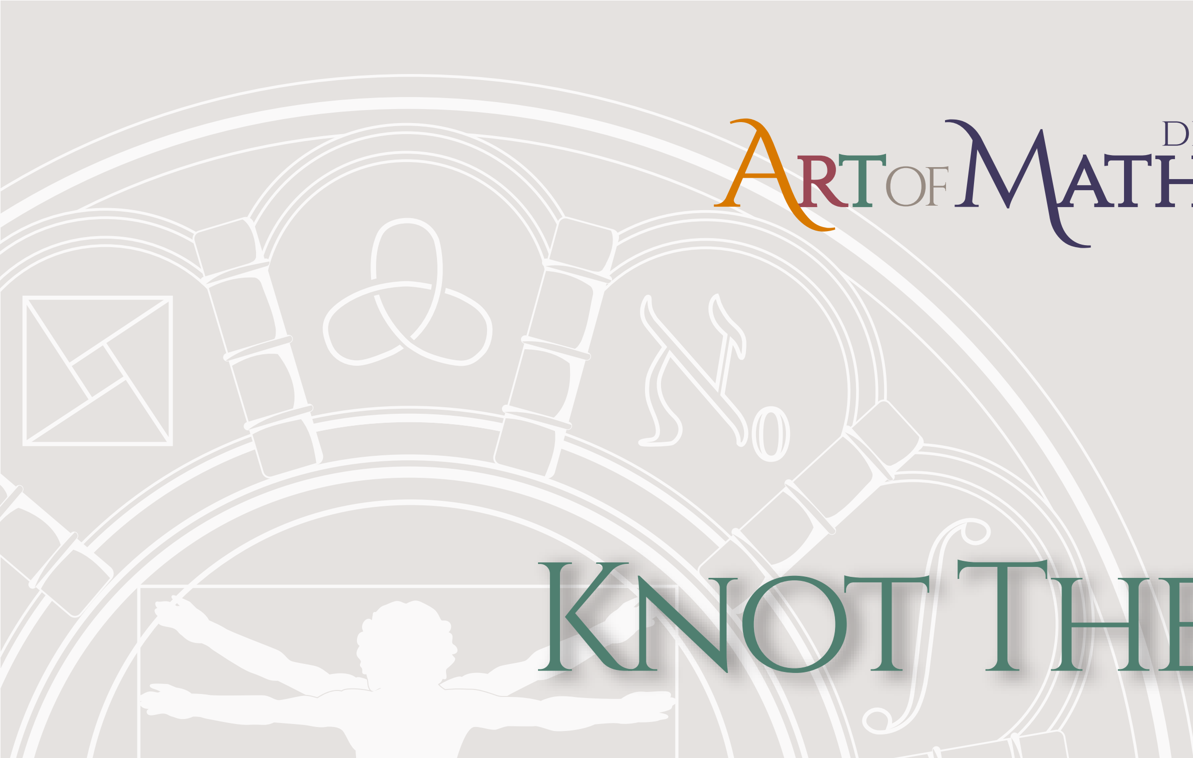
DISCOVERING THE ART OF MATHEMATICS



KNOT THEORY

MATHEMATICAL INQUIRY IN THE LIBERAL ARTS

Philip K. Hotchkiss,
with Volker Ecke, Julian F. Fleron and Christine von Renesse

Discovering the Art of Mathematics

Knot Theory

by Philip K. Hotchkiss

with Volker Ecke, Julian Fleron,
and Christine von Renesse

Acknowledgements

This work is supported by the National Science Foundation under awards NSF0836943 and NSF122915. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

These materials are also based on work supported by Project PRIME which was made possible by a generous gift from Mr. Harry Lucas.

The authors would like to thank their families for continual inspiration, support, and love. He would also like to thank Dr. Robert Hayes of the Office of Academic Affairs for financial and intellectual support and many colleagues on the Westfield State University campus, particularly John Judge, for mathematical guidance and inspiration.

Preface

Mathematics is lots of different things to lots of different people. To many people it is simply arithmetic and is viewed as inaccessible to all but the best and the brightest, but in fact this is not true. Mathematics is a field of study that is represented in almost areas of human activity. For example, it appears in art as perspective and symmetry; in writing as a metaphor; in science as a descriptive/analytic tool; in music as rhythm, sound transmission; in communication in the making of CDs; in medicine in MRIs and CAT scans. People see and experience mathematics almost every day. One area in which the beauty and depth of mathematics is accessible is knot theory.

Knot theory is, simply put, the study of the mathematics of knots. It doesnt seem like there is much mathematics there, but in fact, it is one of the richest, most beautiful, and surprisingly useful areas in mathematics. This book is an attempt to introduce you to knot theory and at the same time expose you to a non-traditional area of mathematics and also give you insight on how mathematicians think.

The format for this book is guided discovery. That is, you will be asked a series of questions that lead you to a specific conclusion. While you will not be creating the questions or the direction of the investigations you will get an idea about how mathematicians think and work. You should not expect these questions to always be easy. Occasionally you will struggle with some of the questions, as professional mathematicians often do. This does not mean you are in over your head, it just means that you will need to spend more time on that question (or series of questions). Use your classmates and professor as resources; its okay (and often helpful) to take a break from the problem but dont give up on the problem. Most of all, **have fun**.

Navigating This Book

Before you begin, it will be helpful for us to briefly describe the set-up and conventions that are used throughout this book.

As noted in the Preface, the fundamental part of this book is the Investigations. They are the sequence of problems that will help guide you on your active exploration of mathematics. In each chapter the investigations are numbered sequentially. You may work on these investigation cooperatively in groups, they may often be part of homework, selected investigations may be solved by your teacher for the purposes of illustration, or any of these and other combinations depending on how your teacher decides to structure your learning experiences.

If you are stuck on an investigation remember what **Frederick Douglass** (American slave, abolitionist, and writer; 1818 - 1895) told us: "If thee is no struggle, there is no progress." Keep thinking about it, talk to peers, or ask your teacher for help. If you want you can temporarily put it aside and move on to the next section of the chapter. The sections are often somewhat independent.

Investigation numbers are bolded to help you identify the relationship between them.

Independent investigations are so-called to point out that the task is more significant than the typical investigations. They may require more involved mathematical investigation, additional research outside of class, or a significant writing component. They may also signify an opportunity for class discussion or group reporting once work has reached a certain stage of completion.

Further investigations, when included are meant to continue the investigations of the area in question to a higher level. Often the level of sophistication of these investigations will be higher. Additionally, our guidance will be more cursory.

Within each book in this series the chapters are chosen sequentially so there is a dominant theme and direction to the book. However, it is often the case that chapters can be used independently of one another - both within a given book and among books in the series. So you may find your teacher choosing chapters from a number of different books - and even including "chapters" of their own that they have created to craft a coherent course for you. More information on chapter dependence within single books is available online.

Certain conventions are quite important to note. Because of the central role of proof in mathematics, definitions are essential. But different contexts suggest different degrees of formality. In our text we use the following conventions regarding definitions:

- An *undefined term* is italicized the first time it is used. This signifies that the term is: a standard technical term which will not be defined and may be new to the reader; a term that will be defined a bit later; or an important non-technical term that may be new to the reader, suggesting a dictionary consultation may be helpful.
- An ***informal definition*** is italicized and bold faced the first time it is used. This signifies that an implicit, non-technical, and/or intuitive definition should be clear from context. Often this means that a formal definition at this point would take the discussion too far afield or be overly pedantic.
- A **formal definition** is bolded the first time it is used. This is a formal definition that suitably precise for logical, rigorous proofs to be developed from the definition.

In each chapter the first time a biographical name appears it is bolded and basic biographical information is included parenthetically to provide some historical, cultural, and human connections.

Contents

CHAPTER 1

Introduction and a Brief History of Knots

> Mathematics is one of the deepest and most powerful expressions of pure human reason, and, at the same time, the most fundamental resource for description and analysis of the experiential world.
>
> **Hyman Bass** (American Mathematician; 1932 -)

Knot theory has become one of the most popular and important areas of mathematics. It has benefited from contributions by professional mathematicians, college students, amateur mathematicians and both professionals and amateurs in other sciences. As with many areas of mathematics, knot theory has its origins in attempts to describe the physical world; but it soon evolved into a rich mathematical field.

Its origins date back to the mid 19^{th} century. In 1858, **Hermann von Helmholtz** (German mathematical physicist; 1821 - 1894) published an article in which he described motion in a ***perfect fluid***; that is, a fluid that did not move or compress when an object moved through it and did not generate any friction as the object moved [E]. Of course, such a fluid does not exist, so Helmholtz's discussion was purely theoretical, but even this type of theoretical analysis important because it gives scientists a place to begin in examining the reality of the problem. In any event, one of Helmholtz's conclusions involved the movement of *vortex rings* and this perfect fluid.

Helmholtz's paper was read by **Peter Guthrie Tait** (Scottish physicist; 1813 - 1901), and Tait built a machine that created smoke rings, which behaved as Helmholtz had predicted. One of Tait's friends and collaborators was **William Thomson** (Scottish mathematician and physicist; 1824 - 1907) who, later on, was better known as Lord Kelvin . Thomson was very intrigued by Tait's machine and the stability of these smoke rings lead him to wonder if perhaps these rings could be used to describe the nature of *matter* [E]. In January 1867 Thomson wrote to Helmholtz detailing his idea:

> The absolute permanence of the rotation [of vortex rings] ... shows that if there is a perfect fluid all through space, constituting the substance of all matter, a vortex ring would be as permanent as the solid atoms assumed by Lucretius [a Roman poet and philosopher, c. 99-55, BCE] and his followers (and predecessors) to account for the permanent properties of bodies (as gold, lead, etc.) and the differences of their characters ... Thus a long chain of vortex rings, or three rings, each running through each of the other, would give each very characteristic reactions upon other such kinetic atoms [T, 513-516].
>
> **William Thomson** (Scottish Mathematician and Physicist; 1824 - 1907)

Thomson eventually conjectured that different knots and *links* in the ***ether***, an inert medium that many scientists thought filled the universe, made up the different *atoms*. Thomson, Tait and **James Clerk Maxwell** (Scottish mathematical physicist; 1831 - 1879), another, began a program of research on Thomson's conjecture. While Thomson and Maxwell began to investigate the manner in which knots could create matter, Tait began to try to catalogue all knots with fewer than eight crossings. [E] (The ***crossing number*** of a knot is the minimal number of times any drawing of the knot has to cross over itself. We will investigate this in a later chapter.)

Tait recognized that different drawings of a knot could have different numbers of crossings. Therefore, he had to be able to determine whether two different drawings represented the same knot. Tait determined a method of describing knots and in 1876 published a table of all knots up to seven crossings, along with the ***minimal diagram*** of the knot; the drawing of the knot with the minimal number of crossings. [HTW] Amazingly, more advanced techniques developed in the 1920's showed that this table (and subsequent tables that Tait produced) were very accurate. [S]

Tait's efforts inspired other people to create their own knot tables. The **Reverend Thomas P. Kirkman** (British Mathematician; 1806 - 1895) sent Tait a table of all knots up to ten crossings. (Interestingly, in a paper entitled The Enumeration, Description, and Construction of Knots of fewer than Ten Crossings, Kirkman gave a 101-word definition of a knot.) **C.N. Little** (American Mathematician; 1858 - 1923), a professor of civil engineering, created a table of knots up to ten crossings and sent them to Tait. [HTW] Little and Tait corresponded and with encouragement by Tait, Little classified ***alternating knots*** (knots where the crossings alternate between over and under as you travel around the knot) up to eleven crossings. Over the next 25 years, Tait, Little and Kirkman managed to catalogue alternating knots up to eleven crossings and *non-alternating* knots up to ten crossings. [HTW].

In 1887 **Albert Michelson** (American Physicist; 1852 - 1931), and **Edward Morley** (American Chemist; 1838 - 1923), conducted a famous experiment that was designed, and failed, to detect the ether. As a result, Thomson's theory that atoms were made from knots in the ether faded and was eventually abandoned by physicists in favor of the current model. [S]. However, mathematicians remained interested in knots and over the next century knot theory evolved into a popular and dynamic area of mathematics.

In the following chapters we will explore some of the basic ideas and techniques in knot theory, some of the ways knots have appeared in art, culture, and some of the recent scientific results involving knot theory.

1. Knot theory arose from a scientific theory (that atoms were knots in the ether) that was eventually abandoned. As scientific knowledge grows old, theories are abandoned in favor of new developments. Often in retrospect, these abandoned theories seem silly. Do some research and describe a scientific theory (other than the theory that atoms were knots in the ether) that has been abandoned and now seems silly. Be sure to address the reasons why the theory was abandoned. You should have several sources (other than Wikipedia) that support your conclusions.

CHAPTER 2

Tangles®

Our main tool in this course is a mathematical toy called a Tangle®. Tangles® are made from little plastic pieces that form a quarter circle and can be snapped together as illustrated in Figure 2.1

Figure 2.1: Tangle®

In addition to being very useful for understanding and studying knots due to their construction, they are very interesting mathematically in their own right. In this first set of Investigations you will study some of the properties of Tangles®. In particular, we will study **simple Tangles®**; that is, those Tangles® that form a single, closed, *unknotted loop* is illustrated in Figure 2.2.

Figure 2.2: A Simple Tangle®

2.1. Counting simple Tangles®

.

To a mathematician, counting simple Tangles® means determining the number of *topologically distinct* shapes one can make with a given number of pieces. We say that two simple Tangles® are ***topologically distinct*** if it is impossible to deform one simple Tangle® into the other without taking it apart.

1. Begin exploring the number of pieces required to make a simple Tangle® and the number distinct shapes that can be made with each number of pieces. Sometimes there will be will be two or more distinct shapes that can be made with a given number of pieces. Remember that a simple Tangle® must form a single, closed, unknotted loop.

2. Catalogue all of the different distinct simple Tangles® that can be made with 10 or fewer pieces. If there are two (or more) topologically distinct shapes for a particular number of pieces, you should describe each shape in your notes with either words or pictures (one way to do this is to use your cell phone to take a picture of each shape and include that your notes) and explain how you **know** that the shapes you have found are topologically distinct.

3. Do you think there is a recognizable pattern in the number of topologically distinct simple Tangles®? Explain.

2.2. Planar Tangles®

.

When confronted with a situation like the one you described in Investigation **3**, mathematicians often try to identify a similar, but (hopefully) easier, problem to solve. As you investigated the number of topologically distinct simple Tangles® in Investigation **2**, you may have noticed that some of the simple Tangles® could be made to lie flat on the table. These flat simple Tangles® are called **planar Tangles®**.

Can we find a pattern in the number of planar Tangles®? The next few Investigations explore this.

<u>Note</u>: **When talking about planar Tangles®, we require that the pieces fit flush together, i.e. no small spaces occur at the spots where the Tangle® pieces connect.**

4. Based on your table in Investigation **2**, what numbers of pieces resulted in a planar Tangles®?

5. Once you have more than 10 pieces, what is the smallest number of pieces required to create a planar Tangle®? Sketch the shapes in your notebook or use your cell phones to take a picture of each shape for your notebook.

6. What is the next smallest number of pieces required to create a planar Tangle®? Sketch the shapes in your notebook or use your cell phones to take a picture of each shape for your notebook.

7. Do you see a pattern in your answers to Investigations **4-6**? If so, describe it.

8. Does the pattern you described in Investigation **7** account for every possible planar Tangle® or can you find planar Tangles®whose number of pieces does not fit with this pattern? Either explain why you believe your pattern accounts for every possible planar Tangle®, or find at least one example of a planar Tangle® whose number of pieces does not fit with your pattern. Draw a sketch these examples in your notebook or use your cell phones to take a picture of each example for your notebook.

9. **Classroom Discussion:** Share your answers to Investigation **8** and either come to an agreement as to why your pattern in Investigation **7** does, in fact, account for every possible planar Tangle®; or if there are examples of planar Tangles® with an "incorrect" number of pieces, pick an example and decide whether this is a valid example or if there is some illegitimacy to the planarness of this Tangle®.

10. Based on your answers to Investigations **4-9**, can you make an educated guess, or what is commonly called a ***conjecture***, about the number of pieces required to make any planar Tangle®?

11. Explain why your conjecture in Investigation **10** should be true and does, in fact, account for every possible planar Tangle®.

Hint: What is it about the shapes of the individual pieces that requires a planar Tangle® to have the number of pieces that agrees with your conjecture in Investigation **10**?

2.3. Counting planar Tangles®

.

At the end of Section 2.2, you described a pattern about the number of pieces required to make a planar Tangle®. The next question a mathematician might ask is, given a specific number of pieces, how many different planar Tangle® shapes can be made with that number of pieces? Before we begin, we need to determine what the word "different" means in this context. In Section 2.1 we used the notion of topologically distinct as our basis for shapes being different. Will that work here?

12. Based on your work for Investigation **5**, are two planar Tangles® that are made with the same number of pieces topologically distinct? Explain.

Based on your answer to Investigation **12**, we need another notion of different. We will call two planar Tangles® **geometrically distinct**, if we can not change one planar Tangle® into the other by simply rotating or flipping it.

Note: The following set of questions are different than the questions in Section 2.1. Now we really **are** interested in the different shapes that can be made to lie flat.

13. For each of the first four number of pieces that make planar Tangles®, determine the number of geometrically distinct planar Tangles® shapes that can be made. Either sketch to use you cell phone to take a picture of each of these shapes for your notebook.

14. Do you think there is a recognizable pattern in the number of geometrically distinct planar Tangles® shapes? Explain.

15. At this point things get more complicated. After the planar Tangles® you considered in Investigation **13**, the next smallest planar Tangle® has 31 geometrically distinct shapes. Find as many of them as you can and include a picture of each shape.

To simplify our writing in what follows, we will call any Tangle® made from n pieces an **n-Tangle®**.

We can use graph paper to draw each planar Tangle® by using the outside edges of the squares to guide our drawing of each Tangle® piece. The main requirement is that the quarter circles match up correctly as they would in a planar Tangle®.

Here are two examples. We can match the planar 4-Tangle® to a single square and the 8-Tangle® to a three squares as follows.

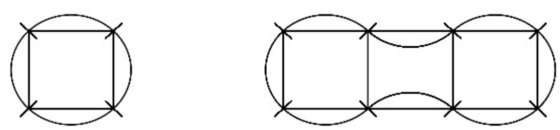


Figure 2.3: Fitting Tangles® to squares

16. In Figure 2.4 there are six shapes made up from seven squares. Some match up to planar n-Tangles® you made for a specific value of n. What is this value of n? Why do some of the shapes match up and not others?

It turns out that finding a formula that gives the number of planar n-Tangles® for each value of n is difficult. In fact, it is unclear whether this mathematical problem has been solved, and **Julian Fleron** (American Mathematician; 1966 -), has conjectured that this is related to a very important and unsolved problem in another area of mathematics, the ***Polyomino Enumeration Problem***. This is a very old problem that essentially tries to count the number of distinct **polyominos**, connected figures one can make with n squares with the requirement that each square share at least one side with another square. For example, the figures in Figure 2.4 are polyominos made from seven squares. Since these are made with seven squares, we call these **seven-polyominos**.

Some of the above mentioned polyominos may look familiar to you if you have ever played the computer game Tetris. However, the polyominos used in Tetris are made with four squares (commonly called *tetrominos*) which are shown in Figure 2.5.

For those who have not played Tetris here is a simple description of how the game works. These pieces drop down the computer screen, which is ten blocks wide, and the object of the game is to rotate the pieces so that you can stack them as efficiently as possible at the bottom of the screen. A screen shot of a partially played game is shown in Figure 2.6.

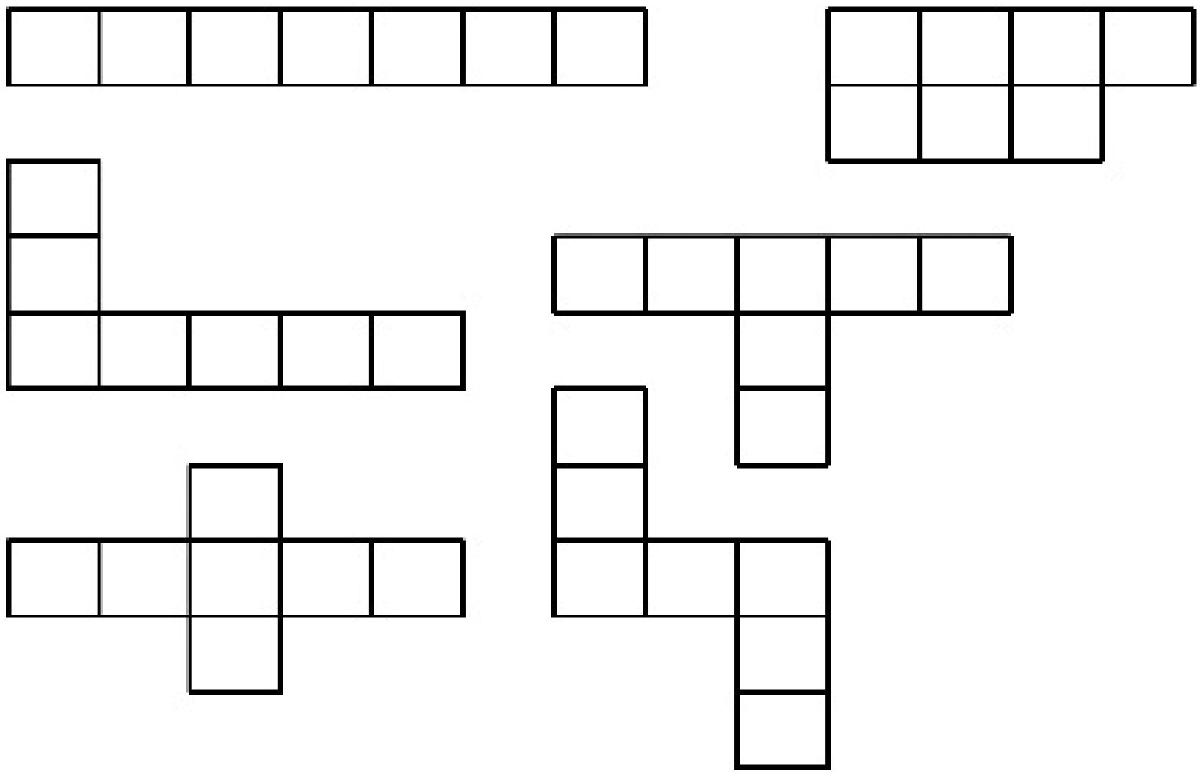


Figure 2.4: Matching planar Tangles® to shapes made from squares

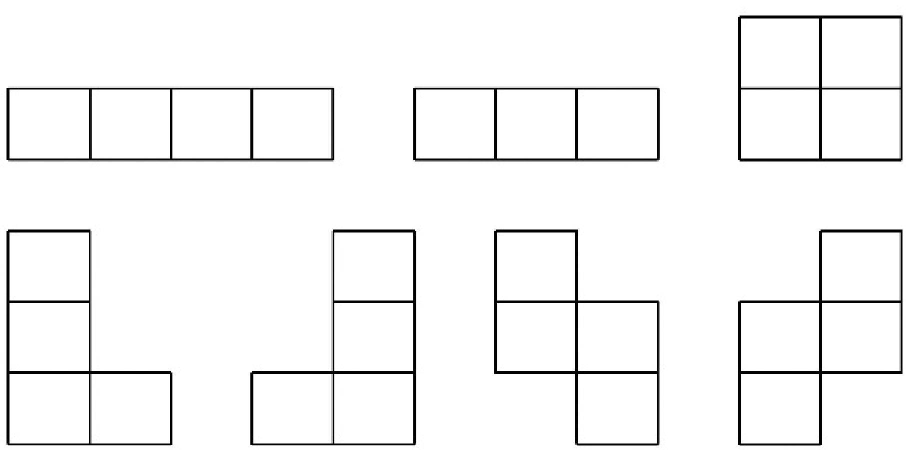


Figure 2.5: Tetris Tetrominos

Every time a row is completely filled it disappears and the players score increases. For example, in the picture in Figure 2.6, if the player rotates the dropping piece 90° counter-clockwise and can drop it into the open spot in the second column, the second row will disappear. The game ends when the pieces are stacked so high that no more can drop down.

