DISCOVERING THE ART OF MATHEMATICS

GEOMETRY

MATHEMATICAL INQUIRY IN THE LIBERAL ARTS

Julian F. Fleron and Volker Ecke,
with Philip K. Hotchkiss and Christine von Renesse

Discovering the Art of Mathematics

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Contents

Preface

This book is a very different type of mathematics textbook. Because of this, users new to it, and its companion books that form the Discovering the Art of Mathematics library[1], need context for the book's purpose and what it will ask of those that use it. This preface sets this context, addressing first the Explorers (students), then both Explorers and Guides (teachers) and finishing with important information for the Guides.

0.1 Notes to the Explorer

"Explorer?"

Yes, that's you - an Explorer. And these notes are for you.

We could have addressed you as "reader," but this is not a book intended to be read like a traditional book. This book is really a guide. It is a map. It is a route of trail markers along a path through part of the vast world of mathematics. This book provides you, our explorer, our heroine or hero, with a unique opportunity to explore - to take a surprising, exciting, and beautiful journey along a meandering path through a great mathematical continent.

"Surprising?" Yes, surprising. You will be surprised to be doing real mathematics. You will not be following rules or algorithms, nor will you be parroting what you have been dutifully shown in class or by the text. Unlike most mathematics textbooks, this book is not a transcribed lecture followed by exercises that mimic examples laid out for you to ape. Rather, the majority of each chapter is made up of Investigations. Each chapter has an introduction as well as brief surveys and narratives as accompaniment, but the Investigations form the heart of this book. They are landmarks for your expedition. In the form of a Socratic dialogue, the Investigations ask you to explore. They ask you to discover mathematics. This is not a sightseeing tour, you will be the active one here. You will see mathematics the only way it can be seen, with the eyes of the mind - your mind. You are the mathematician on this voyage.

"Exciting?" Yes, exciting. Mathematics is captivating, curious, and intellectually compelling if you are not forced to approach it in a mindless, stress-invoking and mechanical manner. In this journey you will find the mathematical world to be quite different from the static barren landscape most textbooks paint it to be. Mathematics is in the midst of a golden age - more mathematics is being discovered now than at any time in its long history. Each year there are 50,000 mathematical papers and books that are reviewed for *Mathematical Reviews*! Fundamental questions in mathematics - some hundreds of years old and others with $ 1 Million prizes - are

[1] All available freely online at `http://artofmathematics.org/books`.

being solved. In the time period between when these words were written and when you read them important new discoveries adjacent to the path laid out here have been made.

"Beautiful?" Yes, beautiful. Mathematics is beautiful. It is a shame, but most people finish high school after 10 - 12 years of mathematics *instruction* and have no idea that mathematics is beautiful. How can this happen? Well, they were busy learning arithmetical and quantitative skills, statistical reasoning, and applications of mathematics. These are important, to be sure. But there is more to mathematics than its usefulness and utility. There is its beauty. And the beauty of mathematics is perhaps its most powerful, driving force. As the famous **Henri Poincaré** (French mathematician; 1854 - 1912) said:

> The mathematician does not study pure mathematics because it is useful; [s]he studies it because [s]he delights in it and [s]he delights in it because it is beautiful.

Mathematics plays a dual role as a liberal art and as a science. As a powerful science, it shapes our technological society and serves as an indispensable tool and as a language in many fields. But it is not our purpose to explore these roles of mathematics here. This has been done in other fine, accessible books. Instead, our purpose is to journey down a path that values mathematics for its long tradition as a cornerstone of the liberal arts.

Mathematics was the organizing principle of the *Pythagorean society* (ca. 500 B.C.). It was a central concern of the great Greek philosophers like **Plato** (Greek philosopher; 427 - 347 B.C.). During the Dark Ages, classical knowledge was preserved in monasteries. The classical **liberal arts** organized knowledge in two components: the ***quadrivium*** (arithmetic, music, geometry, and astronomy) and the ***trivium*** (grammar, logic, and rhetoric) which were united by philosophy. Notice the central role of mathematics in both components. During the Renaissance and the Scientific Revolution the importance of mathematics as a science increased dramatically. Nonetheless, it also remained a central component of the liberal arts during these periods. Indeed, mathematics has never lost its place within the liberal arts except in contemporary classrooms and textbooks where the focus of attention has shifted solely to its utilitarian aspects. If you are a student of the liberal arts or if you want to study mathematics for its own sake, you should feel more at home on this expedition than in other mathematics classes.

