pole. A ray of light passing through point P will intersect with the plane in point Q. This is where the stereographic projection maps the point P.
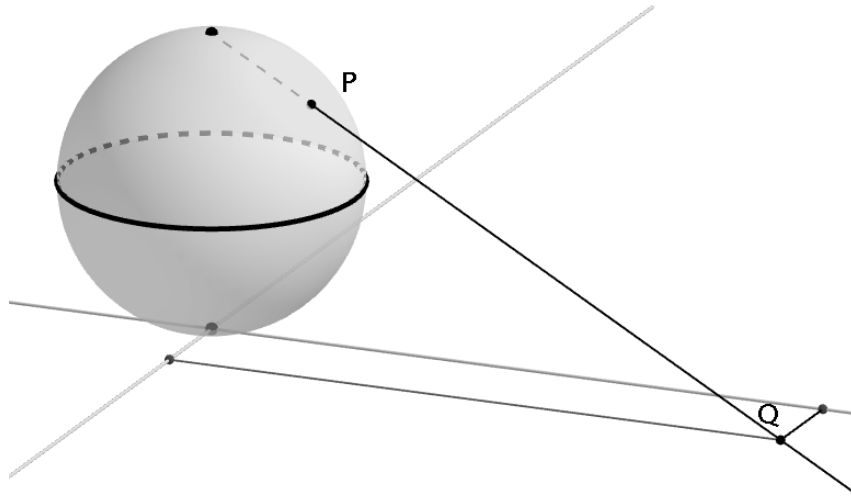
Figure 4.4: Stereographic projection of a point on the sphere onto the plane.

History (Source: Wikipedia) The Stereographic Projection was known to **Hipparchus** (Ancient Greek astronomer, geographer, and mathematician, considered the founder of trigonometry; 90 - 120 BCE), **Claudius Ptolemy** (Greco-Egyptian writer of Alexandria, known as a mathematician, astronomer, geographer, astrologer; 90 - 168 CE) and probably earlier to the Egyptians. It was originally known as the ***planisphere projection***. Planisphaerium by Ptolemy is the oldest surviving document that describes it. One of its most important uses was the representation of celestial charts. The term planisphere is still used to refer to such charts.

François d'Aiguillon gave the stereographic projection its current name in his 1613 work Opticorum libri sex philosophis juxta ac mathematicis utiles (Six Books of Optics, useful for philosophers and mathematicians alike).

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26. INDEPENDENT INVESTIGATION: For the stereographic projection describe in detail what happens to the following geometric objects (see Figure 4.1 for illustrations of these terms):

 a) meridians,

 b) latitudes,

> c) poles,
>
> d) the equator, and
>
> e) great circles.
>
> You could use Lenart Spheres and a flashlight to help with some of these investigations.

Cool Thing: Take a look at these amazing spherical sculptures created with a 3D printer: `https://www.youtube.com/watch?v=lbUOScpu0ws` and `https://www.youtube.com/watch?v=VX-0Laeczgk`. Notice in the sphere sculpture that is covered with circular shapes (about 2 minutes into the first video)? What kinds of shadows to they create in the plane? What conjecture would you make about what circles on the sphere map to in the plane under this projection? Explain.

4.5   Tiny Planets

The "Tiny Planet" App creates another kind of 360 degree panorama; see Figure 4.5. Aren't they amazing!

27. Which projection(s) could have been used to create these panoramas? Explain your thinking.

4.6   Student Project

> **28.** INDEPENDENT INVESTIGATION:
>
> **29.** Take the Google Photo and Tiny Planet Apps and create several 360 degree panoramas. Try Tiny Planets and Rabbit Holes.
>
> **30.** What features in the scene make for interesting or beautiful panoramas? Why?
>
> **31.** Select your favorite panorama and show your artwork to the class.
>
> **32.** Write an artists statement. "I find this panorama beautiful because ..."

(a) "Tiny Planet" of D.A.R. State Forest beach.

(b) "Rabbit Hole" of D.A.R. State Forest beach.

(c) "Tiny Planet" of a Wilson Hall classroom.

(d) "Rabbit Hole" of a Wilson Hall classroom.

Figure 4.5: Four tiny planets of the D.A.R. State Forest beach and the Wilson Hall classroom.

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