This (seemingly) simple game has a connection to another very important unsolved problem that has a $1,000,000 prize for its solution. Three computer scientists, **Erik D. Demaine** (Canadian Computer Scientist; 1981 -), **Susan Hohenberger** (American Computer Scientist; 1978 -), and **David Liben-Nowell** (American Computer Scientist; 1977 -), have determined that the difficulty of a simplified version

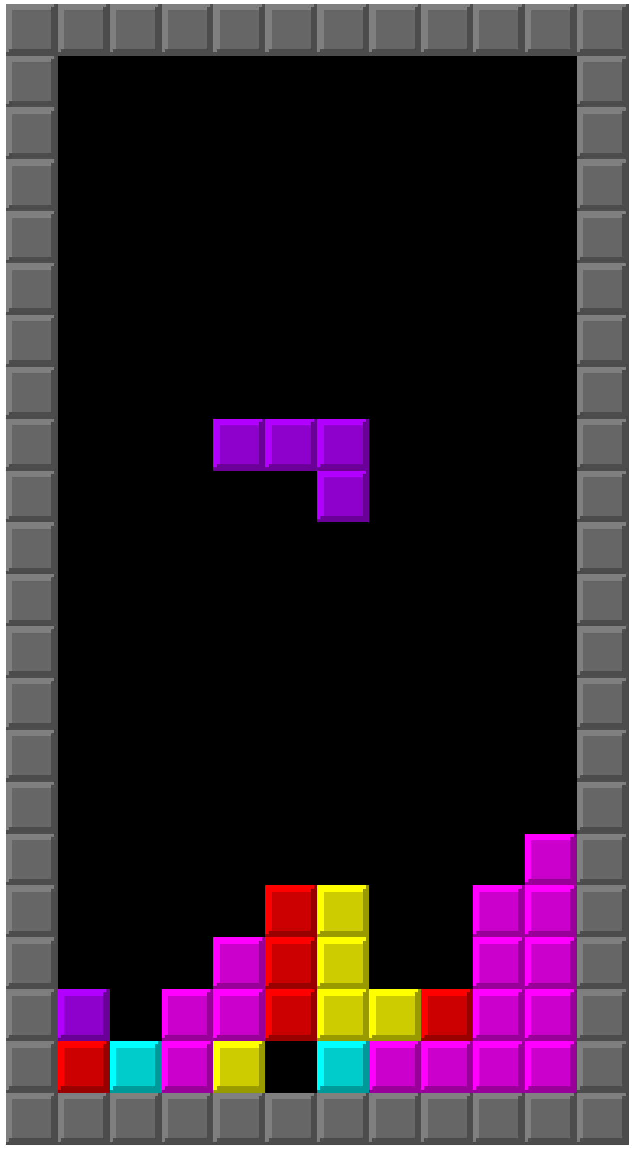


Figure 2.6: Tetris

of this game puts it in a category called NP-complete [Pe]. This essentially means that if someone could find an a procedure (or *algorithm*) that maximizes the players score which can then be programmed into a computer that will run efficiently, then this and other important problems can be easily solved. On the other hand, if some one could prove no such program exists, then no such program exists for any of the related problems. This bigger problem is called the P vs. NP problem and is one of seven problems, called the Millennium Problems that have a $1,000,000 prize sponsored by the Clay Mathematics Institute (To find out more about the Millennium Problems visit [C] or read [D]).

17. Are you surprised that playing with a mathematical toy (Tangles®) is connected to a famous unsolved mathematical problem that has a $1,000,000 prize? Explain.

CHAPTER 3

Knot Projections and Knot Equivalency

In the last chapter, we explored some of the properties of Tangles®. This was important because Tangles® can be very valuable in exploring knots, and they will be our major tool this semester. Now we want to move on to knots. Chapter 1 gave us an overview of the history of knot theory. As we mentioned in that chapter, one of the fundamental problems in knot theory is tabulating knots. However this is not so easy, because the same knot can have many different forms.

To adequately explore knots, we need to define some terms. What is a *knot*? In everyday terms we know what a knot is. We use them everyday when we tie our shoes, or a ribbon on a package. However, these leave two free strands at each end as shown below.

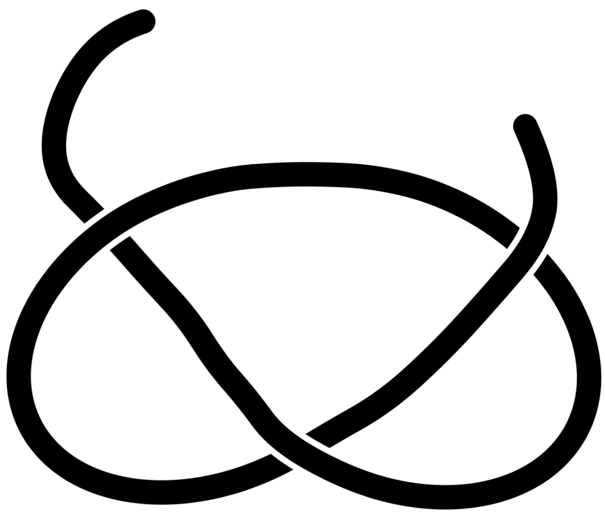


Figure 3.1: Everyday knot

To a mathematician, this is uninteresting, because as long as there are two free ends we can untangle the knot completely, no matter how complicated the original knot was. So to make the situation interesting mathematically, we glue the two ends together to make a closed loop as shown in Figure 3.2.

So informally, a ***knot*** is a knotted closed loop. There are more formal and technical definitions for knots (see [L] for an example of a standard definition), but this will do for us. However, it should be noted that these more technical definitions are necessary in order to prevent bizarre examples from being classified as knots. The reason quotes were used in our definition of a knot is because the simplest knot is a closed loop with no knot as shown Figure 3.3. This is called the **unknot** or **trivial knot**.

One of the advantages of using the Tangles® is that we can use them to make the knot and then use that model to study the properties of the knot.

There are may ways to represent a knot. In Figure 3.4 there are several different representations of the same knot.

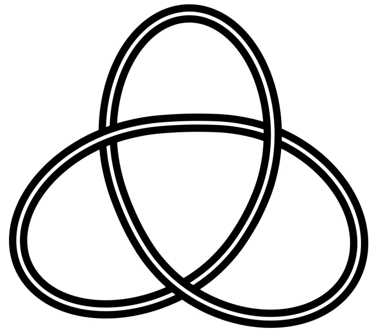


Figure 3.2: A mathematical knot

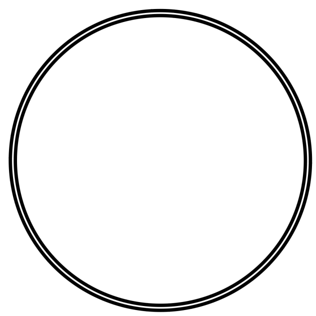


Figure 3.3: The unknot or trivial knot

A picture like the ones in Figure 3.4 is called a ***knot projection***. Most of the projections in this book will be drawn as in the first two pictures. The knots you make with your Tangles® will look like the one in the last picture. Be aware that because the Tangles® are made with hard plastic quarter circles, any knot you make will never look as smooth as the ones in the text.

Figure 3.5 shows a knot that we want to make with our Tangles®. This picture looks complicated so take a few minutes to think about how you would make this knot.

1. Now that youve thought about this for a little bit, make it with your Tangle® and describe how you made the knot.

2. Without playing with the knot you just made, do you think this knot can be untangled; that is, can you make it into the unknot without taking it apart? Why or Why not?

3. Try to untangle the knot. Were you successful? If you were successful describe in words or pictures how you untangled the knot. If you were not successful explain, as best you can, why you could not untangle the knot.

You will need to make drawings of knots throught the text and the easiest projection for you to draw is the one that looks like the picture below in Figure 3.6.

However, we need to be careful in how we draw the knot. For example, if we draw a knot like the one shown in Figure 3.7, we do not know exactly what the knot looks like. In particular, we have problems at the crossings. We cannot tell which part or ***strand***, of the knot goes over the other. To indicate which strand goes underneath and which one goes on top, we draw the strand that goes underneath with a small space for the strand that goes on top as indicated in Figure 3.8.

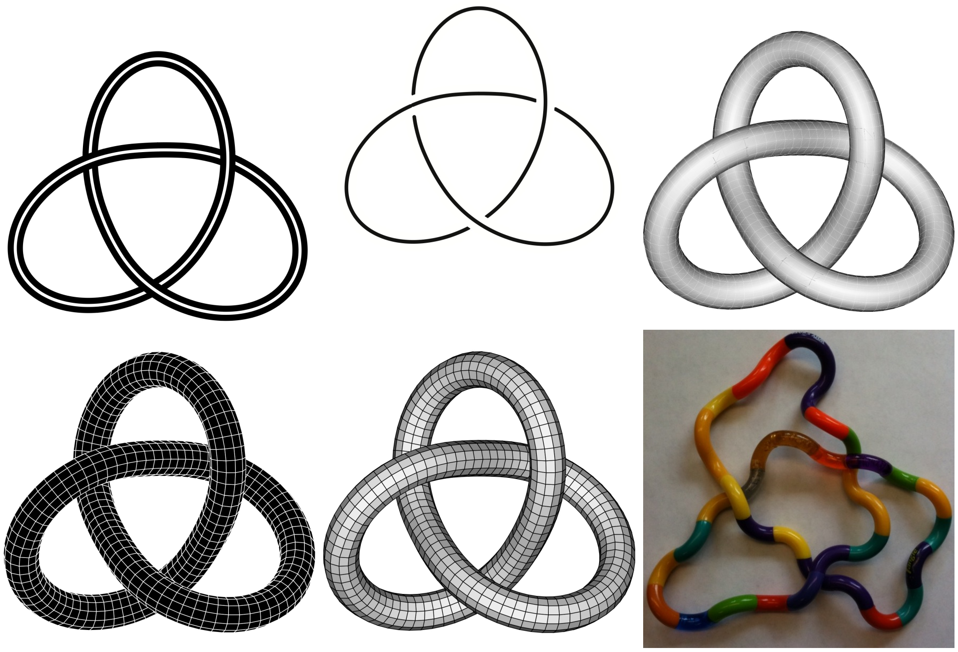


Figure 3.4: Different representations

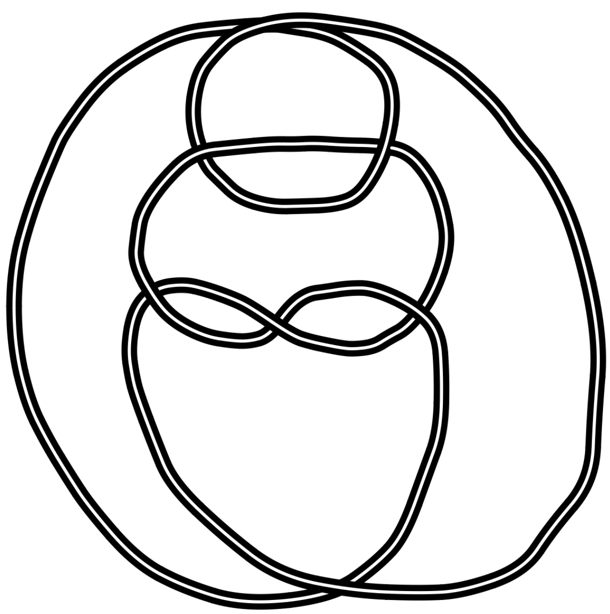


Figure 3.5: A complicated knot

Since there are two ways to draw each crossing and there are seven crossings, so there are

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128$$

different projections associated with the drawing in Figure 3.7. Three of the possibilities are shown in Figure 3.9.

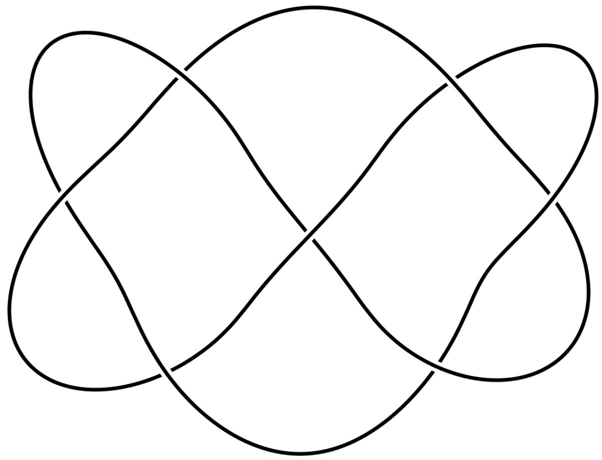


Figure 3.6: A knot projection

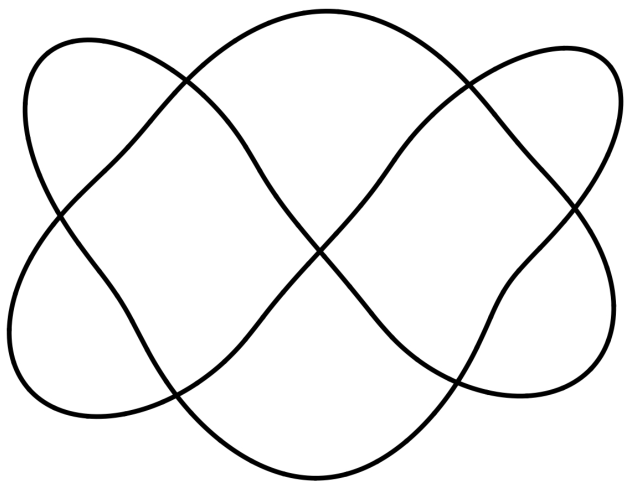


Figure 3.7: An incomplete projection

A natural question to ask at this point is, do these three projections represent three different knots or are they all the same knot?

4. Make each of the three knots shown in Figure 3.9 with your Tangle® and try to deform it into the other two knots without taking the knot apart. What happens in each case? Do any of these projections represent the same knot? Explain.

Hopefully, you determined that two of the knots could be deformed into the unknot; in this case we say that both projections are *equivalent* to the unknot. Like the term knot, the formal definition of knot equivalency is very technical, but for us, we will say two knot projections are ***equivalent*** if we can deform one projection into the other without cutting the knot, untying it; and then tying it into the other knot and gluing the free ends together.

This is essentially the central problem in knot theory:

Figure 3.8: Examples of crossings

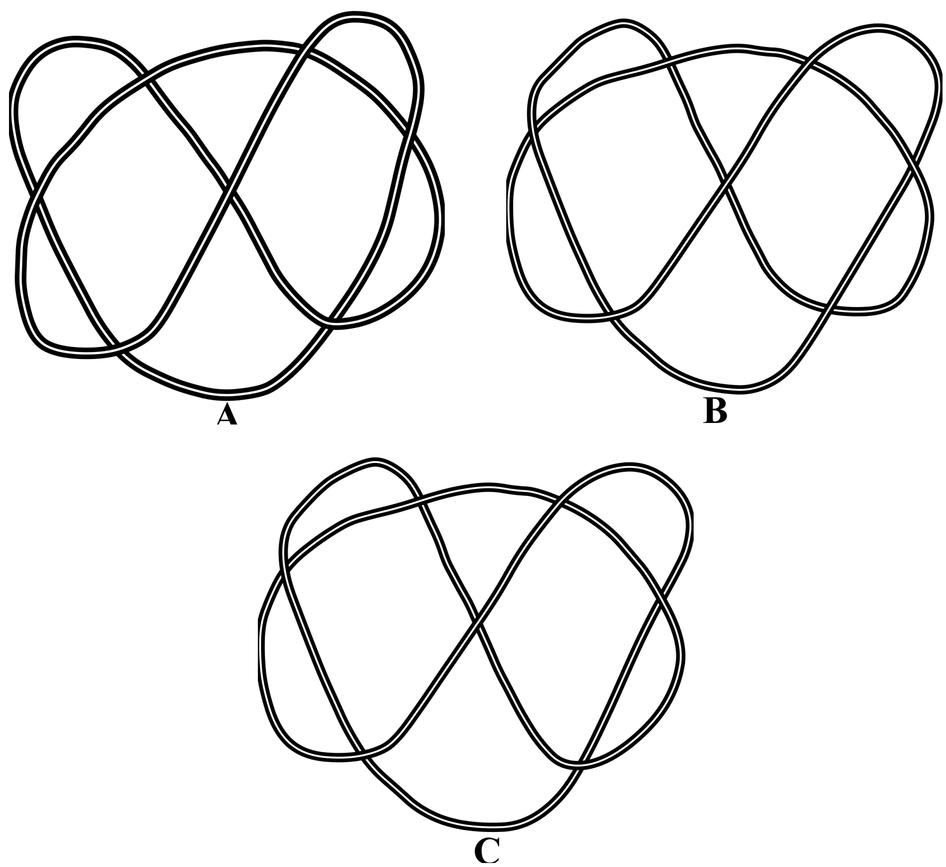


Figure 3.9: Three possible projections

Central Problem. Is there a way to determine whether two knot projections represent distinct knots or the same knot?

For all we know, perhaps every knot projection is trivial; i.e. it is equivalent to the unknot. However, as you saw in Investigation **4**, one of the three knots did not untangle. It simplified to one of the most well known knots called the **trefoil knot**, (from the Latin word **trifolium** which means three-leaved plant) which is shown in Figure 3.10.

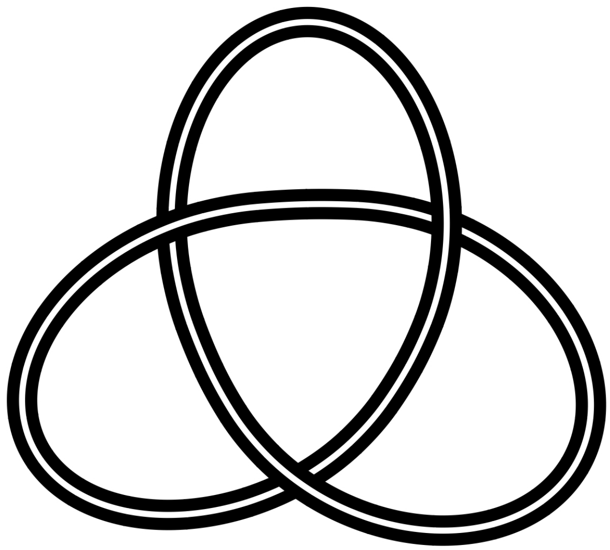


Figure 3.10: The trefoil knot

The question of knot equivalency is much more subtle than it appears. For example, the two knots in Figure 3.11 look very different but in fact, are equivalent.

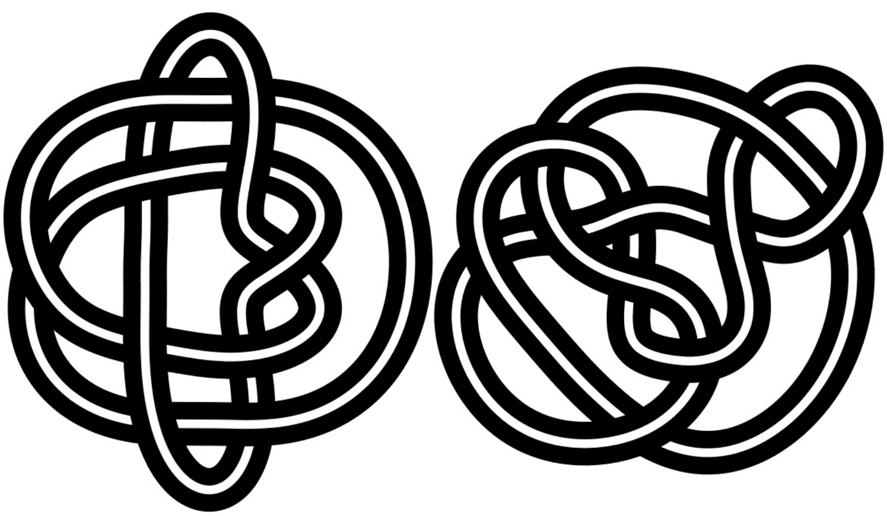


Figure 3.11: The Perko Pair

This fact, which we will dscuss further in Chapter 5, was discovered in the 1973 by **Ken Perko** (American Lawyer and amateur mathematician; 1943 -).

it turns out that there is no way to determine for sure whether or not two knots are equivalent. However, there are techniques that can help determine if two knot projections represent distinct knots. This involves finding some property that every projection of one knot has but is not shared by the other projection. A property that every projection of a knot has is called a ***knot invarient***. We will return to knot invarients in later chapters, but for now we will look at simple ways that one can change the projection of a knot. The next three questions consider the three most basic ways to change a knot projection. These basic moves, called the ***Reidemeister moves*** turn out to be the only types of moves you need to transform a knot projection into any other equivalent projection.

5. In Figure 3.12 are two projections of the trefoil knot. How is the second projection created from the first?

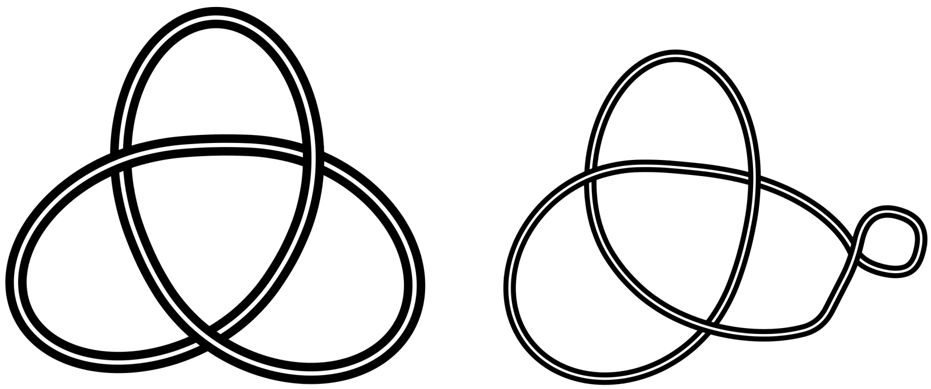


Figure 3.12: Two projections of the trefoil

6. In Figure 3.13 is another projection of the trefoil. How was it created from the original projection of the trefoil?

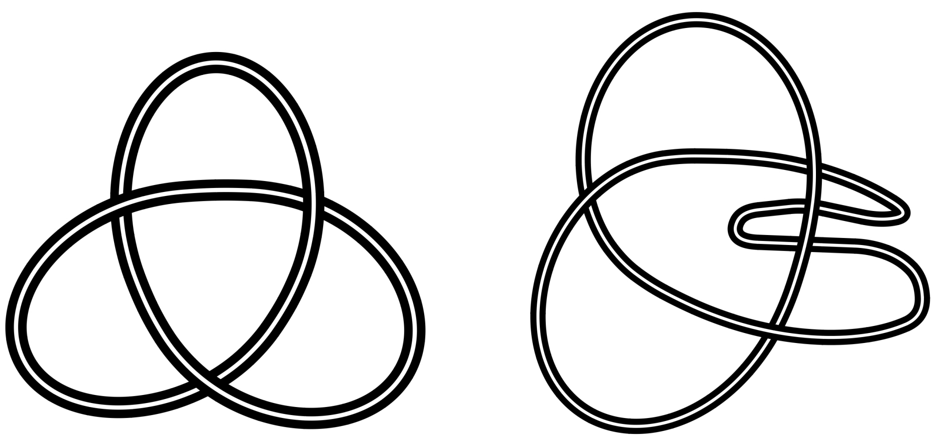


Figure 3.13: Two projections of the trefoil

7. In Figure 3.14 is another projection of the trefoil, how was it created from the original projection of the trefoil?

8. Use your answers to Investigations (5) - (7) to explain how a knot can have infinitely many projections.

As your answer to Investigation **8**) illustrates, a knot can have infinitely many projections. Sometimes it is easy to determine that two projections are equivalent.

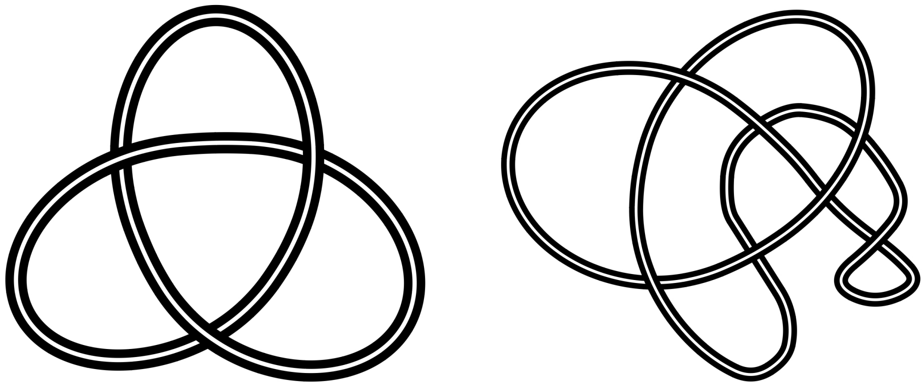


Figure 3.14: Two projections of the trefoil

9. Use your Tangle® to show that the two projections in Figure 3.15 are equivalent. That is, make one of the knots with your Tangles® and then deform it to look like the other. In your notebook either describe in words or make a series of drawings showing how you accomplished this it.