"Surprise, excitement, and beauty? Liberal arts? In a mathematics textbook?" Yes. And more!

In your exploration here you will see that mathematics is a human endeavor with its own rich history of struggle and accomplishment. You will see many of the other arts in non-trivial roles: art, music, dance and literature. There is also philosophy and history. Students in the humanities and social sciences, you should feel at home here too. There are places in mathematics for anyone to explore, no matter their area of interest.

The great **Betrand Russell** (English mathematician and philosopher; 1872 - 1970) eloquently observed:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.

We hope that your discoveries and explorations along this mathematical path will help you glimpse some of this beauty. And we hope they will help you appreciate Russell's claim:

> ... The true spirit of delight, the exultation, the sense of being more than [hu]man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

Finally, we hope that your discoveries and explorations enable you to make mathematics a part of your lifelong educational journey. For, in Russell's words once again:

> ... What is best in mathematics deserves not merely to be learned as a task but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement.

Bon voyage. May your journey be as fulfilling and enlightening as those that have beaconed people to explore the many continents of mathematics throughout humankind's history.

0.2 Navigating This Book

Intrepid Explorer, as you ready to begin your journey, it may be helpful for us to briefly describe basic customs used throughout this book.

As noted in the Preface, the central focus of this book is the **Investigations**. They are the sequences of problems that will help guide you on your active exploration of mathematics. In each chapter the Investigations are numbered sequentially in bold. Your role will be to work on these Investigation individually or cooperatively in groups, to consider them as part of homework assignments, to consider solutions to selected Investigations that are modeled by your fellow explorers - peers or your teacher - but always with you in an active role.

If you are stuck on an Investigation remember what **Frederick Douglass** (American slave, abolitionist, and writer; 1818 - 1895) told us:

> If there is no struggle, there is no progress.

Or what **Shelia Tobias** (American mathematics educator; 1935 -) tells us:

> There's a difference between not knowing and not knowing *yet*.

Keep thinking about the problem at hand, or let it ruminate a bit in your subconscious, think about it a different way, talk to peers, or ask your teacher for help. If you want you can temporarily put it aside and move on to the next section of the chapter. The sections are often somewhat independent.

Independent Investigations are so-called to point out that the task is more involved than the typical Investigations. They may require more significant mathematical epiphanies, additional research outside of class, or a significant writing component. They may also signify an opportunity for class discussion or group reporting once work has reached a certain stage of completion.

The **Connections** sections are meant to provide illustrations of the important connections between the mathematics you're exploring and other fields - especially in the liberal arts. Whether you complete a few of the Connections of your choice, all of the Connections in each section, or are asked to find your own Connections is up to your teacher. We hope that these Connections sections will help you see how rich mathematics' connections are to the liberal arts, the fine arts, culture, and the human experience.

Further Investigations, when included, are meant to continue the Investigations of the mathematical territory but with trails to significantly higher ground. Often the level of sophistication of these investigations will be higher. Additionally, our guidance will be more cursory - you are bushwhacking on less well-traveled trails.

In mathematics, proof plays an essential role. Proof is the arbiter for establishing truth and should be a central aspect of the sense-making at the heart of your exploration. Proof is reliant on logical deductions from agreed upon definitions and axioms. However, different contexts suggest different degrees of formality. In this book we use the following conventions regarding **definitions**:

- An *Undefined Term* is italicized the first time it is used. This signifies that the term is: a standard technical term which will not be defined and may be new to the reader; a term that will be defined a bit later; or an important non-technical term that may be new to the reader, suggesting a dictionary consultation may be helpful.

- An ***Informal Definition*** is italicized and bold- faced the first time it is used. This signifies that an implicit, non-technical, and/or intuitive definition should be clear from context. Often this means that a formal definition at this point would take the discussion too far afield or be overly pedantic.

- A **Formal Definition** is bolded the first time it is used. This is a formal definition that is suitably precise for logical, rigorous proofs to be developed from the definition.