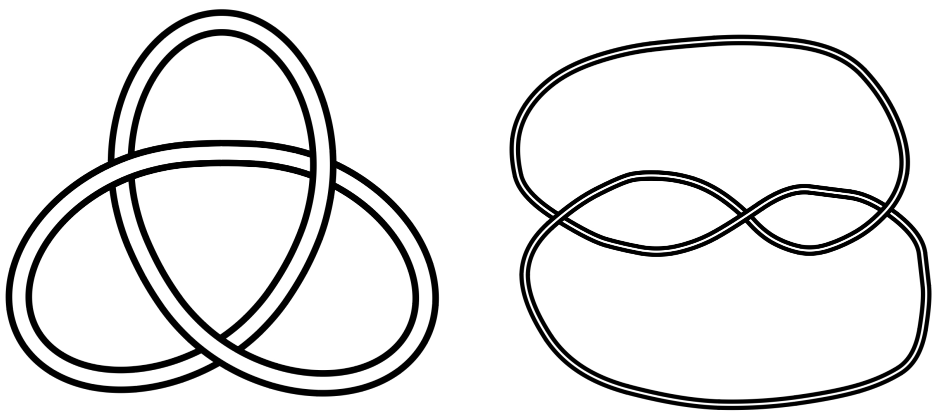


Figure 3.15: Two projections of the trefoil

The trefoil knot, to which we will be referring to extensively throughout this book, has some very interesting properties, one of which is illustrated in the following questions. In Figure 3.16 are two projections of the trefoil.

10. Do you think that these projections are equivalent? Why or why not?

Now make trefoil A with your Tangles® and then see if you can deform it into B.

11. How well did you succeed in turning projection A into B? Remember that the end result needs to look exactly like projection B (including the over and under aspects of the crossings.)

12. Do you think these two projections are equivalent? Explain.

The two projections of the trefoil in Figure 3.16 are ***mirror images*** of each other; that is, if you held one up to a mirror, the image would be the other and vice versa, just like your left and right hand. We call A the **left-handed** trefoil and B the **right-handed** trefoil. **Max Dehn** (German Mathematician; 1878 - 1952), proved in 1914 that the right and left-handed trefoils are not equivalent to each other, but the proof of this is too complicated for this course, so we will accept it as a fact.

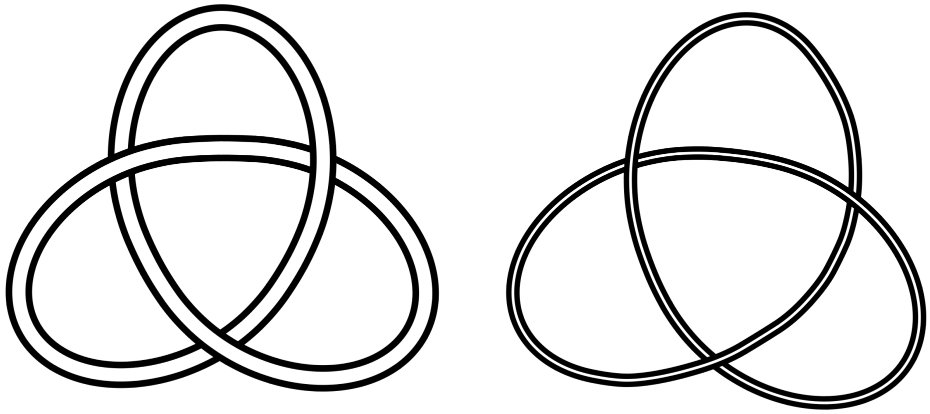


Figure 3.16: Two projections of the trefoil

13. Do you think it is possible for any knot to be equivalent to its mirror image? Explain.

14. Make a knot of your own design with your Tangles® and draw a picture of it in your notebook. Do not make this knot very complicated. Then use a mirror provided by the professor to draw the mirror image of the knot you made in your notebook as well. Now try to deform the knot you made into its mirror image. How successful were you in your attempt to deform your knot into its mirror image? If you were successful in deforming the knot into its mirror image either describe how you accomplished this in words or by a series of drawings in your notebook. If not, indicate whether or not you believe it is possible to deform the knot into its mirror image and explain why you believe this.

15. Now repeat the process in Investigation **14** several times with different knots of your own design and attempt to deform them into their mirror images. How successful were you in your attempt to deform your knots into their mirror images? If you were successful in deforming the knot into its mirror image either describe how you accomplished this in words or by a series of drawings in your notebook. If not, indicate whether or not you believe it is possible to deform the knot into its mirror image and explain why you believe this.

In Figure 3.17 are two projections of a knot called the***figure-eight knot***.

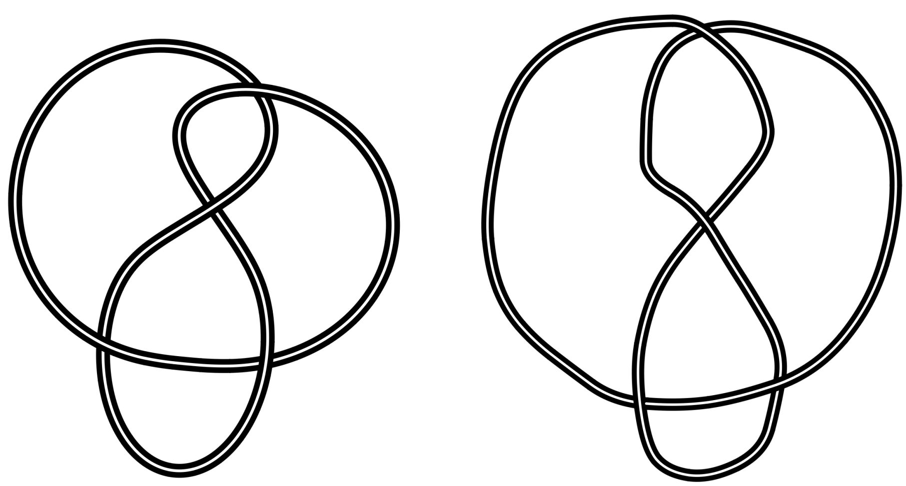


Figure 3.17: Two projections of the figure-eight knot

16. Why do you think this knot is called the figure-eight knot?

17. Use your Tangle® to make one of the two projections in Figure 3.17 and try to deform it into the other projection. How successful were you in your attempt to deform the figure-eight knot into its mirror image? If you were successful in deforming the knot into its mirror image either describe how you accomplished this in words or by a series of drawings in your notebook. If not, indicate whether or not you believe it is possible to deform the knot into its mirror image and explain why you believe this.

In Investigations (14) - (17) you attempted to determine whether several knots were equivalent to their mirror images. You may, or may not, have been successful in this endeavour. We say a knot is **chiral** if it is <u>not equivalent</u> to its mirror image and **amphichiral** if it <u>is</u> equivalent to its mirror image.

18. Of the knots you considered in Investigations (14) - (17), which ones are chiral? Which ones are amphichiral?

This notion of chirality is very important in chemistry. The physical configuration of many molecules is chiral in the sense that they can not be deformed into their mirror images. We use the terms left-handed and right-handed to distinguish between the mirror images of chiral molecules. This difference turns out to be very important.

For example, all creatures use only right-handed sugars and left-handed amino acids [Pi]. Most drugs have a left and a right-handed form. The pain reliever ibuprofen is chiral and the left-handed version is one hundred times more powerful than the right-handed version. However, it is much too expensive and difficult to create pills using only the left-handed version, so currently all ibuprofen pills contain an equal amount of left and right-handed molecules [Pi]. Sometimes the mirror images of a chemical are very different. The chemical Darvon is a painkiller and its mirror image, the appropriately named Novrad, is a cough medicine.

Another (in)famous illustration involves the drug thalidomide. In the 1950s and 60s a German pharmaceutical company, Chieme Grnenthal, began marketing the drug thalidomide as a sedative for morning sickness in pregnant women. Unfortunately if thalidomide was taken between the 26th and 30th day of the pregnancy then birth defects were possible, and many infants were born with various birth defects after their mothers had taken thalidomide. The types of birth defects include abnormally short limbs with toes and fingers sprouting from other parts of the body, flipper-like arms, eye and ear defects or malformed internal organs such as unsegmented small or large intestines. Thalidomide is chiral, where the right-handed version is a strong tranquilizer and the left-handed version causes birth defects. An added complication is that even if a dosage consisting entirely of the right-handed version was administered to a pregnant woman, birth defects would still result because her body would induce a chemical reaction that would change it into the left-handed version [Pi].

There are important applications of the trefoil (and other knots) in other areas as well. **Phoebe Hoidn** (Swiss ???; 19?? -) and **Andrzej Stasiak** (Polish ???; 19?? -) of the University of Lausanne, Switzerland and **Robert Kusner** (American Mathematician; 1960 -) of the University of Massachusetts of Amherst have studied the properties of the trefoil and other knots made from electrically charged filaments. It turns out that when the knot is charged it tightens into a very small region on a perfect circle.

This property shows promise of helping scientists better understand the properties of electrons. [P].

In the 1970s **William Thurston** (American Mathematician; 1946 -) proved that the *complement* of the figure-eight knot has a special structure called *hyperbolic*. By ***complement*** we mean that we put a thickened up version of the knot inside a three-dimensional ball, and the complement of the knot is this ball with the thickened up knot removed. This was a major discovery and indirectly lead Thurston to make a famous and important conjecture in the 1980s called the *Geometrization Conjecture*. In 2003 **Grigori Perelman** (Russian Mathematician; 1966 -) announced that he had proved Thurstons Geometrization Conjecture. An important consequence of this is that Perelman had also proved the *Poincaré Conjecture*, one of the $1,000,000 Millennial Prize Problems (see [C] or [D]). This work won Perlemann the 2006 ***Fields Medal***, the mathematical communitys version of the Nobel Prize. The Fields Medal, which comes with a $13,400 prize, is awarded every four years to a mathematician for important work done before he or she turned 40. However, Perelman is a very reclusive mathematician and he did not attend the award ceremony for the Fields Medal, has not claimed the monetary award or the $1,000,000 Millennium Prize for solving the Poincare Conjecture. Upon hearing this **Stephen Colbert** (American Political Satirist; 1964 -), of Comedy Centrals *Colbert Report*, did a wonderful segment about "Donut Math". To see this clip go to

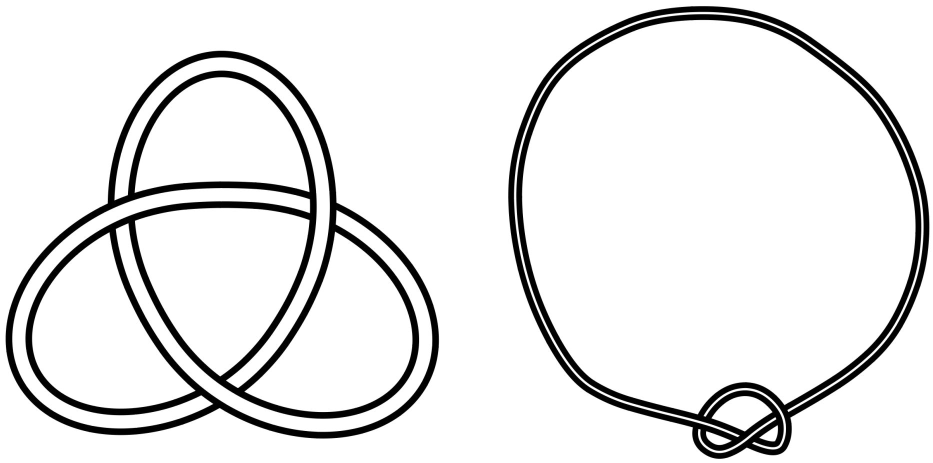


Figure 3.18: What happens when a trefoil knot is charged

the Comedy Central web page http://www.comedycentral.com and search the *Colbert Report* video clips for "Cheating Death-Fields Medal".

19. Make the trefoil knot (shown below in Figure 3.19) with your Tangles®, and try to deform it into the unknot. Describe what happens.

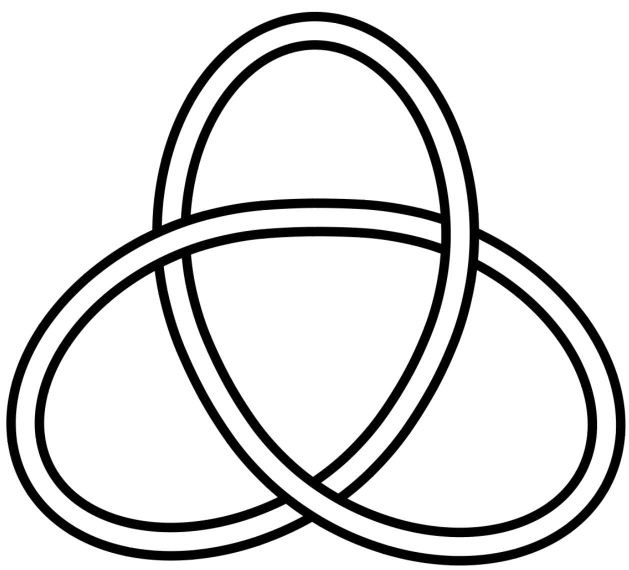


Figure 3.19: Is this the unknot?

20. How long did you try to deform the trefoil into the unknot?

21. Based on your answers to Investigations (19) and (20), do you think that the trefoil knot is, or is not, equivalent to the unknot? Explain.

22. Have you established, **beyond any doubt**, your answer to Investigation **21**? Explain.

The determination of whether the trefoil is distinct from the unknot will have to wait until Chapter 7.

CHAPTER 4

The Dowker Notation

How do we describe a knot? In particular, given a knot projection, is there a way to describe it so that someone could draw an equivalent projection?

4.1. Determining the Dowker Notation from the knot projection

1. Write out a set of directions so that someone could use them to draw a projection that is equivalent to the knot shown in Figure 4.1 without seeing a picture of the knot.

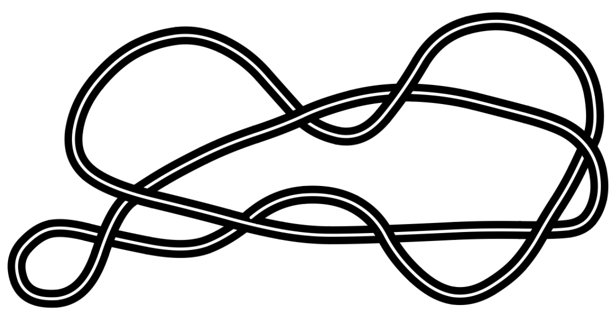


Figure 4.1: Unknown Knot

In the late 1970's **Hugh Dowker** (Canadian Mathematician; 1912 - 1982), came up with a notation to describe a projection. We will not use his exact notation, but a version that is a bit easier with which to work.

First, we give the knot an **orientation**; that is, we choose a direction in which to travel around the knot, and a starting point, P, as shown below.

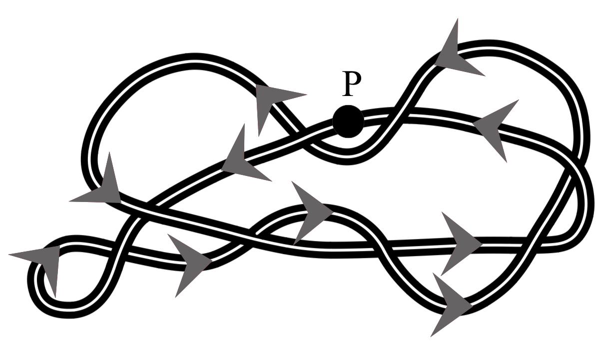


Figure 4.2: Choice of Orientation

We then move around the knot beginning at P in the direction indicated by the orientation, labeling each of the crossings by the next integer as we move along the knot until we have returned to the point at which we started.

An important aspect of this labeling is that the each numbers will be assigned either a + or a − sign, depending on the crossing. If we pass through the crossing on the *overstrand* (as shown below) we make the number positive (+).

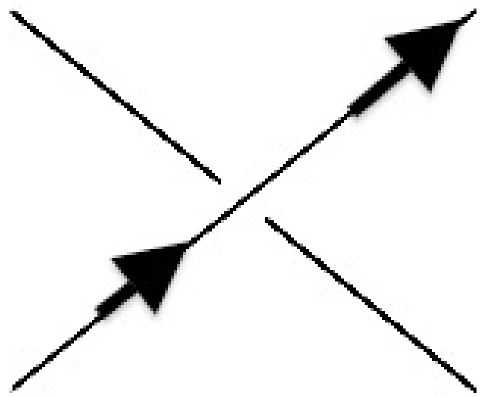


Figure 4.3: Overcrossing

If we pass through the crossing on the *understrand* (as shown below) then we make the number negative (−).

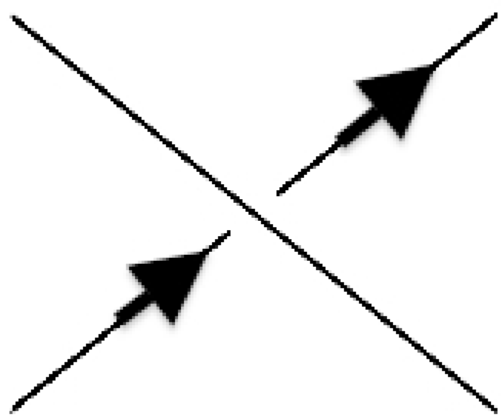


Figure 4.4: Undercrossing

The first crossing we come to is on an overstrand so it will be labeled +1 as shown in Figure 4.5.

The next crossing will be assigned a 2; so we need to determine if it is +2 or −2. Since we are on the understrand, it will be a −2 as illustrated in Figure 4.6.

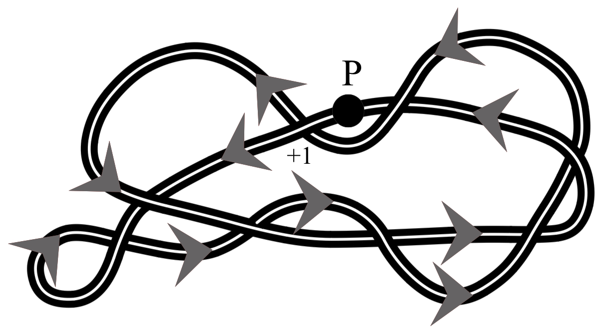


Figure 4.5: First Crossing Label

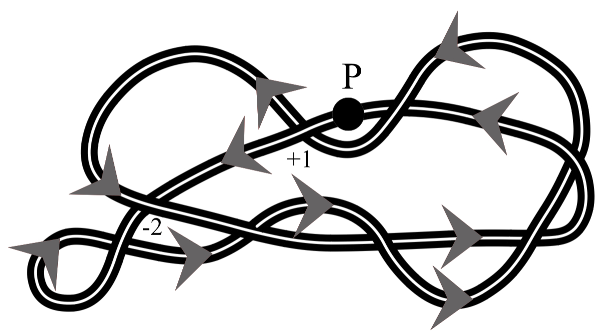


Figure 4.6: Second Crossing Label

We now move to the next crossing, since we are on the overstrand the number, 3, will be positive as in Figure 4.7.

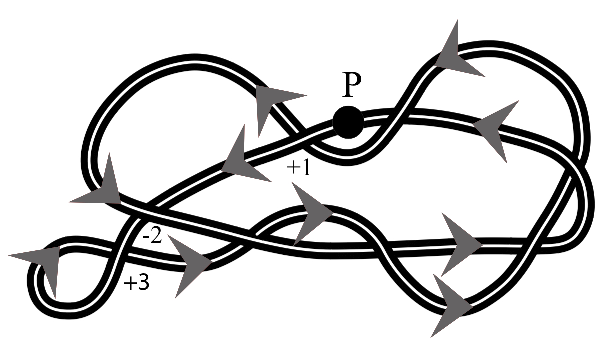


Figure 4.7: Third Crossing Label

The next crossing will be assigned the number 4. In this case, we are passing through the crossing on the understrand, so we assign it a -4 (Figure 4.8).

2. Determine the numbers for the next five crossings.

The next crossing we come to after Investigation **2** is the one at which we began; but since we are passing through the crossing on the understrand, we are not yet back to the point at which we started, and so we will assign a -10 to this crossing in addition to the 1 as shown in Figure 4.9.

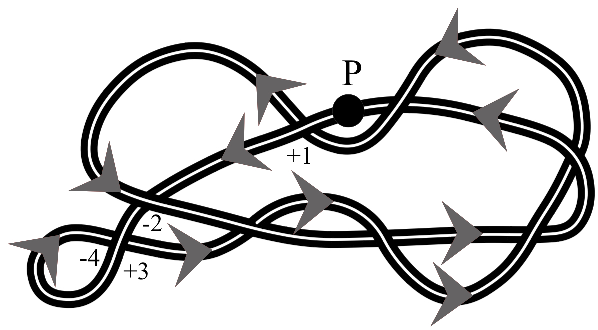


Figure 4.8: Third Crossing Label

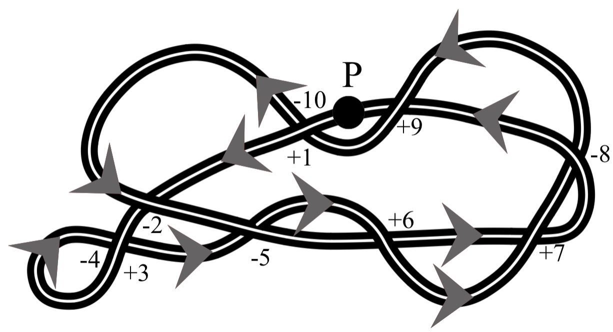


Figure 4.9: Crossing Labels up to -10

3. Assign numbers to the remaining crossings until you get back to the point P.
4. Explain why every crossing has two numbers.
5. Repeat the numbering process described above for the knot shown in Figure 4.10.

6. Ingoring the plus and minus signs, what do you notice about the pair of numbers at each crossing?

In the next few questions you will work through the explanation of why your observation from Investigation **6** is true. We will sketch out the ideas of the proof by looking at an example. This example will illustrate the notion of ***proof by contradiction***. That is, we will assume the observation is false and then find that this leads to an impossible situation.

We begin by supposing there was a crossing at which both numbers were odd, say the numbers $+3$ and -7. Now consider the loop of the knot formed by traversing the knot from this crossing back to itself as shown below in Figure 4.11.

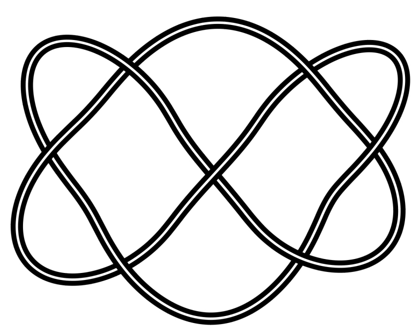


Figure 4.10: A 7-Crossing knot

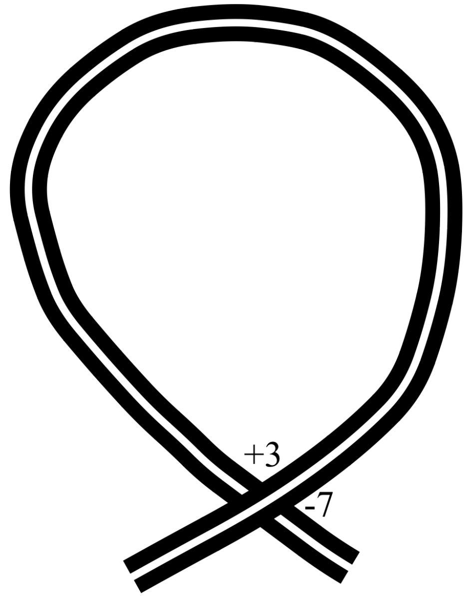


Figure 4.11: The loop formed by a crossing with two odd numbers

7. How many additional crossings must have been omitted from the loop in Figure 4.11 in order to result in a labeling of +3 and −7 at the only crossing shown? Explain.

8. Draw a picture of this loop with additional crossings added in that is consistent with your answer to Investigation **7**. Since the over and under aspects of the crossings are not important for this particular argument, you may draw them all with the loop as the overstrand.

9. Is it possible to give an orientation of the entire knot, **from which this drawing is only a piece**, so that all of the strands you added in Investigation **8** can be oriented so that they are all going from inside the loop to outside the loop or vice versa? Explain.

10. Based on your answer to Investigation **9** and the fact that the knot must be ***connected*** (that is, the knot is in one piece), what has to happen with the strands on the interior of the loop? Is this possible for the picture you drew in Investigation **8**? Explain.

11. Briefly explain why your answers to Investigations (7) - (10) illustrate why the two numbers at any crossing must consist of one even number and one odd number.

12. Now that we have this labeling, what information do you think it can tell the person about the knot? Moreover, why should the information conveyed by only the labels be enough for someone to construct an equivalent projection of the knot? Explain.

The manner in which we convey this labeling information is called the ***Dowker notation*** for a knot.

13. Independent Investigation: Pair up with another classmate and each of you should choose 5 knots at random from the knot table in the back of the book. In this table the knots are listed by the number of crossings. For example the notation 5_2 denotes the second knot in the list with 5 crossings.

For this investigation, do not choose knots with more than 7 crossings. Then use the labeling method we have described above to label each knot. Determine a way to organize the label information so that someone could reconstruct the knot without seeing the projection; then test your method of organization by asking your classmate to redraw the knot based on the information you have given them. After your partner has reconstructed each knot, they should make the knot with their Tangle® and see if it is equivalent to the knot you chose.

14. How succsessful were you in communicating information about the knots to your partner? How successful were you at reconstructing the knots given to you by your partner?

There are lots of ways to represent the Dowker notation and if you were successful in doing so then you may skip ahead to Investigation **15**. If you would like some help in reorganizing your information then continue on.

Going back to our first knot in Figure 4.1, your complete labeling should have looked like this:

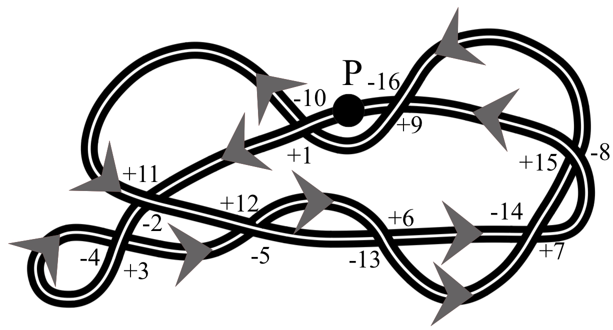


Figure 4.12: Complete Labeling of our unknown knot

4.1.1. Deriving The Notation. Because each crossing has two numbers, we want to take advantage of that in creating a way to convey that information. Let us illustrate several ways to do this for our unknown knot and its labeling shown in Figure 4.12.

4.1.1.1. *Ordered Pair Notation.* This knot has eight crossings. The simplest way to communicate the crossings is to list the number pairs for the cossings. This is the style for the Dowker notation that we will, for the most part, use throughout this book.

$$(+1,-10),(-2,+11),(+3,-4),(-5,+12),(+6,-13),(+7,-14),(-8,+15),(+9,-16)$$

Note that each crossing is listed only once and in this case they are organized by the lowest number of the pair; that is, the crossings are listed in the order they are first labeled.

4.1.1.2. *Numerical Order Table Notation.* For this notation we list each crossing exactly once in a table with the lowest number of the pair on the top row.

$+1$	-2	$+3$	-5	$+6$	$+7$	-8	$+9$
-10	$+11$	-4	$+12$	-13	-14	$+15$	-16

4.1.1.3. *Even-Odd Table Notation.* This is a variation of the most common form of the Dowker notation. In this notation the fact that every crossing has an even and an odd number is used. Instead of listing the crossings in the order they are labeled, they are listed with the odd numbers in order on the top row and the corresponding even numbers listed below.[*] We will use this notation in Section 4.3.

$+1$	$+3$	-5	$+7$	$+9$	$+11$	-13	$+15$
-10	-4	$+12$	-14	$+16$	-2	$+6$	-8

15. Find a Dowker notation for the knot in Investigation **5**.

16. Find a Dowker notation for each of the two knots in Figure 4.13.

[*]The most common form of the Dowker notation lists just the even numbers whose order is determined by the odd numbers. For this knot the notation would be $-10\ \ -4\ 12\ \ -14\ 16\ \ -2\ 6\ \ -8$. It is understood that the corresponding odd numbers would be 1, 3, 5, 7, 9, 11, 13, 15 and if the even number has no sign the odd number is negative and if the even number is negative the odd number is positive.

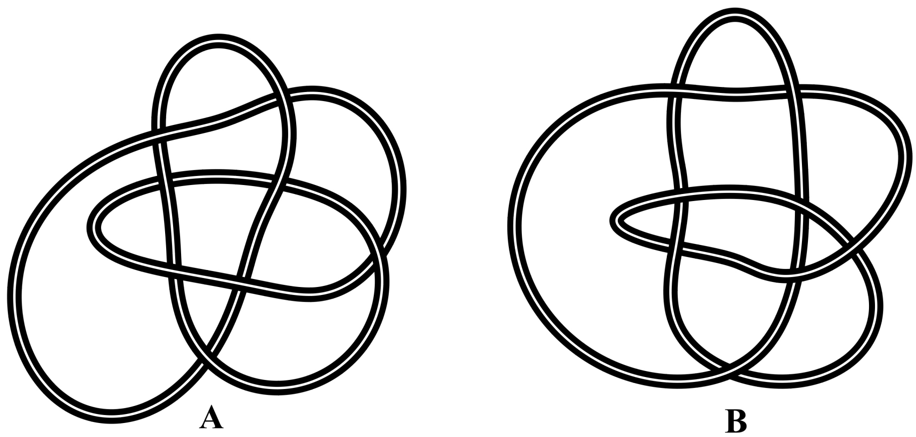


Figure 4.13: Two 8 crossing knots

17. Redo Investigation **16**, but this time start at a point on the opposite side for each knot.

18. Based on your answers to Investigations (16) and (17) is the Dowker notation unique? That is, is there only one notation for each knot? Explain.

4.2. Reconstructing the knot from the Dowker Notation

In Section 4.1, we saw that we could describe a knot by a list of ordered pairs (see page 26), and some of you may have been able to reconstruct a knot projection from the Dowker notation. In this section we will explore in more detail how to do this reconstruction.

Suppose somebody gave us the Dowker notation

$$(-1, +6), (-2, +5), (-3, +8), (+4, -9), (+7, -10)$$

19. How many crossings are there in the knot we are interested in reconstructing? Explain.

20. Draw each crossing in your notes and use your Tangles® and some masking tape to create each crossing and the labels that go with it. For example, the first crossing listed in the above notation might look like the drawing in Figure 4.14.

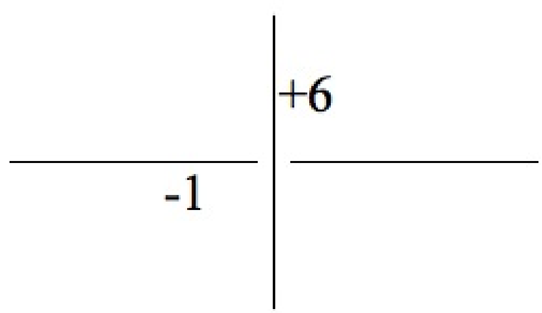


Figure 4.14: The first crossing

21. Now determine how to connect these five crossings to form the knot. Remember that when you connect strands from the crossings you must connect pieces that have consecutive numbers and you cannot create any new crossings when you form the knot.

22. Draw a projection of your knot in your notes and confirm that the Dowker notation for your picture is the same as the one with which we started. You need to make sure you begin at the crossing labeled with -1 and $+6$, and your orientation matches that of the orientation implied in the notation.

23. Use the ideas above to draw a projection of the knot with the notation

$$(-1, +4), (+2, -7), (-3, +8), (-5, +10), (+6, -9)$$

24. Draw a projection of the knot with the notation

$$(+1, -6), (-2, +7), (+3, -8), (-4, +9), (+5, -10)$$

25. Draw a projection of the knot with the notation

$$(-1, +8), (+2, -7), (-3, +10), (+4, -13), (-5, +12), (+6, -11), (-9, +14)$$

26. Draw a projection of the knot with the notation

$$(-1, 6), (+2, 9), (+3, -14), (-4, +11), (-5, +16), (+7, -12), (-8, +13), (+10, -15)$$

27. In Investigations (23) (25) you drew projections of knots where all the even numbers had the same sign (+ or −). What do you notice about the pattern of the crossings in these knots as you move along the knot following the numbers in numerical order?

28. The types of knots in Investigations (23) (25) are called alternating knots. Explain why you think this term is used.

4.3. When is the Dowker notation drawable?

A natural question for a mathematician to ask about the Dowker notation at this point, is whether every possible Dowker notation results in a knot projection that is drawable. In this set, we will explore this question. To make our lives easier we will only consider notations with all negative odd numbers and all positive even numbers. We begin by determining all possible Dowker notations for a three crossing knot projection.

29. What are the set of numbers that will be used for any potential three crossing Dowker notation?

Before we start trying to come up with all the possible notations, we would like to determine how many possible notations there are. To do this, we need an efficient way of counting the all the possible notations with three crossings. The notation we have been using, the Ordered Pair Notation, is not conducive to easily counting the total number of notations. In this case, the Even-Odd Table Notation (page 26) is a better choice.

We know that the crossings will have odd labels -1, -3 and -5 and then count the ways we can match up the even numbers. One way to do this is with a *branching diagram* which is shown in Figures 4.15 and 4.16. Each horizontal row will corresponds to the odd numbers for the three crossings and the choices we have for the even labels based on the choices we have already made. Since there are three possible even numbers to use as labels, we have three choices for the crossing labeled with -1. This means our diagram will have three initial branches as shown below.

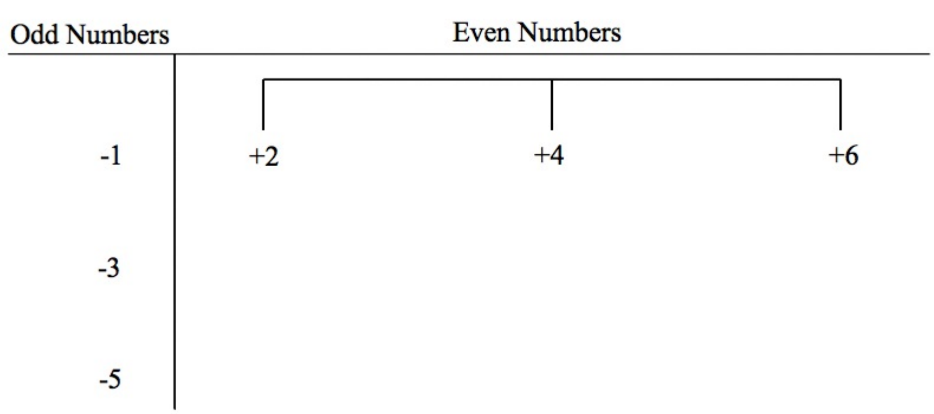


Figure 4.15: The first branches

30. After we have made our choice for the crossing with odd number -1, how many remaining choices do we have for the crossing with odd number -3? Explain.

31. Explain how the branching diagram in Figure 4.16 illustrates our total possible choices for the crossings labeled with -1 and -3.

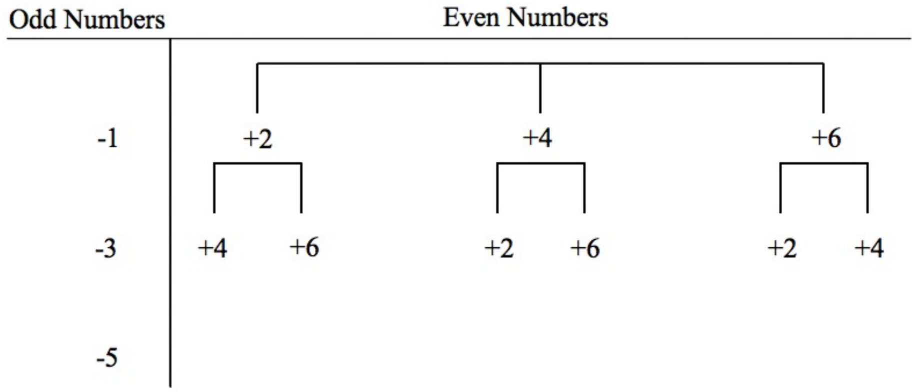


Figure 4.16: The first and second branches

32. After choosing the even numbers for the crossing labeled with -1 and then the crossing labeled with -3, how many choices remain for the crossing with label -5? Explain.

33. Add this level to the branching diagram in Figure 4.16. To then determine the total number of possible Dowker notations, we count the number of ends at the botom of our diagram. For a knot with three crossings how many possible Dowker notations are there?

The branching diagram also allows us to write down, in an organized way all the possible Dowker notations for a knot with three crossings. The left hand side gives us the odd numbers, -1, -3, and -5 and to get the corresponding even numbers for each notation, we read down each branch starting at the top, so for example the Dowker notation (using the Even-Odd Table Notation) corresponding to the last branch on the right would be

$$\begin{array}{ccc} -1 & -3 & -5 \\ +6 & +4 & +2 \end{array}$$

This can be easily rewritten in the Ordered Pair Notation if you wish as follows:

$$(-1, +6), (+2, -5), (-3, +4)$$

34. Write down the remaining possible Dowker notations for a three crossing knot and then try to reconstruct the knot projection for each possible notation. Are all of them drawable (that is, does each notation result in a complete knot projection)? If not, which ones are not drawable and why are you unable to complete the drawing?

Now that we've determined what happens with all possible Dowker notations for knots with three crossings, we want to do the same analysis for knots with four crossings.

35. Use a branching diagram to determine the total number of possible Dowker notations for knots with four crossings.

While the number of of possible Dowker notations for knots with four crossings which you determined in Investigation **35** is not very large, it is too time consuming to try and draw all the resulting projections. Instead we will take a random sample of these possible notations and see what happens when we try to draw the resulting projections.

36. Choose 10 possible Dowker notations at random and try to draw each of the possible projections. Are all of them drawable? If not, which ones are not drawable and why are you unable to complete the drawing?

37. Based on your answer to Investigation **36** do you think all of the possible Dowker notations for knots with four crossings are drawable? Explain.

38. Based on your answers to Investigations (34) and (37) do you think every possible Dowker notation for a knot (no matter how many crossings) is drawable? Explain.

Now we want see what happens with all possible Dowker notations for knots with five crossings. As was the case with knots that have four crossings, there are too many possible Dowker notations to draw them all. In fact, there are too many possibilities to even use a branching diagram to count them all. Fortunately there is another method to count the number of possible notations. To figure this out, let us look back to our answer to Investigations (33) and (35). They key here is to look at the number of choices we had for the even number at each crossing.

39. When determining the number of possible Dowker notations for knots with three crossings, how many choices did you have for the even number for the crossing labeled with -1? How many choices did you have for the even number for the crossing labeled with -3? How many choices did you have for the even number for the crossing labeled with -5?

40. Make a conjecture about how you can combine your answers from Investigation **39** to get the total number of possible Dowker notations for knots with three crossings.

Hint: Think about multiplication.

41. Now we want to see if we can extend the conjecture you made in Investigation **40** to counting all possible Dowker notations for four crossing knots. How many choices did you have for the even number for the crossing labeled with -1? How many choices did you have for the even number for the crossing labeled with -3? How many choices did you have for the even number for the crossing labeled with -5? How many choices did you have for the even number for the crossing labeled with -7? Does the extension of the conjecture you made in Investigation **40** give the correct number of possible Dowker notations for four crossing knots? If not, revise your conjecture so that it gives the correct numbers of possible Dowker notations for three and four crossing knots.

42. Based on your answer to Investigation **41** how many possible Dowker notations are there for knots wth five crossings?

43. Create 10 possible Dowker notations for knots with five crossings.Remember you need to use all the numbers -1, -3, -5, -7, -9, $+2$, $+4$, $+6$, $+8$ and $+10$; each odd number needs to be paired with an even number and you can only use each number once. After you have picked your 10 possible Dowker notations and try to draw each of the resulting projections. Are all of them drawable? If not, which ones are not drawable and why are you unable to complete the drawing?

Hint: When you are creating your 10 posible Dowker notations, you should avoid notations where one or more crossings are labeled with consecutive numbers. For example, you should not use the possibility

$$(-1, 8), (+2, -3), (+4, -9), (-5, +6), (-7, +10)$$

because the pairs $(2, 3)$ and $(5, 6)$ contain consecutive numbers (ignoring the positive and negative signs).

It turns out that many of the possible Dowker notations for knots with five crossing are not drawable. As the number of crossings increase, the percentage of non-drawable notations increases. This turned out to be a problem when mathematicians tried to catalogue knots in an orderly fashion with more than 10 crossings, which will be considered in the next chapter.

CHAPTER 5

Composite Knots and Knot Enumeration

5.1. Composite Knots

The few knots we have studied so far are relatively simple knots. We can use these knots to build more complicated knots, called *composite knots*. This construction is illustrated using the trefoil knot and the figure-eight knot in the three steps below.

Step 1: Use your Tangles® to make one copy of each of the knots in Figure 5.1. (You will need half of your Tangle®for one knot and half for the other knot).

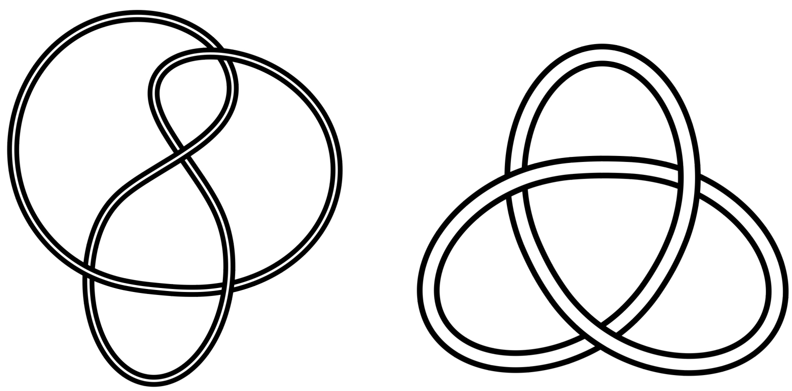


Figure 5.1: The figure-eight knot and the trefoil

Step 2: Now remove a link from the right most loop of the figure-eight knot and a link from the left most loop of the trefoil as shown in Figure 5.2.

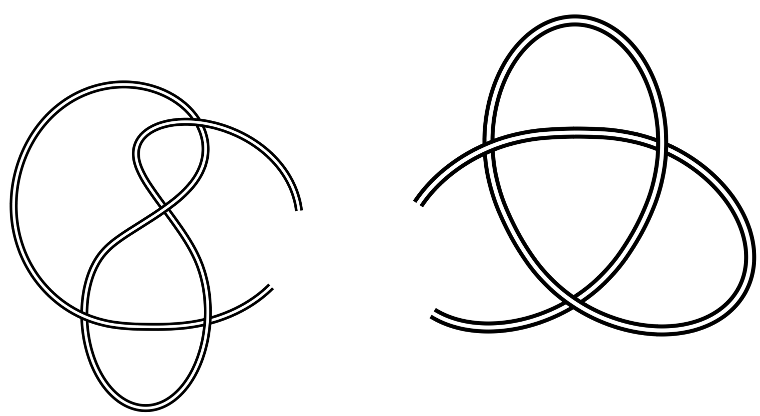


Figure 5.2: The figure-eight and the trefoil knots with missing links

Step 3: Now join the two top loose ends and the bottom loose ends as shown in Figure 5.3.

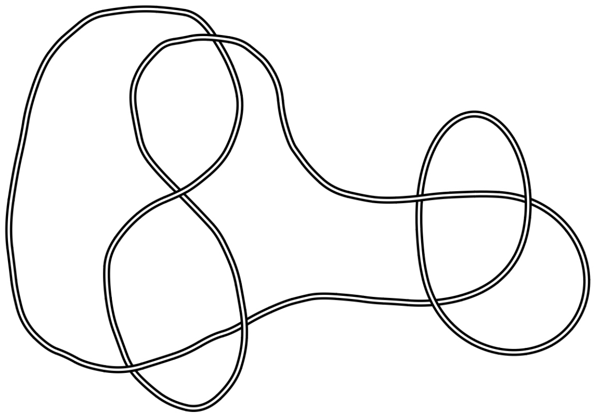


Figure 5.3: The figure-eight and the trefoil joined together

This is called the ***composition*** of the figure eight knot and the trefoil knot.

Notation: If we denote the figure-eight knot by F and the trefoil knot by T, then we denote the composition by $F\#T$.

1. Let J and K be the two knots shown in Figure 5.4. Use your Tangles® to make $J\#K$ and then draw a copy of $J\#K$ in your notebook.

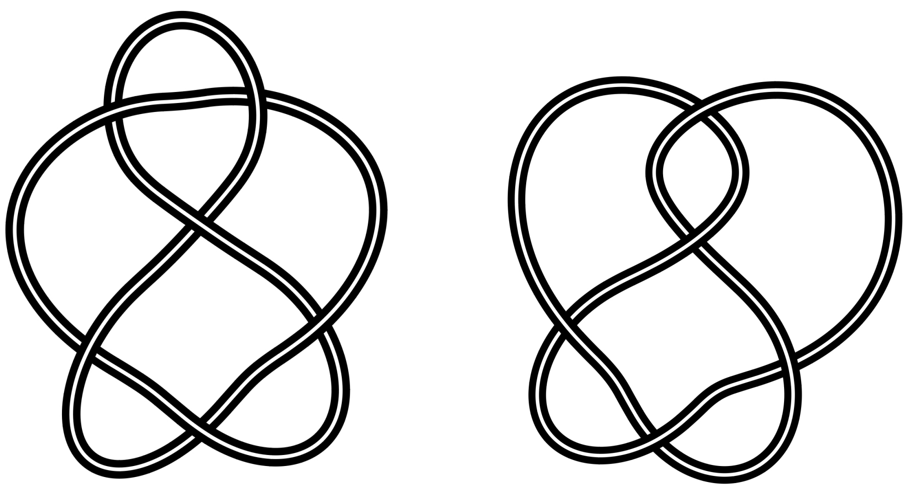


Figure 5.4: The knots J and K.

2. Make the composition, $T\#U$, of the trefoil, T, and the unknot, U, which are shown in Figure 5.5 with your Tangles®. What is the result?

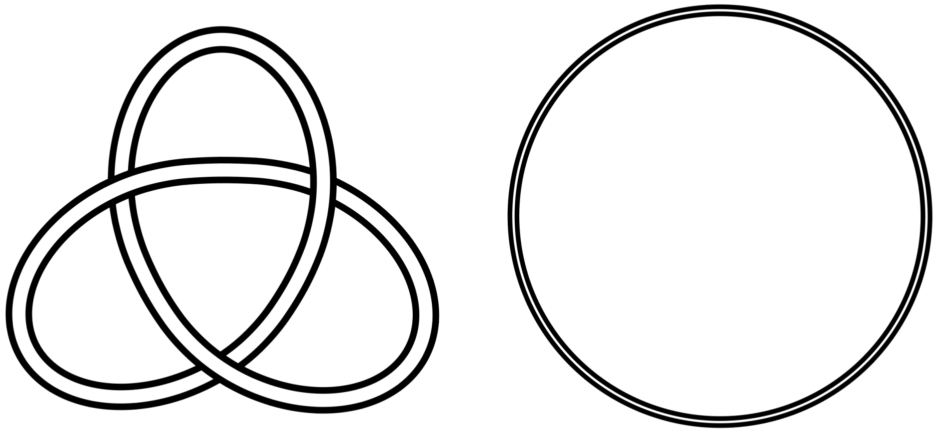


Figure 5.5: The knots T and U.

3. Make the composition, $U\#F$, of the unknot, U, and the figure-eight knot, F, which are shown in Figure 5.6 with your Tangles®. What is the result?