In each chapter the first time a **Biographical Name** appears it is bolded and basic biographical information is included parenthetically to provide historical, cultural, and human connections.

In mapping out trails for your explorations of this fine mathematical continent we have tried to uphold the adage of **George Bernard Shaw** (Irish playwright and essayist; 1856 - 1950):

> I am not a teacher: only a fellow-traveler of whom you asked the way. I pointed ahead – ahead of myself as well as you.

We wish you wonderful explorations. May you make great discoveries, well beyond those we could imagine.

0.3 Directions for the Guides

Faithful Guide, you have already discovered great surprise, beauty and excitement in mathematics. This is why you are here. You are embarking on a wonderful journey with many explorers looking to you for bearings. You're being asked to lead, but in a way that seems new to many.

We believe telling is not teaching. Please don't tell them. Answer their questions with questions. They may protest, thinking that listening is learning. But we believe it is not.

This textbook is very different from typical mathematics textbooks in terms of structure (only questions, no explanations) and also of expectations it places on the students. They will likely protest, "We're supposed to figure this out? But you haven't explained anything yet!" It is important to communicate this shift in expectations to the students and explain some of the reasons. That's why we have written the earlier sections of this preface, which can help do the explaining for us (and for you).

You need support as well. A shift in pedagogy to a more inquiry-based approach may be subtle for some, but for many it is a great leap. Understanding this we have assembled an online resource to support teachers in the creation and nurturing of successful inquiry-based mathematics classrooms. Available online at `http://artofmathematics.org/classroom` it contains a wealth of information - in many different forms including text, data, videos, sample student work - on many critical topics:

- Why inquiry-based learning?
- How to get started using our books...
- A culture of curiosity
- Learning contracts
- Grouping students
- Choosing materials - Mixing It Up
- Asking good questions
- Creating inquiry-based activities
- Making mistakes
- Cool things
- Proof as sense-making
- Homework stories
- Exams
- Posters
- Assessment: Student Solution Sets
- Evaluating the effectiveness of inquiry-based learning
- ... and much more ...

We wrote the books that make up the Discovering the Art of Mathematics library because they have helped us have the most extraordinary experiences exploring mathematics with students who thought they hated mathematics and had been disenfranchised from the mathematical experience by their past experiences. We are encouraged that others have had similar experiences with these materials. We love to hear success stories and are also interested in hearing about things that might need to be changed or did not work so well. Please feel free to share your stories and suggestions with us: `http://artofmathematics.org/contact`.

0.3.1 Notes for Guides on the Geometry Exploration

Three-Dimensional Students with Two-Dimensional Mindsets

As Johannes Kepler tells us:

> Where there is matter, there is geometry.

We live in a three-dimensional world. Despite this, the overwhelming focus of the experiences students have gained in contemporary American mathematics classrooms are two-dimensional. We have, using A Square's words, confined our students to limited dimensionality.

What (very) little classroom experience our students have with three-dimensional geometry, has rigidly locked them into this tiny box which constitutes what passes for three-dimensional geometry. Ask them to name a three-dimensional geometric object. They will say “sphere”, “cube”, “box”, and “cylinder”. None will say a human body, a car, a building, a tree, a strand of DNA, or the Louvre.

We strongly recommend that you repeatedly make connections to the real world as your students explore dimensions.

Our students need to think about the geometry of a tree. How a compound mitre saw cuts crown moulding. How a CAT scan images the body. How an animator creates life-like digital movies.. . .

Having them do this will help liberate them - as A Square implored us to.

And it will help them embrace Kepler's maxim.

Dimension: This Book's Theme

With Kepler's words in mind, choosing topics to include in a book on Geometry is a crucial task if one is to keep the book at a reasonable size. So the focus, no pun intended, is critical. We have chosen to focus on Dimension in this book.

This theme gets right at the heart of what we see as one of our students' biggest geometric weakness - their impoverished experience with three-dimensional geometry despite being three-dimensional beings. The subject matter is mathematically rich, but it also has very deep connections to so many areas of interest and experience of our students - art, technology, architecture, visualization and perception.

Of course, there are lots of other choices teachers can and do make. Luckily, there are quite a number of resources that are available for those interested in plotting a different course or supplementing what we have done.