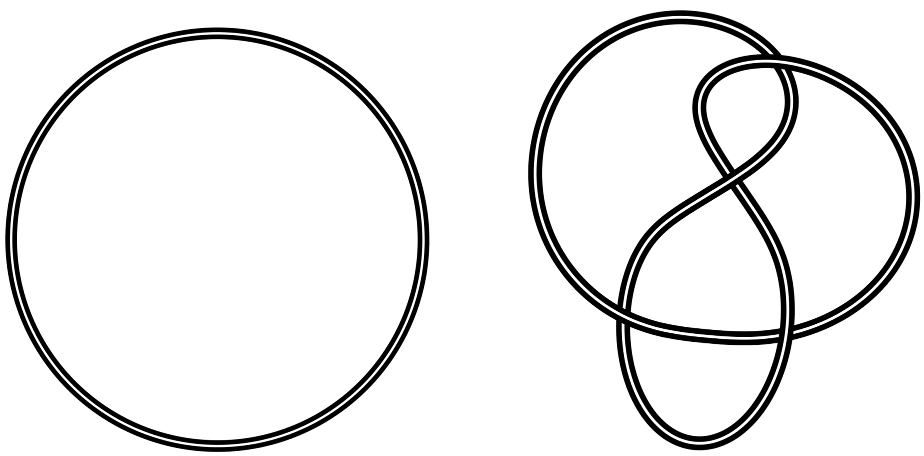


Figure 5.6: The knots U and F.

4. If H is any knot, what will be the result of the composition $H\#U$ where U is the unknot? Explain.

5. In Figures 5.7-5.9 are several pairs of nontrivial knots. For each pair, use your Tangles®to make them and then their composition. Are any of the composition knots equivalent to the unknot? Explain.

a.

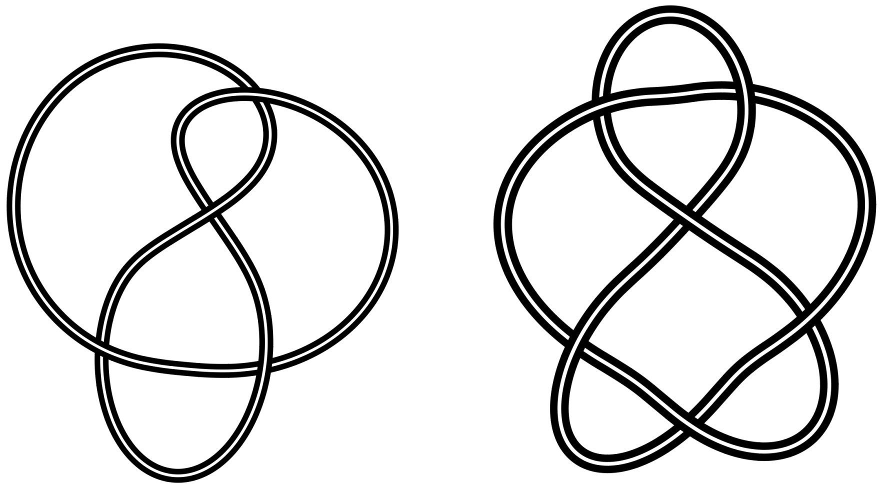


Figure 5.7: The knots F and J.

b.

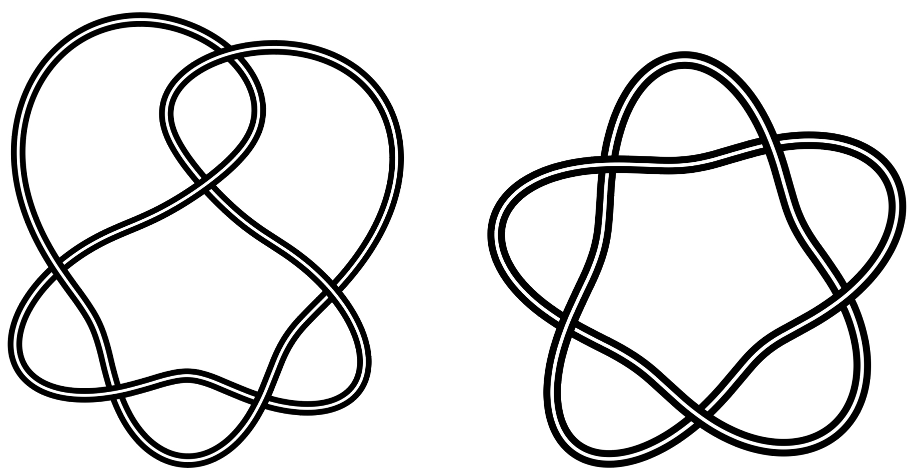


Figure 5.8: The knots M and N.

c.

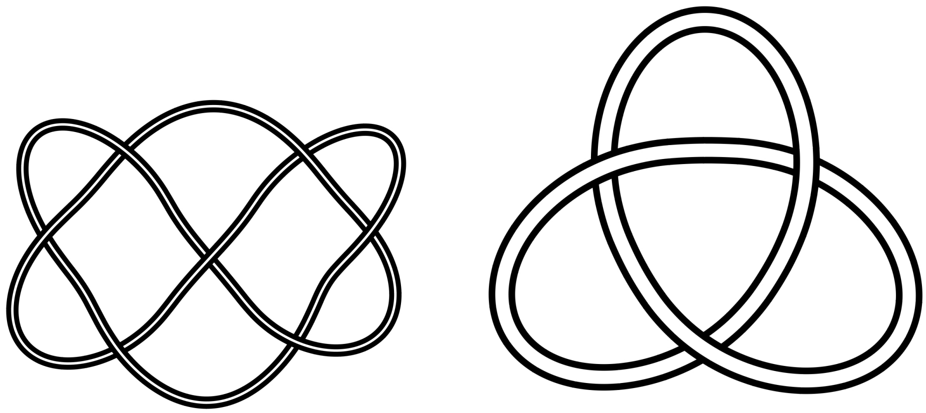


Figure 5.9: The knots L and T.

Consider the composition, $F\#T$, of the figure-eight knot F and the trefoil knot T from the beginning of this section:

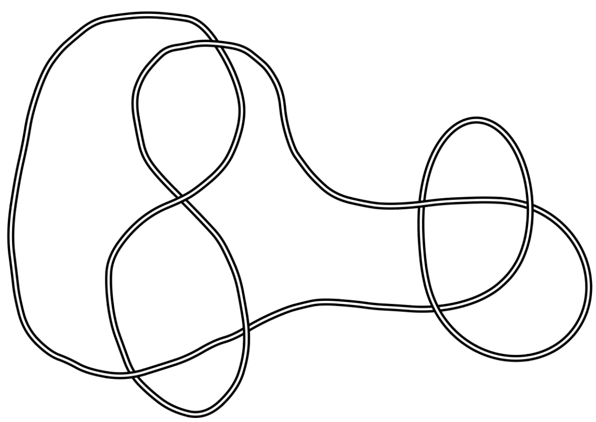


Figure 5.10: The Composite of the Figure-eight and the Trefoil, $F\#T$

Suppose instead we had we made the composition in the reverse order, $T\#F$ as shown in Figure 5.11.

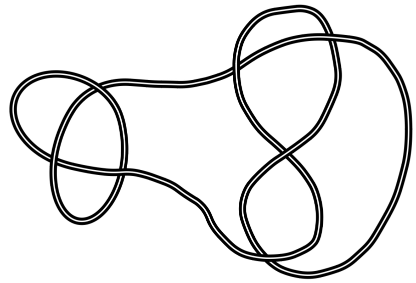


Figure 5.11: The Composite of the Trefoil and the Figure-eight, $T\#F$

The next few questions will explore the relationship between $F\#T$ and $T\#F$.

6. If we shrink T in the first composition $F\#T$ to a tiny knot as show in Figure 5.12

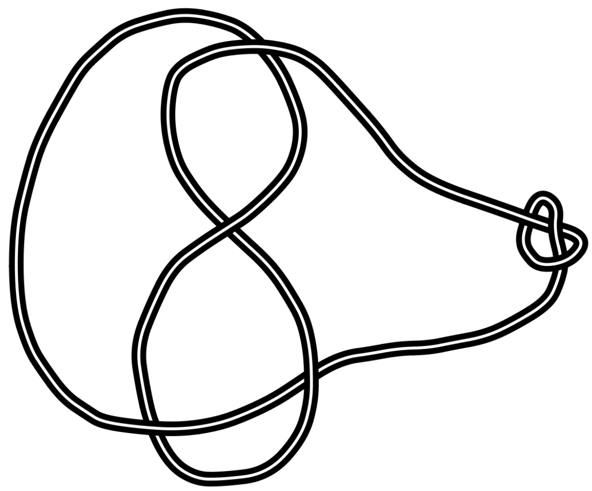


Figure 5.12: The Composite of the Figure-eight and a small Trefoil, $F\#T$

is this still equivalent to the composite knot $F\#T$? Explain.

7. Now suppose we slide T around F as in Figure 5.13

is this knot still equivalent to $F\#T$? Explain.

8. Based on your answers to Investigations **6-7** is $F\#T$ equivalent to $T\#F$? Explain.

9. Does the order in which we do the composition matter? That is, if J and K are two knots, are $J\#K$ and $K\#J$ distinct knots or are they equivalent? Explain.

5.2. Prime Knots

In the previous section we established an arithmetic for knots using composition. That is, for any two knots K and J, then $K\#J$ is another knot such that $K\#J = J\#K$; and if U is the unknot then for any knot K, $U\#K = K = K\#U$. This arithmetic should remind you of multiplication in the whole numbers, $\{1,2,3,4, \dots\}$, with the unknot U playing the role of 1. Just like the whole numbers under multiplication, there is no division in knot composition, because if K is a nontrivial knot, then there is no nontrivial knot M such that $K\#M = U$ (as was suggested in Investigation **5**). The proof of this is very complicated and beyond the scope of this course. We will accept this fact without proof.

This analogy with whole numbers extends even further. In the whole numbers the prime numbers, $\{2,3,5,7,11,\dots\}$, are numbers that can not be factored except as 1 times itself. These numbers form the building blocks of the whole numbers because every whole number can be written uniquely as a product of

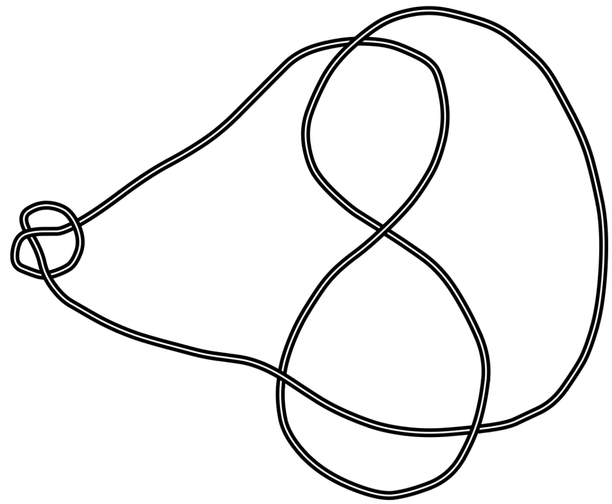


Figure 5.13: The Composite, $F\#T$, of the Figure-eight with the small Trefoil slid around to the other side

prime numbers. For example, $60 = 2 \times 2 \times 3 \times 5$. Similarly, we can "factor some knots into simpler knots. As illustrated in the following questions.

10. For each of the composite knots in Figure 5.14, the ***granny knot*** and the ***square knot***, determine what two knots were composed to make each knot

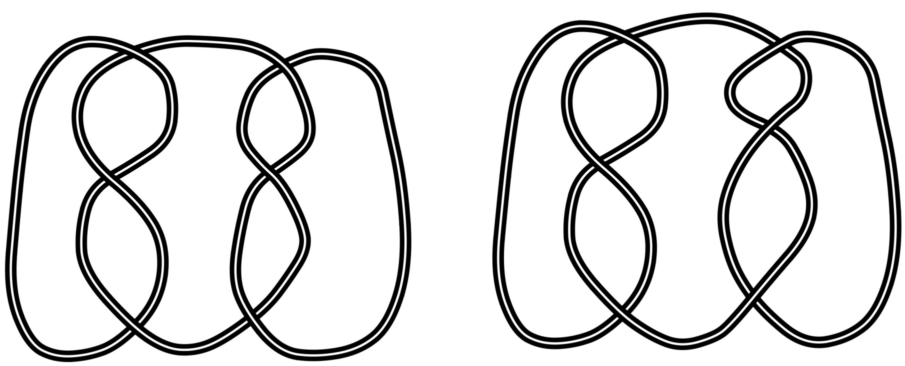


Figure 5.14: The Granny Knot and the Square Knot

11. The granny knot and the square knot are known to be distinct. Explain why your answer to Investigation **10** supports this fact.

Hint: Recall what we said about the trefoil knot and its mirror image in Chapter 3.

Borrowing terminology from number theory, we say that knots that are not composed of two or more simpler knots are called ***prime knots***. Like the prime numbers, the prime knots form the building blocks for more complicated knots. The unknot, the trefoil and the figure-eight knot are examples of prime knots, whereas the granny and square knots are composite knots.

5.3. The Crossing Number of a Knot

In Chapter 4 we considered the Dowker notation of a knot. This notation is handy for describing a particular projection. However, as we saw, there were some problems involved with this notation. The most important problem to us is that the Dowker notation is not unique; that is, the same projection can have more than one notation, depending on where you start. This is a big problem because one of the major goals of knot theory is to create as complete a catalogue of knots as possible. This task is essentially impossible, for technical reasons, so instead mathematicians have worked on cataloguing prime knots. To do this effectively we need a notation that is unique for each prime knot. Several different notations have been developed, but the most common notation is one that goes back to Peter Guthrie Tait and is based on the *crossing number* of the knot. In this section we will examine this notion.

We should point out that even though the Dowker notation is not suitable for creating a catalogue of prime knots, it will provide a useful tool for us to make sure we account for all possible knots with a given number of crossings. In order to count the knots accurately, you might find the Even-Odd Table Notation (page 26) the most useful in this section.

12. Use the Dowker notation to write down notations for all possible projections with one crossing. For this question you will need to consider the different over/under aspects of the crossing. Draw these possibilities in your notebook. How many of these knot projections are equivalent to the unknot? Explain.

13. Are there any **nontrivial knots** that have a projection with exactly one crossing? Explain. Recall that nontrivial means the knot is **not** equivalent to the unknot (see page 9 for the definition of a trvial knot).

14. Use the Dowker notation to write down notations for all possible projections with two crossings. Draw these possibilities in your notebook. How many of these knot projections are equivalent to the unknot? Explain.

15. Are there any nontrivial knots that have a projection with exactly two crossing? Explain.

16. Look back at some of the knots we have made and discussed in previous chapters, are there any nontrivial knots that have a projection with exactly three crossings? Explain.

As we saw in Chapter 4 the crossings, in a very real sense, determine the projection and in turn have some bearing on the specific knot created. A natural question to ask at this point is what role does the number of crossings play in determining the knot? Recall that in Investigation **8** from Chapter 3 you determined that a knot can have a projection with an arbitrarily large number of crossings. Now we would like to know if each knot has a projection with as few crossings as possible.

17. Independent Investigation: Your teacher is going to give each group a pre-made knot made out of string. How many crossings does this knot have? Is this the least number of crossings required to make this knot? That is, are there crossings in this knot that can be eliminated without changing the knot? If there are extraneous crossings, determine how many crossings are actually necessary to make this knot, then draw a projection with this number of crossings in your notebook. If no crossings can be eliminated, explain how you determined this.

Now compare your results with the other groups in the class. Does each group have the same number of crossings in their knot? More importantly, do you all have the same knot? How do you know?

18. There is nothing special aboout the knot in the above Indepent Investigation; use the ideas from your investigation to explain why **any** knot has a minimum number of crossings required to make that specific knot. That is, for each knot there is a certain number of crossings that any projection of that knot must have.

We say that the **crossing number** of a knot is the least number of crossings in any projection of that knot.

19. What is the crossing number of the unknot? Explain.

20. What is the crossing number of the knot you received in the above Independent Investigation? Explain.

The crossing number of a knot has been the traditional way prime knots have been classified since Tait published his first classification in 1876. To understand his classification we begin with knots with as few crossings as possible.

21. How many **distinct** knots are there with 0 crossings? Explain.

22. Based on your answer to Investigation **13**, how many distinct knots are there with 1 crossing? Explain.

23. Based on your answer to Investigation **15**, how many distinct knots are there with 2 crossings? Explain.

These were relatively easy. After this, the counting gets harder. There are 48 possible configurations which could form a potential prime three crossing knot, and there are too many of these to draw. We need to find a way to reduce this number the number of configurations we need to consider. The Dowker notation (Chapter 4) will help us here.

24. To begin our analysis, consider the following three Dowker notations for possible three crossing knots. For each notation, draw the knot and then explain why all of these notations result in a projection that is equivalent to the unknot.

a. $\begin{matrix} +1 & +3 & -5 \\ -2 & -4 & +6 \end{matrix}$ **b.** $\begin{matrix} -1 & +3 & -5 \\ +4 & -2 & +6 \end{matrix}$ **c.** $\begin{matrix} -1 & -3 & +5 \\ +6 & +2 & -4 \end{matrix}$

25. All of the projections in Investigation **24** are very similar in the way that they are drawn. Look back at the Dowker notations and explain how this can be observed in the pairings of the numbers in the Dowker notation.

26. Based on your answers to Investigation **25**, explain what kind of pairings of numbers can we ignore in trying to create a non-trivial three crossing knot.

27. To now determine how many distinct prime knots are there with 3 crossings, we can proceed as follows:

a. Use your answer to Investigation **26** to write out all the remaining possible Dowker notations for three crossing knots that avoid the pairings described in Investigation **26**.

b. Use your Tangles® to determine how many distinct 3-crossing knots there are.

28. Are any of the knots in Investigation **27b** mirror images of each other?

Because knots that are mirror images of each other are identical except for the crossings, we will count mirror image knot pairs as one knot.

29. Given that mirror image pairs count as one knot, how many distinct prime 3-crossing knots are there?

We now move on to four crossing knots.

30. Create several four crossing Dowker notations that have only **one** crossing with the type of number pairing that you described in Investigation **26**. Either draw the corresponding projection or make them with your Tangles®. Are all four crossings necessary? Explain.

31. Repeat the steps Investigation **27** to determine all distinct 4 crossing knots.

32. Are any of the knots in Investigation **31** mirror images of each other?

33. How many prime 4-crossing knots are there?

34. Use your answers to Investigations **21-33** to complete the following table.

Crossing Number	Number of Knots
0	
1	
2	
3	
4	

Table 5.1: Numbers of Distinct Prime Knots up to Four Crossings

35. Do you see any patterns in Table 5.1? Explain.

Tabulating prime knots becomes even more complicated after this point. Below is a table detailing the number of distinct knots (ignoring mirror images) for crossing numbers $5 - 16$.

The number of knots for $0 - 9$ crossings were discovered in the 1890's by Peter G. Tait and verified by**James Waddell Alexander** (American Mathematician; 1888 - 1971) and **Garland Baird Briggs** (American Mathematician; ?? - ??) in the 1920's. In Appendix A is a table showing all the knots with nine or fewer crossings.

The number of prime knots with 10 crossings was first announced by C.N. Little as 166. However, in 1973, **Ken Perko** (American Lawyer, amateur mathematician; 1943 - ??) discovered that the two 10 crossing knots shown in Figure 5.15, now called the ***Perko Pair***, were actually the same:

36. Are you surprised an amateur mathematician discovered these two knots were equivalent 75 years after they were proclaimed as distinct? Explain.

Crossing Number	Number of Knots
5	2
6	3
7	7
8	21
9	49
10	165
11	552
12	2,176
13	9,988
14	46,972
15	253,293
16	1,388,705

Table 5.2: Numbers of Distinct Knots for Crossings Numbers 5 - 16

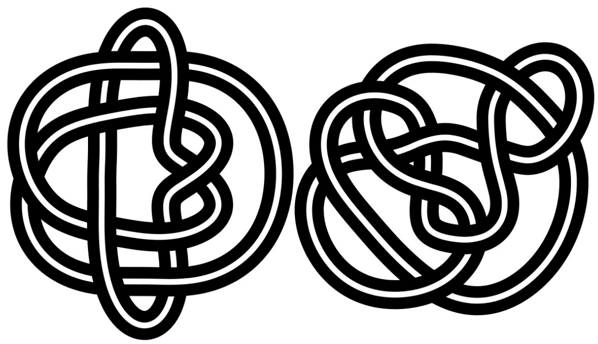


Figure 5.15: The Perko Pair

37. Amateur mathematicians have made many important contributions to mathematics. Find several articles, either in print or on the web, that detail contributions by contemporary amateur mathematicians (i.e. people from the 20th century). For our purposes, an amateur mathematician would be someone who does not have a Ph.D. in mathematics and whose primary job is not in mathematics.

In 1970 **John Conway** (American Mathematician; 1939 -) announced that he had tabulated all the prime knots with 11 or fewer crossings in 1981 **Alain Caudron** (French Mathematician; ?? - ??) produced the first correct list of all prime knots with 11 or fewer crossings. Caudron's list corrected several errors in Conway's list. From 1981 – 1997 **Morwen Thistlewaite** (English Mathematician; ?? - ??) used the Dowker notation and a computer to determine the number of prime knots with 12 – 15 crossings. In 1998 Thistlewaite, **Jim Hoste** (Mathematician; ?? - ??) and **Jeff Weeks** (American Mathematician; ?? - ??) also used the Dowker notation and a computer to determine the number of prime knots with 16 crossings.

Unsolved Problem: How many 17 crossing prime knots are there?

5.4. Further Investigations

F1. Show that the Perko pair knots are equivalent by making the first with your Tangle® and then rearranging it to look like the second. Make drawings illustrating how you accomplished this.

CHAPTER 6

Knot Games

There are many games that people can play involving ideas from knot theory. In this chapter, we will consider a few of these and then examine the mathematics of these games.

6.1. Unlink the Strings

With a partner, cut two pieces string of similar length and tie a loop at each end of both stings with a slip knot as shown in Figure 6.1.

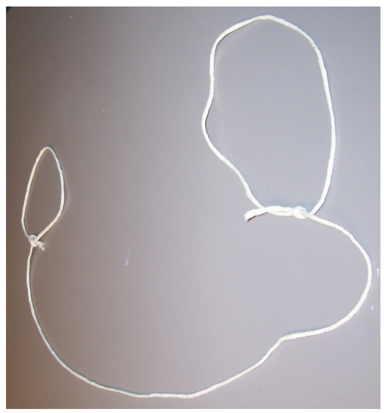


Figure 6.1: A piece of string with a loop at each end

Now each person should place one loop of their string around each wrist, however, the strings should be linked as shown in Figure 6.2.

Figure 6.2: Initial position for game

The goal of this game is to unlink the strings with out removing the loops from each persons wrist as shown below.

Figure 6.3: End position of game

1. How successful were you at unlinking the strings? Why do you think this is possible?

6.2. The Human Knot

The next knot game requires you to form a human knot and then try to untangle the resulting knot or link, without breaking the knot or link. There are several different ways to play this game. We will use one that has a bit of structure to it.

With the people at your table and people from a neighboring table, form circle with five or more people. We will create a knot as follows:

I. If your group has an even number of people, each person should grasp the right hand of a person not their neighbor.
If your group has an odd number of people, everyone except for one should grasp the right hand of a person who is not their neighbor. The remaining person should use their right hand to grasp the left hand of another person. This step is illustrated in Figure 6.4

Figure 6.4: Setting up the human knot

II. Now everybody else should grasp the left hand of another person different from the person whose right hand they grasped.
III. Now try to untangle the resulting knot.This step is illustrated in Figure 6.5

2. How successful was your group at being able to untangle yourselves? Explain.
3. If you were not successful, did your group form a nontrivial knot? How do you know? Can you identify the knot you did form?

Figure 6.5: Untangling the human knot

The goal of this game, remember, is to untangle yourselves to form the trivial knot. A natural question to ask at this point is, how likely is it that the original knot is equivalent to the unknot? In the next few questions, you will explore this in more detail. The main ideas for this section were presented by Laura Hutchinson, an undergraduate student at Union College in Schenectady, NY, at the Fourteenth Annual Hudson River Undergraduate Mathematics Conference in April 2007 [Hu].

6.2.1. The Human Knot Game with Four People. Since is would be time consuming to continue playing this game with more and more people, we will use our Tangles®. We begin with the case there are four people playing. To examine the possibilities in this case and the following, it will help to have an easy way to illustrate the different ways a group can form a human knot. We will denote each person in the human knot by a dot, and a line between two dots indicates that the two corresponding people are holding hands. Note that the way the arms cross over and under each other are important in determining the knot, so when we draw the diagrams we need to be clear when arms cross over and when the cross over. One possible configuration for four people is indicated in Figure 6.6.

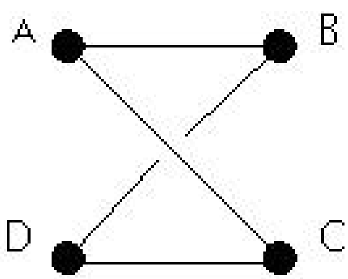


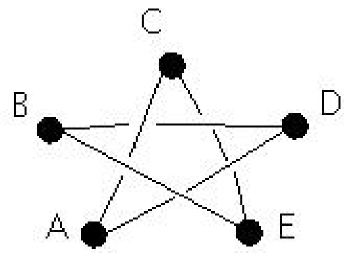
Figure 6.6: One possible configuration for four people forming a human knot.

4. Draw the four possible diagrams for the initial state when four people are playing (including the possibility in Figure 6.6) in your notebook. Then make each knot with your Tangles® and attempt to untangle it. Which initial states, if any, are equivalent to the unknot? Explain.

5. If only four people play the Human Knot Game, can they ever form a nontrivial knot? Explain.

6.2.2. The Human Knot Game with Five People. Next, we want to explore the possibilities if five (5) people play the game. For example, in Figure 6.7 are two distinct possibilities that can be formed with the same pairs of hands being grasped.

6. Use your Tangles® to determine if the two initial configurations in Figure 6.7 are equivalent to each other and/or the unknot. Explain.



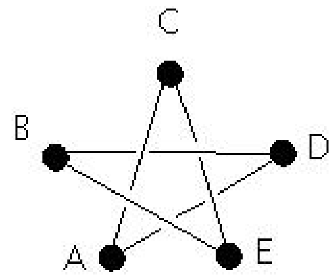


Figure 6.7: Two possible configurations for five people forming a human knot.

As you would expect, the over-under crossings complicate the analysis. To deal with this complication, we will first ignore the over-under aspect of these configurations and simply find all of the possible configurations.

Let us label the people as vertices: A, B, C, D, and E. In this way the graphs in Figure 6.7 would have a labeled graph which looks as follows when we ignore the under and over aspects of the crossings:

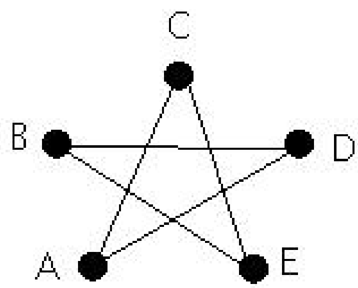


Figure 6.8: A five person configuration for a human knot, ignoring the over and under aspects of the crossing.

Before we examine all the possibilities arising from the configuration in Figure 6.8, we wish to examine some simpler cases. In the questions below you will be drawing diagrams similar to the in Figure 6.8. In all diagrams you should label the vertices as in Figure 6.9:

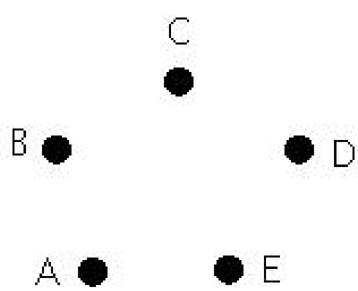


Figure 6.9: Our labeling convention for the five person human knot configurations.

7. If we ignore the over and under aspects of the crossings, how many different configurations are there where A is holding hands with B? Draw each of these in your notebook. Note that configurations that would be equivalent as knots need to be counted as distinct configurations here.

8. If we ignore the over and under aspects of the crossings, how many different configurations are there where A is holding hands with C? Draw each of these in your notebook.

9. If some of the configurations in Investigation **8** are identical to those from Investigation **7**, indicate which ones are redundant.

10. If we ignore the over and under aspects of the crossings, how many different configurations are there where A is holding hands with D? Draw each of these in your notebook.

11. If some of the configurations in Investigation **10** are identical to those from Investigations **7** or **8**, indicate which are redundant.

12. We have not checked those configurations where A is holding hands with E to find any final configurations that may not have been accounted for. Either do this or explain why our list is complete already.

13. In Investigations **7-12** you should have found 12 different configurations. Explain why eleven of these configurations will always lead to an unknot regardless of the way the over and under aspects of the crossings are formed.

Hint: How many crossings are needed to create a nontrivial knot?

The remaining configuration may or may not lead to an unknot, depending on the over and under aspects of the crossings. We will use the Dowker notation to describe and distinguish the various possibilities. (If you need a refresher on the Dowker notation, see Chapter 4.) Since the Dowker notation requires an orientation, we put one on the graph beginning at the vertex A, as shown in Figure 6.10. This gives us the "path $ACEBDA$.

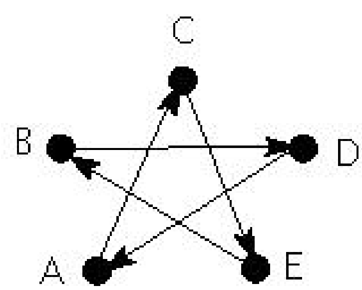


Figure 6.10: Our orientation on the remaining configuration.

Beginning at A we will follow the paths and label the crossings as we go over them. Recall that with the Dowker notation, the numbers assigned to the over-crossing are given + sign and the numbers assigned to an under-crossing are given − sign. Since we have not yet determined the over and under crossings we will initially ignore the + and the − signs.

14. In your notes draw the diagram in Figure 6.10 and label the crossings as you come to them. Note that there are five crossings so you should use the numbers $1 - 10$.

Now if you labeled the diagram correctly, the Dowker notation for each of the 32 possible knots will be $(\pm 1, \pm 6), (\pm 3, \pm 8), (\pm 5, \pm 10), (\pm 7, \pm 2), (\pm 9, \pm 4)$ (depending on how the crossings are chosen).

We have two choices (depending on whether the odd number is positive or negative) for each of the five crossings; thus, there are $2\times 2\times 2\times 2\times 2 = 2^5 = 32$ possible initial configurations. We can use the Dowker notation to enumerate these possibilities in an organized fashion and then examine whether or not they are equivalent to the unknot. For example, one possible notation is $(-1, +6), (-3, +8), (+5, -10), (-7, +2), (-9, +4)$ We could then either modify the crossings in the drawing in Figure 6.10 to show the corresponding projection as shown in Figure 6.11:

or we could draw the corresponding projection using the ideas from Chapter 4 as shown in Figure 6.12:

15. Use the Dowker notation to determine the 32 possible initial configurations and then for each configuration, draw a projection using either, a modified version the labeled diagram as in Figure 6.11, or the ideas from Chapter 4 that corresponds to the Dowker notation as in Figure **??**.

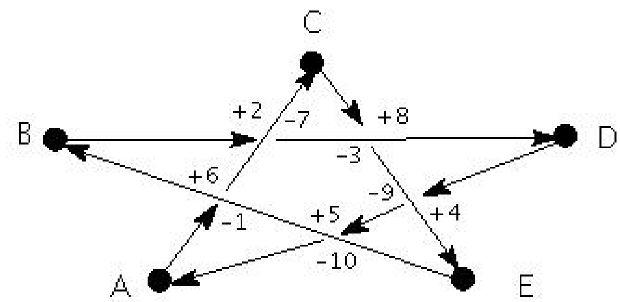


Figure 6.11: The projection corresponding to the notation $(-1,+6),(-3,+8),(+5,-10),(-7,+2),(-9,+4)$.

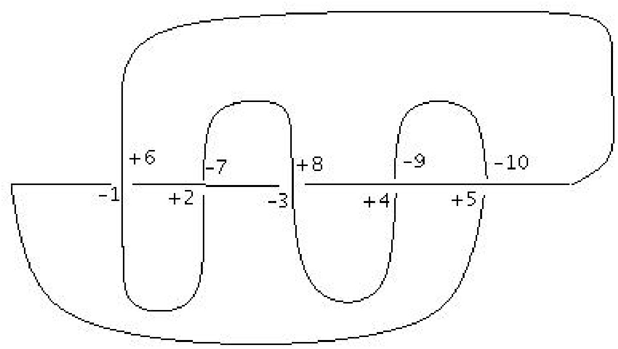


Figure 6.12: Another projection corresponding to the notation $(-1,+6),(-3,+8),(+5,-10),(-7,+2),(-9,+4)$.

Note: Try to find a method to determine these configurations in an orderly fashion. Remember the pairing of the numbers remains the same in each configuration. All you are changing are which numbers are positive and which are negative. In addition, you can use the notion of mirror image to reduce the number configurations you need to find.

16. Use your Tangles® to construct each of the configurations from Investigation **15** and then determine which configurations are equivalent to the unknot and which are not. For those that are not equivalent to the unknot, determine the knot to which the configuration is equivalent.

17. Based on your answers to Investigation **16**, what is the probability that when five people join hands in one of the 32 possible initial configurations corresponding to the graph in Figure 6.8, they can untangle themselves? Explain.

18. Use the probability from Investigation **17** and your answer to Investigation **13** about the additional 11 possible configurations to determine the probability that a random chosen human knot will be equivalent to the unknot.

CHAPTER 7

Distinguishing Knots: Tricolorability and Reidemeister Moves

In Chapter 3 we mentioned, but did not proved (despite developing strong circumstantial evidence), that the trefoil knot is not equivalent to the unknot. Since then we have explored several aspects of knots but have not discussed any tools for determining whether two different looking projections are really different knots. It turns out that this is a very difficult question; in fact, at this point in time there is no general method that will determine if two projections represent the same knot or distinct knots. However, there are some tools that can help us determine when two projections represent distinct knots. In this chapter we will explore one such tool, the notion of *tricolorability*. This will help us finally show that the trefoil and the unknot are, in fact, distinct knots.

Before we begin, a quick note on knot projections. In previous chapters, our knot projections looked like the knot in Figure 7.1.

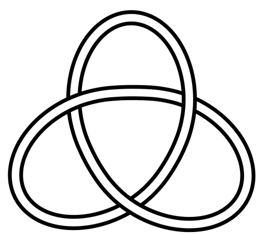


Figure 7.1: Our previous way of illustrating a knot projection

In this chapter we will be using a different type of projection which is illustrated in Figure 7.2

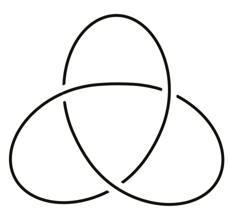


Figure 7.2: Our way of illustrating a knot projections in this chapter

7.1. Tricolorability

The notion of tricolorability involves, as you might expect, coloring a knot projection using three different colors. However, there are important conditions on how the coloring may be done. The first condition involves which parts of the projection are to be colored. A ***strand*** of a knot projection is a section of the projection from one under-crossing to another. For example, in Figure 7.3 the projection of the trefoil has three strands while the projection of the unknot only has two strands.

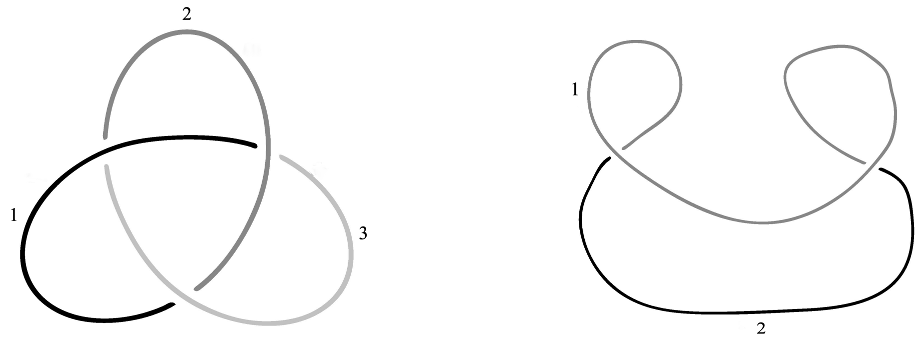


Figure 7.3: Strands for a projection of the trefoil and a projection of the unknot

We say that a knot projection is *tricolorable* when three different colors are used in coloring the entire knot such that each strand is colored one of the three colors and that at each crossing either:

A. All three colors are used or

B. Only one color is used.

1. In Figure 7.4 are three knot projections which may or may not have been tricolored correctly. For each projection explain why the tricoloring is either correct or incorrect.

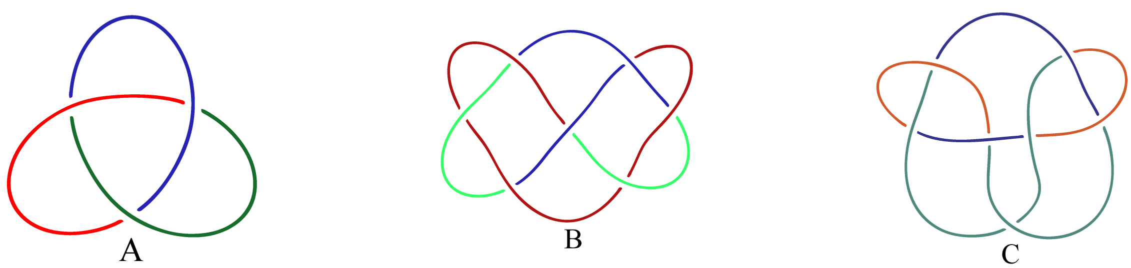


Figure 7.4: Three projections with potential tricolorings.

Just because one tricoloring is not correct, it does not follow that the projection is not tricolorable. If it is tricolorable, you many need to make several attempts before you find an appropriate coloring. The trick here is to begin at a crossing where three strands meet and color each strand a different color and then try to extend the coloring to the entire knot so that it satisfies either condition A or B from above at each crossing. We illustrate this process using a projection of 8_5 knot, which is known to be tricolorable, beginning at the crossing labeled A as shown in Figure 7.5.

We now look at the crossings that have one or two colors present and try to determine how to color the other strand(s) at that crossing.

2. Which of the crossings labeled B, C, D, E, F, G or H would be the best crossing at which to extend the coloring? Explain and indicate which strand and what color you will use.

3. After coloring the strand you identified in Investigation **2**, indicate the next crossing and strand you where you will extend the coloring.

4. Continue the process described in Investigations **2-3** until the knot is tricolored; remembering that there may be one or more crossings where only one coloring will appear.

5. The knot projection in Figure 7.4 that has an incorrect tricoloring is, in fact, known to be tricolorable. Using the process described in Investigations **2-4** find a correct tricoloring.

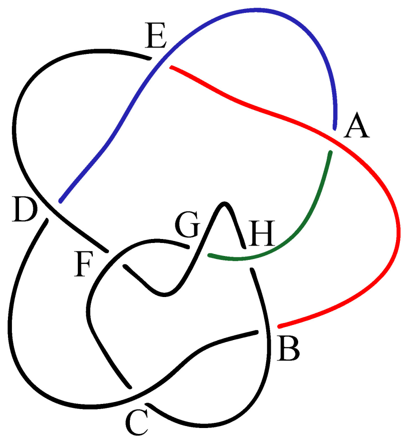


Figure 7.5: Beginning tricoloring of a projection of the 8_5 knot

6. The figure 8 knot, shown in Figure 7.6 is known to be not tricolorable. Try starting the coloring process described in Investigations **2-4** at each of the four crossings, then describe why each attempt results in an improper coloring.

<u>**Note:**</u> When determining that a knot projection is not tricolorable, you need to say more than "I tried and wasn't able to tricolor the knot." It is important that you explain why **every** attempt at tricoloring will fail.

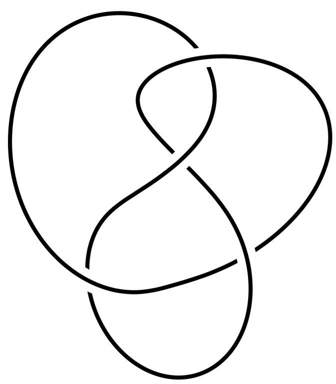


Figure 7.6: The figure-eight knot

7. Explain why the standard projection of the unknot, shown in Figure 7.7, is not tricolorable.

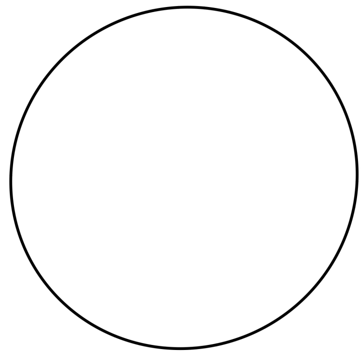


Figure 7.7: The unknot

8. Two non-standard projections of the trefoil and the unknot are shown in Figure 7.8. Is either projection tricolorable? Explain.

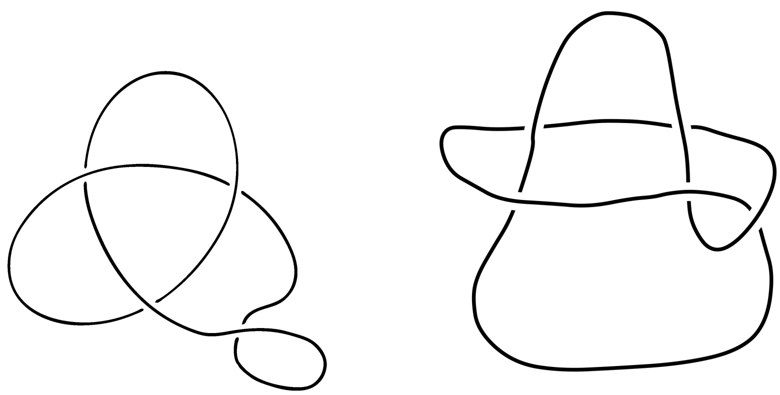


Figure 7.8: Non-standard projections of the trefoil and the unknot.

9. Another pair of non-standard projections for the trefoil and the unknot are shown in Figure 7.9. Are either of these tricolorable? Explain.

10. Based on your answers to Investigations **1**, **8** and **9**, do you think every projection of the trefoil knot can be tricolored? Explain.

11. Based on your answers to Investigations **7-9** do you think any projection of the unknot can be tricolored? Explain.

Your answers to Investigations **10** and **11** should suggest that if one projection of a knot is tricolorable then any projection is tricolorable. This is, in fact, true and we will explain why in the next section. Thus, tricolorability is a ***knot invariant***. That is, tricolorability is not dependent on the projection.

12. Explain why Investigations **10** and **11** combined with the discussion above allow us to conclude that the trefoil knot and the unknot are distinct knots.

13. Explain why Investigations **6** and **10** combined with the discussion above allow us to conclude that the trefoil knot and the figure-eight knot are distinct knots.

14. Do Investigations **6** and **7** allow us to conclude that the unknot and the figure-eight knot are equivalent? Explain.

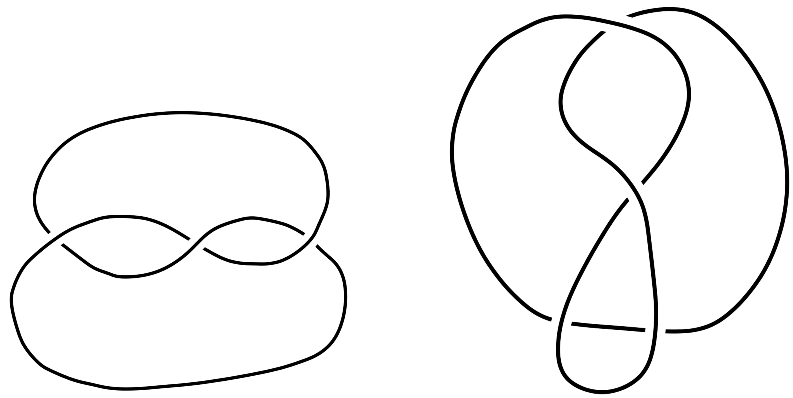


Figure 7.9: Another pair of non-standard projections of the trefoil and the unknot.

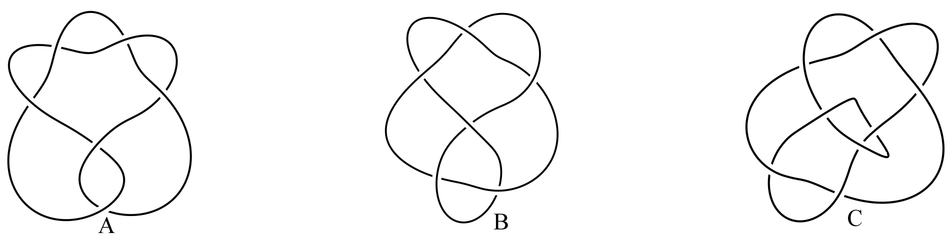


Figure 7.10: Three knot projections.

15. Only two of the knots in Figure 7.10 are tricolorable. Determine which ones are tricolorable, provide an appropriate coloring for the two that are tricolorable.

7.2. Reidemeister Moves

In the previous section we mentioned, after looking at several examples, that tricolorability is a knot invariant. That is, if one projection of a knot is tricolorable, then so is every other projection of that knot. However, we did not prove this. To establish that tricolorability is a knot invariant, we need a way to show that if one projection of a knot is tricolorable then we can get to any other equivalent projection in a way that preserves tricolorability.

In Chapter 3 we considered the notion of equivalency and we said that two knots were equivalent if we could deform one knot into the other without taking it apart and retying it. For example, we determined that the two projections of the trefoil shown in Figure 7.11 were equivalent.

Of course, not every group used exactly the same deformation, and this is okay, because each group could replicate their work and you could show someone else how you did it. However, to actually prove that tricolorability (and other properties) is an invariant, we need to find a procedure that has a more rigid structure. What we need is a procedure that has a well-defined sequence of steps that can be repeated by other people. Fortunately for us, such a procedure exists. In 1926 **Kurt Reidemeister** (German Mathematician; 1893 - 1971) proved that if two knot projections are equivalent, then it is possible to get from one projection to the other in a finite sequence of moves using only three basic types of moves, called (appropriately enough) **Reidemeister moves** which are shown in Figure 7.12.

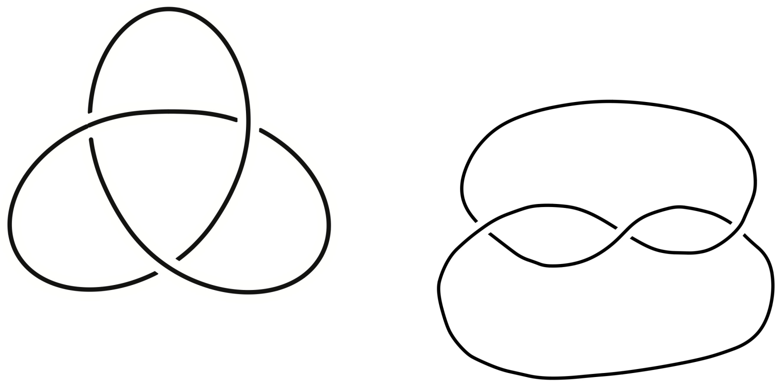


Figure 7.11: Two projections of the trefoil.

16. In Investigations **5-7** from Chapter 3 you looked at three ways of transforming the trefoil knot. Go back to these Investigations and identify which Reidemeister move was used in each Investigation.

These three types of moves are all you need to be able to show two knot projections are equivalent. In Figure 7.13 is an example of a sequence of Reidemeister moves which show that two projections of the trefoil knot are equivalent. Note that all three moves can go in either direction.

In Figure 7.14 on page 55 is another example of a sequence of Reidemeister moves which show that two projections of the figure-eight knot are equivalent.

17. As indicated in Figure 7.14, each step in the sequence of Reidemeister moves requires three or four individual Reidemeister moves. For each step, use your Tangles® do determine the missing Reidemeister moves and draw the corresponding sequence of projections illustrating the given Reidemeister moves. Note that the Reidemeister moves occur in the order listed for each step.

18. In Figure 7.15 are two projections of the trefoil knot.

Explain in words how you can get from one projection to the other. (I dont want you to use the Reidemeister moves here. That will come in the next Investigation.)

Now we would like to find a sequence of Reidemeister moves to go from the projection on the left in Figure 7.15 to the other. This is a bit trickier. To get you started we will illustrate the first three Reidemeister moves.

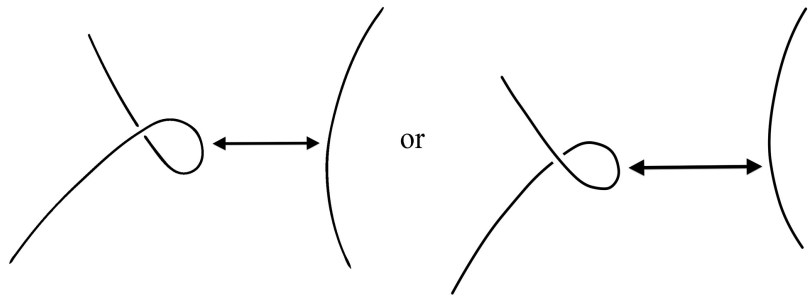
19. Complete the series of pictures (begun in Figure 7.16) illustrating the remaining Reidemeister moves needed to show that the two projections of the trefoil knot in Investigation **18** are equivalent.

20. Which procedure was easier, Investigation **18** or **19**? Explain.

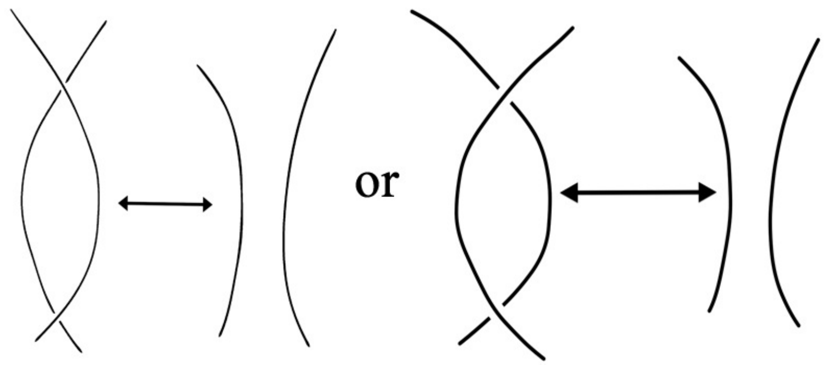
As you can see, finding the Reidemeister moves taking one projection to another is more complicated than just deforming one projection into the other. However, as was mentioned above, Kurt Reidemeister was able to prove in 1926 that two knot projections were equivalent if and only if there was a finite combination of these three moves that deformed one knot projection into the other. This proof is too difficult and long to cover in this class, so we will accept this fact without proof.

Theorem 7.1 (Reidemeister, 1926). *Two knot projections are equivalent if, and only if, there is a finite sequence of Reidemeister moves that deforms one projection into the other.*

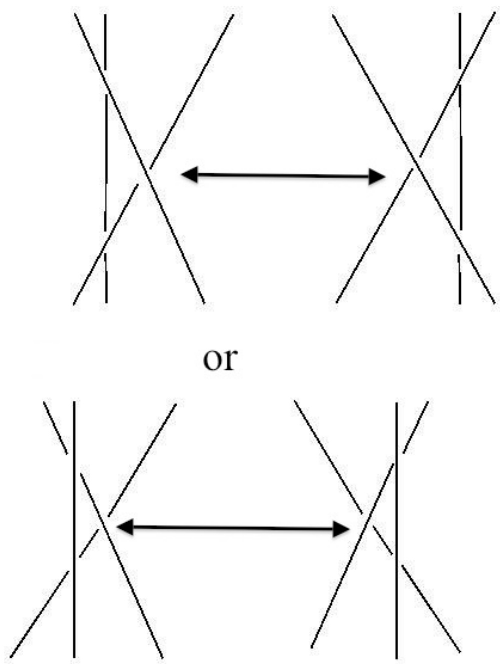
In practice if we have two equivalent projections, it is difficult to find a sequence of Reidemeister that changes one knot projection into the other; but that is okay, because in practice we do not do this. The real power and use of these moves lies in the fact that we can use these moves to establish knot invariants. The following questions illustrate how this works.



(a) Type I (twist - untwist)



(b) Type II (poke - unpoke)



(c) Type III (slide)

Figure 7.12: The Three Reidemeister Moves.

21. Recall that the trefoil knot is tricolorable. Choose an appropriate tricoloring of the trefoil shown below.

22. Perform a Type I move on the trefoil and draw the resulting projection. Is this new projection tricolorable? If so, color your drawing appropriately; if not explain why.

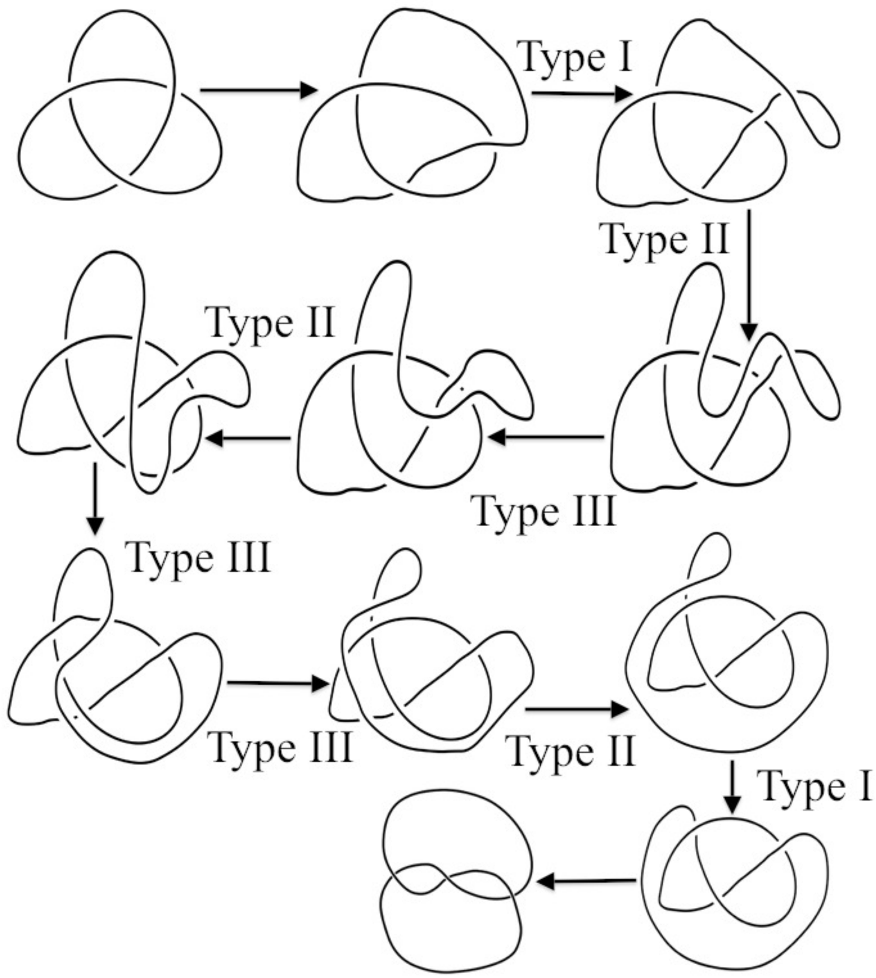


Figure 7.13: A sequence of Reidemeister moves showing two projections of the trefoil knot are equivalent

Hint: Look at your answer to Investigation **8**.

23. In Figure 7.18 a Type II move has been performed on the original trefoil projection from Investigation **21**. Is this new projection tricolorable? If so, color your drawing appropriately; if not explain why.

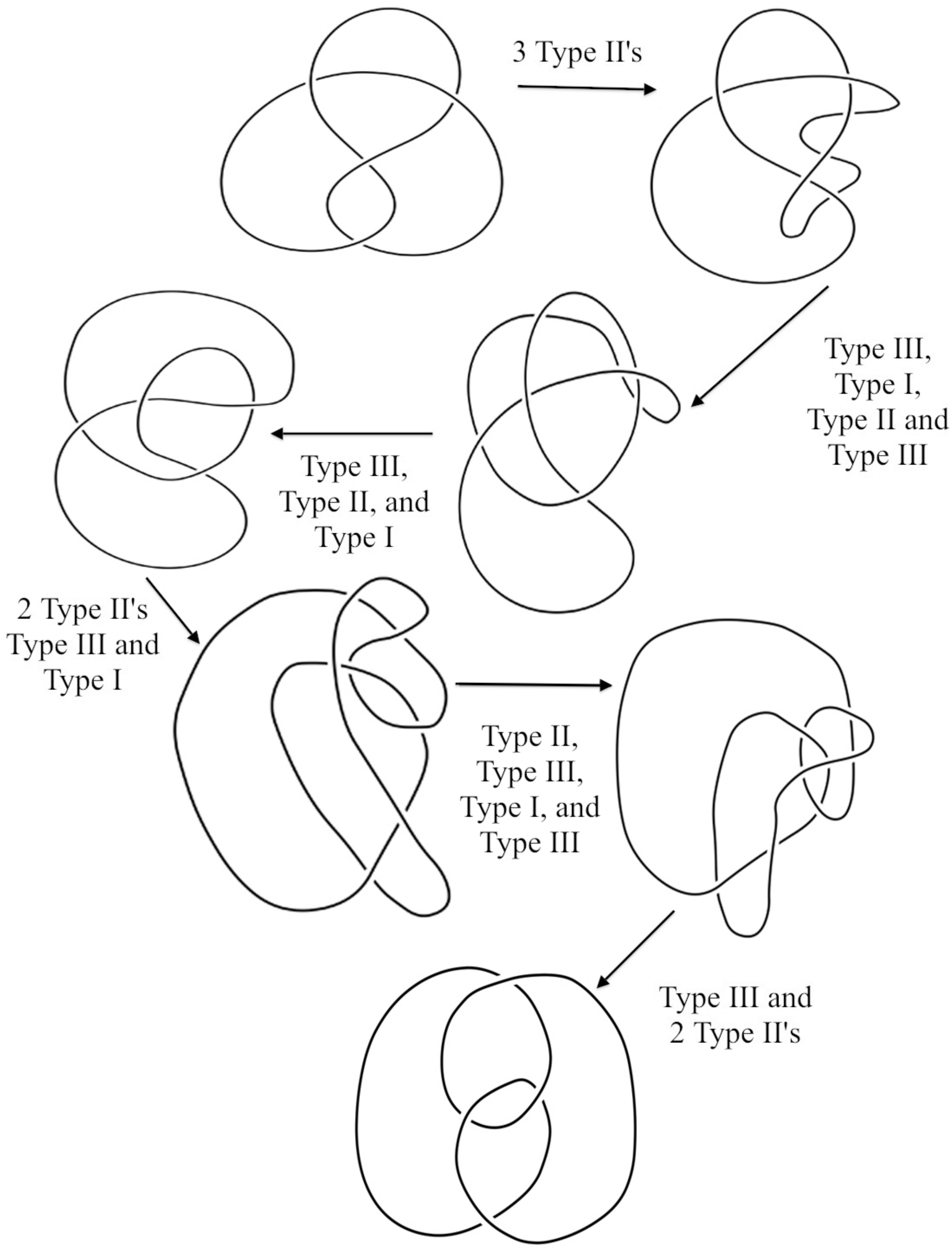


Figure 7.14: A sequence of Reidemeister moves showing two projections of the figure-eight knot are equivalent

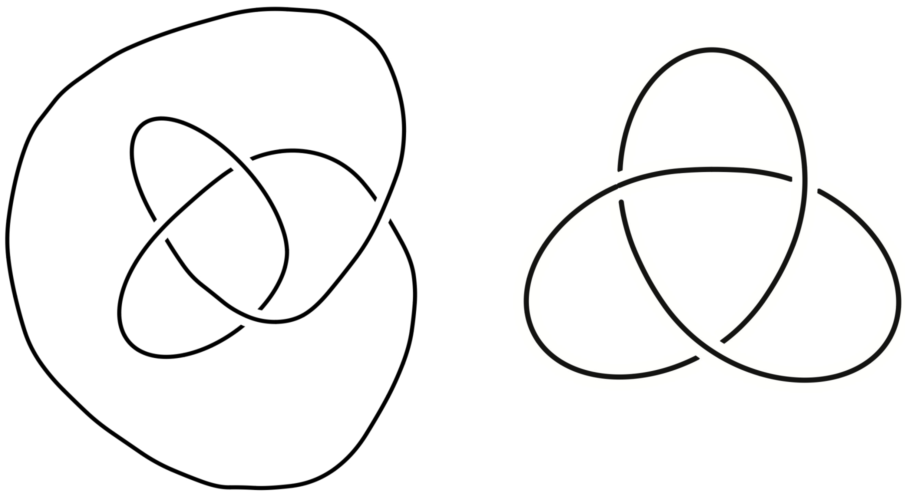


Figure 7.15: Two projections of the trefoil knot.

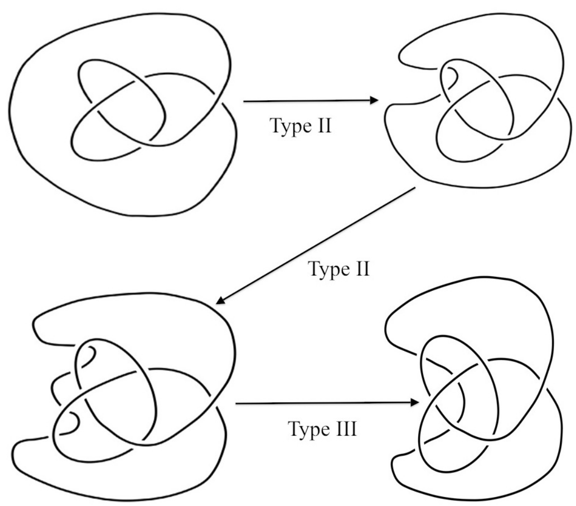


Figure 7.16: The first three Reidemeister moves of the sequence that shows the two projections of the trefoil knot in Figure 7.15 are equivalent

24. In Figure 7.19 another two Type II moves have been performed on the trefoil projection from Figure 7.18. Is this new projection tricolorable? If so, color your drawing appropriately; if not explain why.

25. In Figure 7.20 a Type III move has been performed on the trefoil projection from Figure 7.19. Is this new projection tricolorable? If so, color your drawing appropriately; if not explain why.

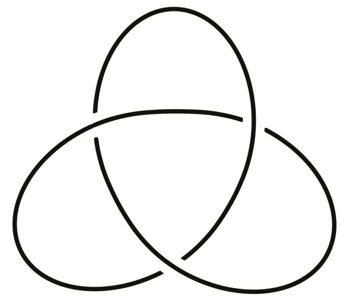


Figure 7.17: The trefoil knot.

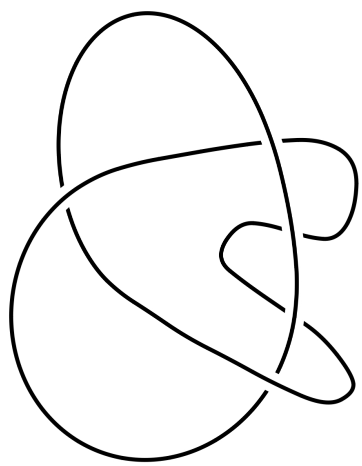


Figure 7.18: The trefoil knot after a Type II move.

26. Suppose we did a finite number Reidemeister moves in succession on a trefoil knot.

- **a.** Is the resulting projection equivalent to the trefoil knot? Explain.
- **b.** Based on your answers to Investigations **22-25** will the resulting projection be tricolorable? Explain.

27. There is nothing special about the trefoil knot. Suppose we know one projection of a knot K is tricolorable. Explain why Theorem 7.1 and your answers from Investigations **21-26** suggest that any other projection of K is tricolorable.

Your answer to Investigation **27** is the general idea of the proof that tricolorability is a knot invariant, it also illustrates the usefulness of the Reidemeister moves.

The power of tricolorability as a knot invariant lies in its ability to definitively say two knots are not equivalent. The Investigations that follow illustrate how this works.

28. Use your answers to Investigations **6** and **27** above to decide if the figure-eight knot equivalent to the trefoil knot. Explain.

29. Can we use tricolorability to show that two knots are the same? That is, if knots K and T are either both tricolorable or not tricolorable, can we conclude that K and T are equivalent? Explain.

Hint: Consider the unknot and the figure-eight knot.

Unfortunately the Reidemeister moves are just as problematic. The Reidemeister moves can only tell us that two knot projections are equivalent; they cannot tell us if the projections represent distinct knots.

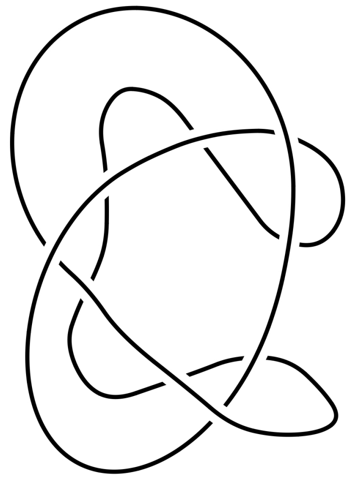


Figure 7.19: The trefoil knot after two more Type II moves.

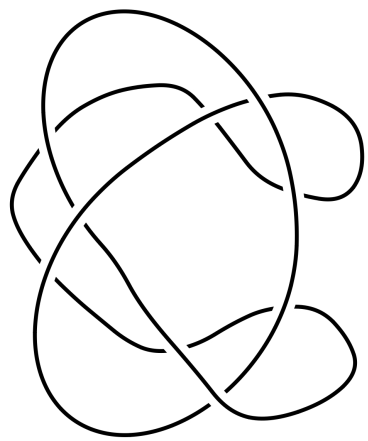


Figure 7.20: The trefoil knot after a Type III move.

Just because we cannot find a sequence of Reidemeister moves taking one knot projection to another that does not mean no such sequence exists; it might be that another two or three moves will work.

One of the aspects of the Reidemeister moves that makes them difficult to work effectively with is the fact that it is almost impossible to determine in advance how many Reidemeister moves it will take to get from one projection of a knot to another projection of the same knot. However, there was a remarkable theorem proved several years ago by **Joel Hass** (American Mathematician; ?? - ??) and **Jeffery C. Lagarias** (American Mathematician; ?? - ??) in 2001 that determines the maximum number of Reidemeister moves needed to untangle a projection of the unknot.

Theorem 7.2 (2001, Hass, Lagarias, (HL)). *For each integer $n > 0$, any projection of the unknot with n crossings can be transformed to the standard trivial knot projection (a circle) using at most $2^{(10^{11})\times n}$ Reidemeister moves.*

What does this mean? This gives the maximum number of Reidemeister moves you will need to untangle a knot projection that is equivalent to the unknot; in reality the number of Reidemeister moves needed will be much less. For example consider the knot projection in Figure 7.21.

30. According to Theorem 7.2 what is the maximum number of Reidemeister moves needed to untangle the projection in Figure 7.21?

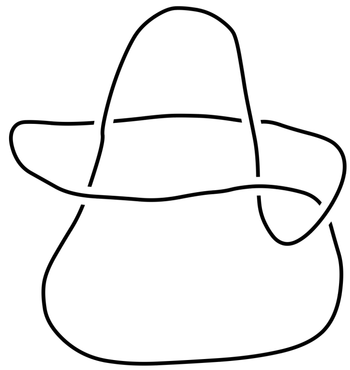


Figure 7.21: A projection of the unknot with 5 crossings.

31. Use a series of drawings to show that the projection in Figure 7.21 can be untangled using just three Reidemeister moves.

One of the surprising aspects of Theorem 7.2 is the size of the upper bound for the maximum number of Reidemeister moves. The number $2^{10^{11}\times n}$ will be a a huge number for any value of n. The authors of Theorem 7.2 conceded that they could have found a smaller upper bound for the maximum number of Reidemeister moves, but that number would have been much more complicated to express. Before we discuss this theorem, we will try to get a sense of how big $2^{10^{11}}$ is. For the sake of this discussion we will not worry about the value of n.

32. What happens when you try to use a calculator to find a value for $2^{10^{11}}$? Explain.

To get a better understanding of how big $2^{10^{11}}$ really is, we will compare it to some other big numbers.

33. Without determining this, if you determined the number of seconds you were alive, do you think that would that be larger or smaller than $2^{10^{11}}$? Why?

34. Compute the number of days you have been alive.

35. Use your answer to Investigation **34** to compute the number of hours you have been alive (for ease of computation, assume you were born at midnight on the morning of your birthday and you are doing this computation at midnight tonight).

36. Determine the number of minutes you have been alive.

37. Determine the number of seconds you have been alive.

38. Was your conjecture in Investigation **33** correct? Explain.

39. The current estimated age of the universe is 13.7 billion years old. Without computing it, do you think that the age of the universe <u>in seconds</u> is bigger or smaller than $2^{10^{11}}$? Explain.

40. Ignoring leap years, determine the estimated age of the universe in seconds. Was your conjecture in Investigation **39** correct? Explain.

<u>Hint</u>: $2^{10^{11}} \approx 10^{30,102,999,566}$

41. Scientists have estimated the number of atoms in the universe. Do you think this number is more or less than $2^{10^{11}}$? Explain.

42. Outside of class, use the internet or a science textbook to find the estimated number of atoms in the universe. How accurate was your conjecture in Investigation **41**?

As you can see, $2^{10^{11}}$ is much more than many large numbers, so while Theorem 7.2 is not a practical result, it illustrates an important point in mathematical research. When mathematicians are looking at procedures like transforming one knot projection to another using Reidemeister moves, the actual length of the procedure is not important, what matters is that we can put a cap on the length of the procedure. This is sufficient to a theoretical mathematician, it means the process can be completed in a finite amount of time. However, to an applied mathematician, engineer, computer scientist or anyone who wishes to practically use such a procedure, the actual length is, of course, very important.

We would like to conclude this Chapter, by making one final observation. Theorem 7.2 concerns the ***unknotting problem***:

THE UNKNOTTING PROBLEM. Is there a way to determine whether a given knot projection is equivalent to the unknot?

This problem is a special case of the **Central Problem** in knot theory which we stated on page 13 in Chapter 3. The **UnKnotting Problem** has, in fact, bern solved. However, the solution is not very practical in the sense that it can not be done efficiently. In 1999 Joel Hass, Jeffery Legarias and **Nicholas J. Pippenger** (American Mathematician; 1943-present - p)roved that this problem is in the class of problems, NP which we discussed on page 7 in Chapter 2.

CHAPTER 8

The Unknotting Number of a Knot

8.1. Changing a Knot Projection into the Unknot

As we mentioned in Chapter 3 it is difficult to determine if a projection of a knot is equivalent to the unknot. A natural question to ask at this point is, instead of trying to determine if a knot projection is equivalent to the unknot, is it at least possible to change a knot into the unknot by changing some, but not all, of the crossings? It turns out that the answer is yes, and the procedure is relatively easy. We will investigate this using the knot 8_{20}, which is shown in Figure 8.1.

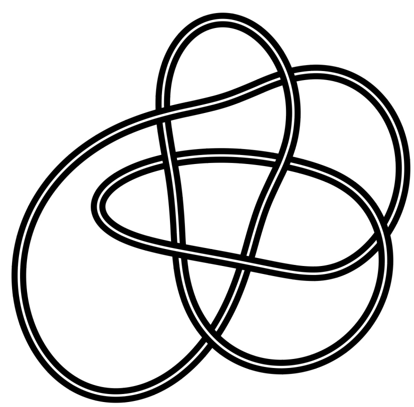


Figure 8.1: The 8_{20} Knot

As we did in Chapter 4, we pick a starting point, P, on the projection (not at a crossing) and an orientation as shown in Figure 8.2.

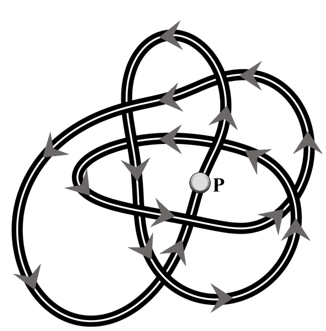


Figure 8.2: The 8_{20} Knot with an Orientation

We start traveling along the projection at the point P in the direction indicated by the orientation. We want to change only <u>some</u> of the crossings but not all, because if we change all of the crossings, we just get the mirror image of the original knot projection. What we would like is a set of directions that we can use for any knot projection which tell us which crossings to change and which to leave alone; so that by following these directions, we change enough crossings (and probably more than we need) to turn the projection into the unknot.

1. Describe a set of rules which tell you which crossings to change and which to leave alone that you think will change the projection of the 8_{20} knot into the unknot; and then try it out on the projection in Figure 8.2 by making the new projection with your Tangle® and see if it is equivalent to the unknot.

Hint: As you travel around the projection from point P in the direction given by the orientation, try to think of a simple rule to decide whether or not to change a crossing. Also consider how you can determine when you are done and how to ensure that you do knot change a crossing twice.

2. Did your rules from Investigation **1** work? Explain. If they did not work, rework them and try again.

3. Once you have successfully unknotted the 8_{20} knot, try your procedure from on several other knots in the Appendix to be sure it works for other knot projections. If your procedure doesn't always work, keep reworking it until it does.

4. Explain why doing your procedure on any knot projection will always result in a projection of an unknot.

8.2. The Unknotting Number

In Chapter 5, we used with the crossing number of a knot, which gives a measure of the knots complexity, to help us tabulate knots. In this chapter, we will consider a related measure of a knots complexity, the unknotting number of a knot.

5. Consider the trefoil knot shown in Figure 8.3.

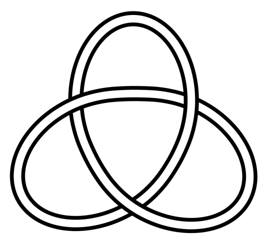


Figure 8.3: The Trefoil

Use the ideas from Chapter 7 to explain why the trefoil is not equivalent to the unknot.

6. Make the trefoil with your Tangle® and then change exactly one of the crossings. What happens?

7. The figure-eight knot is shown in Figure 8.4 . Make this knot with your Tangle® and then change one of the crossings. What happens?

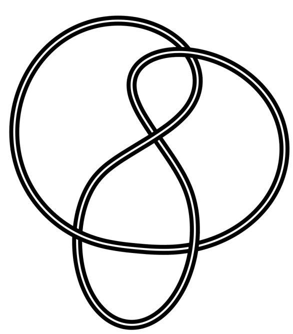


Figure 8.4: The Figure-Eight Knot

8. The 5_2 knot is shown in Figure 8.5. After making this knot with your Tangle®, can you find a crossing that when changed results in an unknot?

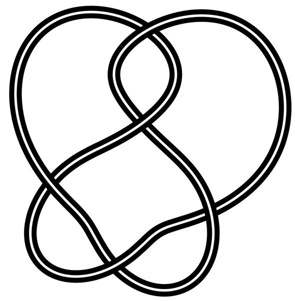


Figure 8.5: The 5_2 Knot

9. If you change a crossing on any knot projection, what do you think will happen? Explain.
10. The 5_1 knot is shown in Figure 8.6. After making this knot with your Tangle®, can you find a crossing that when changed results in the unknot?

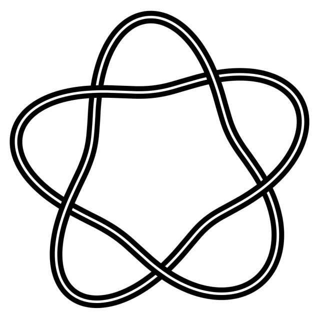


Figure 8.6: The 5_1 Knot

11. Why do you think this happened? Which knot do you get when only one crossing of the 5_1 knot is changed?
12. How many crossings do you think you need to change in the 5_1 knot in order to get the unknot? Explain.
13. Remake the 5_1 knot with your Tangle® and test your conjecture. What happens?

The ***unknotting number*** of a knot K, denoted $U(K)$, is the fewest number of crossing changes in **any** projection that will make the (new) knot projection equivalent to the unknot.

14. Explain why $U(0_1)$, the unknotting number of the unknot, has to be zero.
15. Based on your answers to Investigations **6-8**, what is the unknotting number of the trefoil, the figure-eight knot, and the 5_2 knot?
16. Above we said that the unknotting number of a knot, K was the fewest number of crossing changes in **any** projection of the knot that results in a projection. Explain why for the trefoil, the figure-eight knot and the 5_2 knot, we do not need to check any other projections to know their unknotting number.
17. What do your answers to Investigations **10-13** suggest is the value of $U(5_1)$, the unknotting number of the 5_1 knot? Explain.
18. Do your answers to Investigations **10-13** and **17** prove that we know the value of $U(5_1)$? Explain.

Mathematicians have been able to prove that **every** projection of the 5_1 knot requires you to change at least two crossings so in fact, $U(5_1) = 2$.

However, it is difficult to determine the unknotting number of a knot. The exact unknotting number is unknown for many knots. For example, the unknotting number for the knot 10_{11} shown in Figure 8.7 is either 2 or 3, but no one has been able to either find a projection where changing only two crossings results in the unknot, or prove that such a projection does not exist.

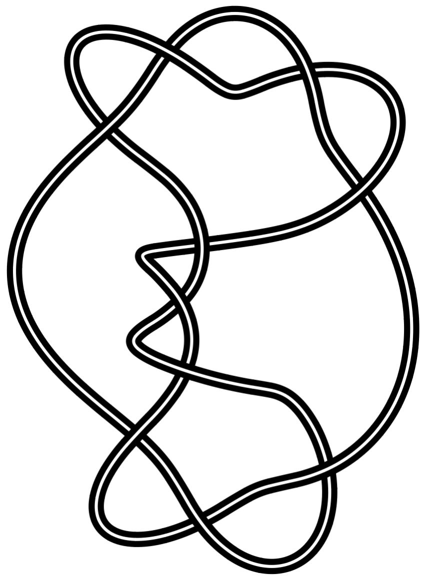


Figure 8.7: The 10_{11} Knot

Another example is the $11a_{354}$ knot shown in Figure 8.8 for which the unknotting number is either 2, 3 or 4.

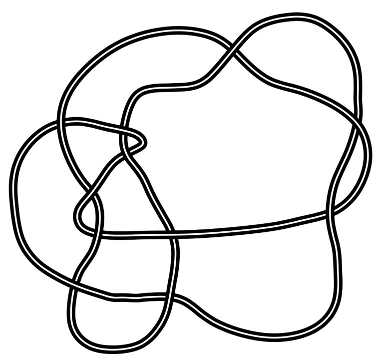


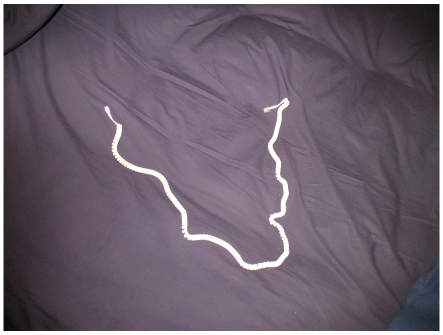
Figure 8.8: The $11a_{354}$ Knot

19. Use the projection of the $11a_{354}$ knot in Figure 8.8 to verify that $U(11a_{354}) \leq 4$. That is, show that with this projection there are four crossings you can change so that the resulting projection is equivalent to the unknot. In your notes, draw the knot and indicate which four crossings you changed.

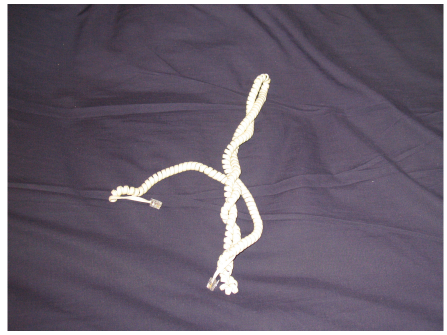
20. Explain why your work in Investigation **19** does not prove that $U(11a_{354}) = 4$.

One problem in determining the unknotting number of a knot is the fact that sometimes a projection with more crossings requires changing fewer crossings. To illustrate this problem, consider the knot below which has 10 crossings.

While determining the unknotting number for a knot may be very difficult, your body knows how to do it. Your DNA is coiled extremely tightly in your cells, which is called *supercoiling*. To understand supercoiling, consider a telephone cord. It is normally coiled as shown in Figure 8.9a, after a while though it coils up more as shown in Figure 8.9b.



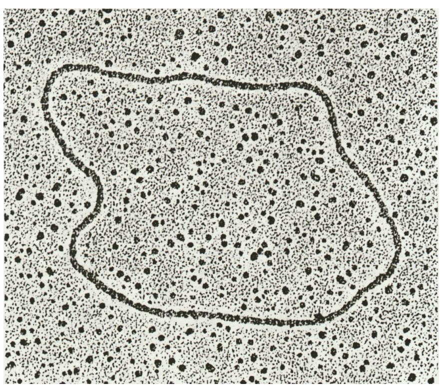
(a) Normal coil



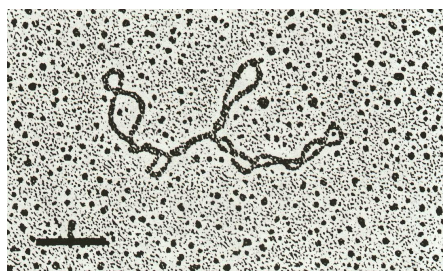
(b) Supercoiled

Figure 8.9: Illustration of Supercoiling with a Telephone Chord

Supercoiling is what allows DNA to fit inside a cell. In Figure 8.10a is a picture of uncoiled circular DNA and in Figure 8.10b is a picture of what it looks after supercoiling takes place. Notice how much smaller the supercoiled DNA is.



(a) Uncoiled DNA



(b) Supercoiled DNA

Figure 8.10: Illustration of Supercoiling in DNA

The picture in Figure 8.11 is a photograph taken with an Electron Microscope of an E.coli bacteria cell that has been opened. The cell is at the center and the thin strands are from the DNA. Without supercoiling the DNA could not possibly fit into the cell.

So what does this have to do with knots? When DNA becomes supercoiled it becomes very tangled (possibly even knotted) and this creates difficulties during replication. To deal with this supercoiling every organism has enzymes called *Topoisomerase I* and *II*. Topoisomerase II is the enzyme that allows replication to take place by changing crossings at appropriate times and place. It still is unclear how this enzyme knows where and when to change crossings.

Another problem in determining the unknotting number is the fact that sometimes a projection with more crossings requires changing fewer crossings. To illustrate this problem, consider the two projections shown in Figure 8.12

It is known that every projection of this knot has at least 10 crossings. It is also known that you need to change at least three of the crossings of the projection in Figure 8.12a to make the unknot and that you only need to change two crossings of the projection in Figure 8.12b to make the unknot.

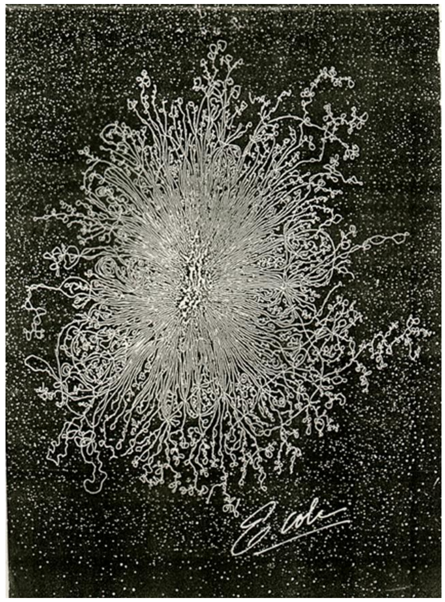
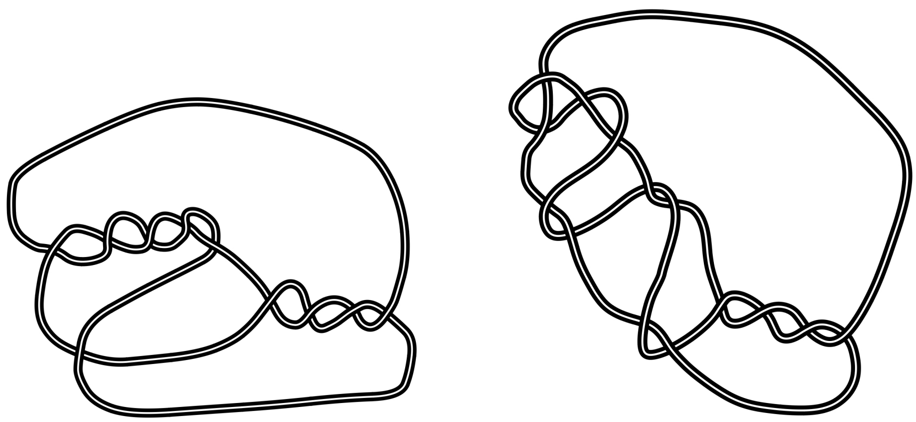


Figure 8.11: Picture of the DNA of an E.Coli bacteria



(a) Projection with 10 crossings

(b) Projection with 14 crossings

Figure 8.12: Two projections of the same knot

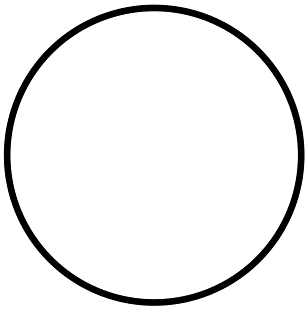
21. Are you surprised that the projection in Figure 8.12b, which has more crossings, requires fewer crossing changes than the one in Figure 8.12a in order to make the unknot? Explain.

8.3. Further Investigations

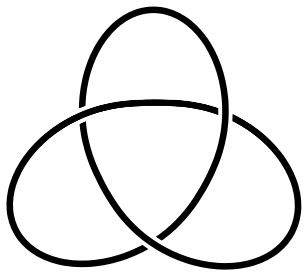
F1. Make the projection in Figure 8.12a with your Tangles® and determine which three crossings need to be changed in order to create the unknot; then make a drawing of the projection and indicate which crossings you changed.

F2. Make the projection in Figure 8.12b with your Tangles®and determine which two crossings need to be changed in order to create the unknot; then make a drawing of the projection indicate which crossings you changed.

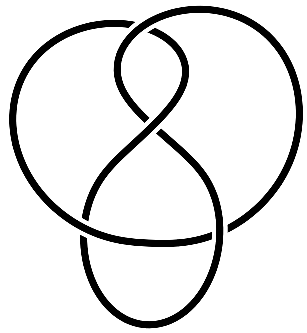
Table of Knots Up to 8 Crossings



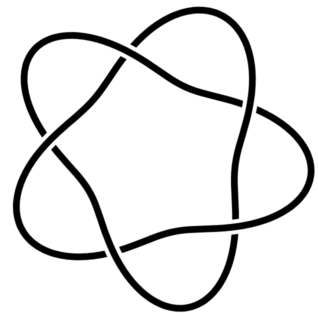
0



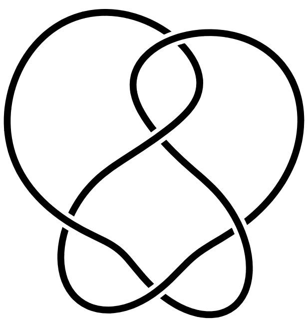
3_1



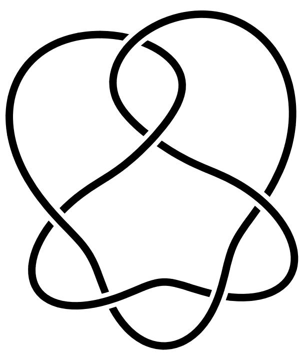
4_1



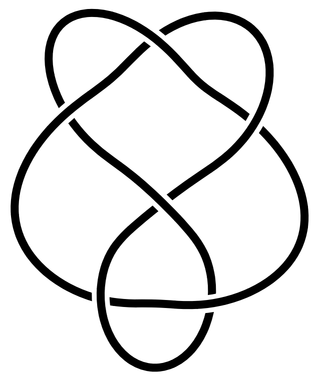
5_1



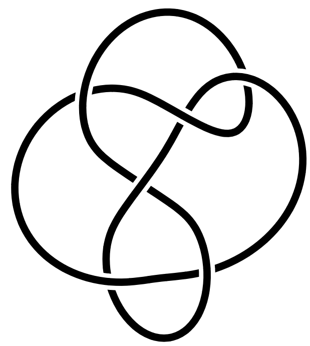
5_2



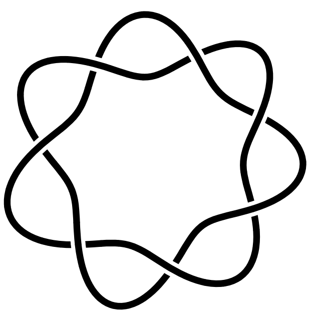
6_1



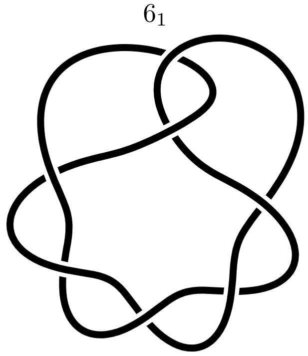
6_2



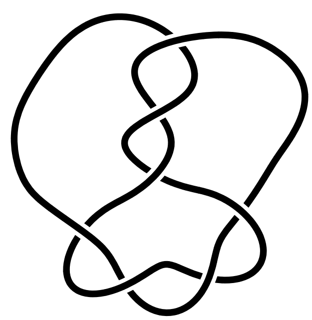
6_3



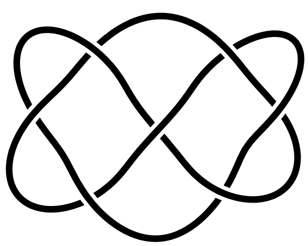
7_1



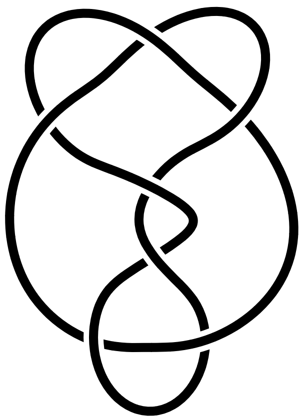
7_2



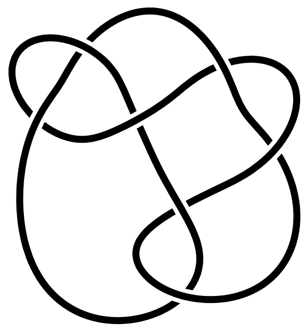
7_3



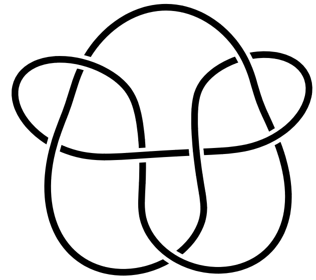
7_4



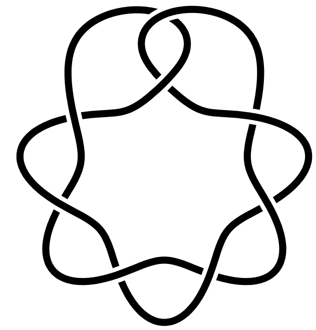
7_5



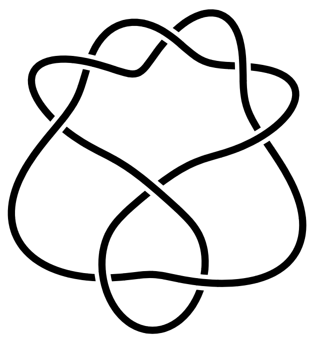
7_6



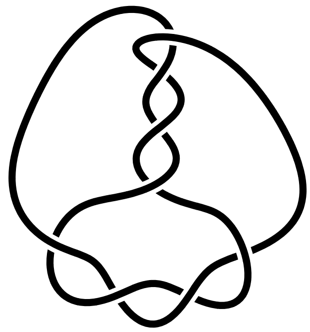
7_7



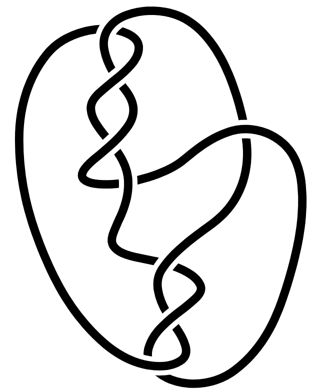
8_1



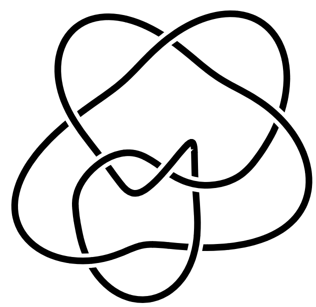
8_2



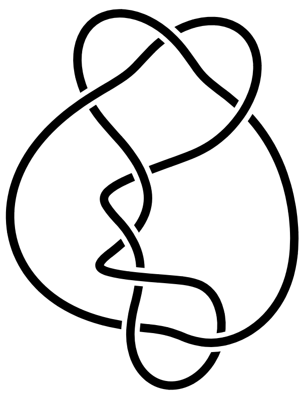
8_3



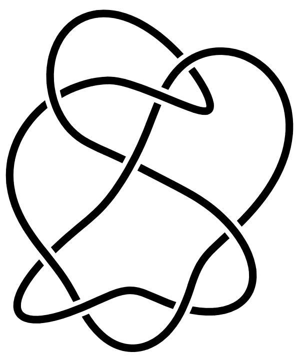
8_4



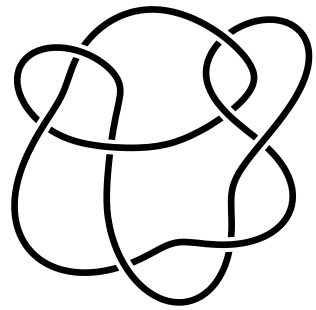
8_5



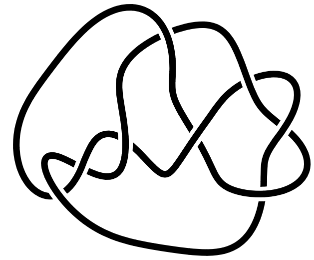
8_6



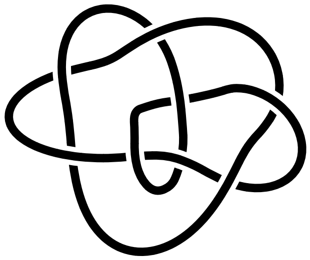
8_7



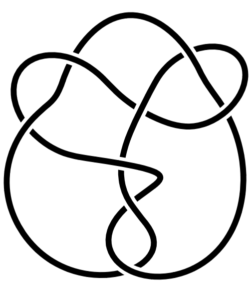
8_8



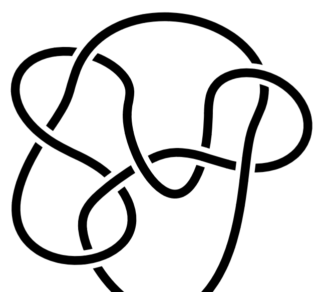
8_9



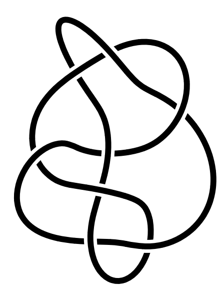
8_{10}



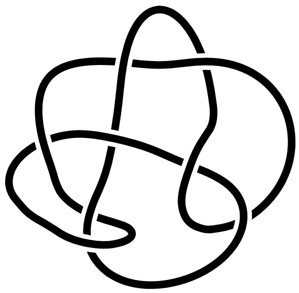
8_{11}



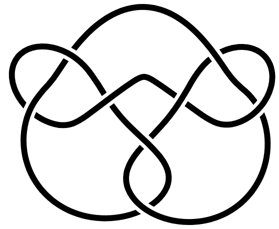
8_{12}



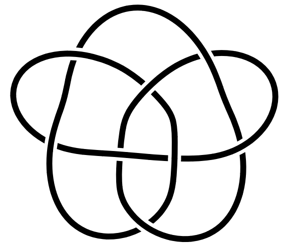
8_{13}



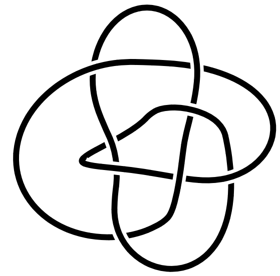
8_{14}



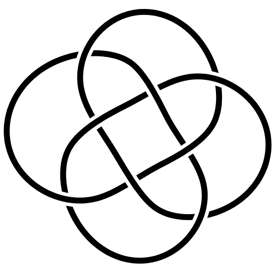
8_{15}



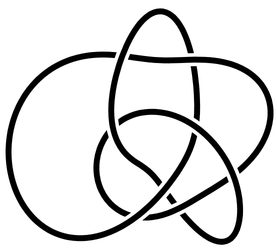
8_{16}



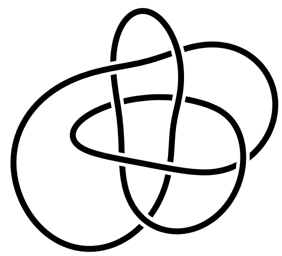
8_{17}



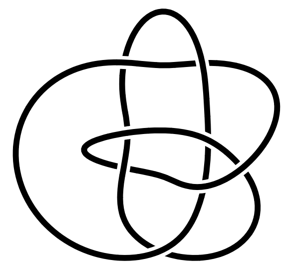
8_{18}



8_{19}



8_{20}



8_{21}

Index