There are a few very good books that have a distinctive inquiry-based approach that we can recommend:

- Groups and Symmetry: A Guide to Discovering Mathematics by David W. Farmer is a short (102 page), inexpensive ($16) paperback which in provides a wonderful introductory inquiry-based tour of symmetry - from rigid motions through tessellations, finite figures, Frieze patterns, wallpatterns, and into some basic group theory. We have found the book to be written for a slightly higher level of students than those that we have written for. But the material is excellent.

- Symmetry, Shape and Space: An Introduction to Mathematics Through Geometry by L. Christine Kinsey and Teresa E. Moore is a full-length (494 pages) that is very much informed by inquiry-based learning. It contains 13 chapters, loosely connected. Several chapters are highly recommended: Tesselations, Two-Dimensional Symmetry, Polyhedra, and Shape. They are excellent. Several others have non-trivial overlap with our materials here and, to be honest, we like our approach better.

- Viewpoints: Mathematical Perspective and Fractal Geometry in Art by Marc Frantz and Annalisa Crannell is a full-length text with a very strong inquiry approach that is based on wonderful workshops the authors have run for years. It is strongly guided by links to Visual Art. Outstanding.

Other Approaches to Geoemetry

Interesting foci, other than our theme of dimension, for inquiry-based learning courses on geometry for mathematics for liberal arts students include:

- **Non-Euclidean Geometry** - The chapters on "Space" from Symmetry, Shape and Space can be supplemented with materials from Jeff Week's wonderful Shape of Space, Timothy G.

Freeman's fine Portraits of the Earth: A Mathematician Looks at Maps from the Mathematics Across the Curriculum Project, Donal O'Shea's very readable account of the only solved Millenium Prize Problem - The Poincare Conjecture: In Search of the Shape of the Universe, a collection of *Lénárt spheres*, the wonderful online book Math and the Art of M. C. Escher available at `http://euler.slu.edu/escher/index.php/Math_and_the_Art_of_M._C._Escher` and the wonderful online applet nonEuclid available at `http://www.cs.unm.edu/~joel/NonEuclid/NonEuclid.html` provide enough resources for a great course on non-Euclidean goemetry.

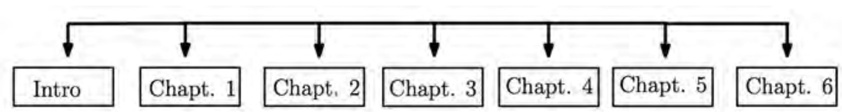
- **Transformational Geometry** - Groups and Symmetry, the chapters "Tessellations" and "Two-dimensional symmetry" from Symmetry, Shape, and Space, the wonderful online book Math and the Art of M. C. Escher available at `http://euler.slu.edu/escher/index.php/Math_and_the_Art_of_M._C._Escher`, and myriad print and online resources, applets, and art provide ample fodder for a course on transformational geometry.

- **Mathematics and Art** - Viewpoints: Mathematical Perspective and Fractal Geometry in Art, Paul A. Calter's Squaring the Circle: Geometry in Art and Architecture form the Mathematics Across the Curriculum Program, and Tony Robbin's thought-provoking Shadows of Reality: The Fourth Dimension in Relativity, Cubism, and Modern Thought can be supplemented with many beautiful art books (e.g. Julian Beever's Pavement Chalk Artist, Al Seckel's Masters of Deception: Escher, Dali & the Artists of Optical Illusion, and the books on fiber arts by sarah marie belcastro and Carolyn Yackel) to make a wonderful course on mathematics and art.

- **Applied Geometry** - We have run a very successful course on mathematical applications of goemetry where the focus was the interplay between the dimensions. Students used the free C.A.D. software Google SketchUp to build both architectural models and photoimaged .kmz models of buildings in our town while learning about many other applications of dimensional interplay. In some ways this is a greatly expanded version of the 3 in this book. More information can be found at `http://www.westfield.ma.edu/ecke/flatland/Home.html` and `http://revisioningwestfield.westfield.ma.edu/`.

0.3.2 Chapter Dependencies

Guides are encouraged to pick and choose topics freely, from this book and others in the Discovering the Art of Mathematics series, depending on their interests and those of their students. There are no chapter dependencies in this book:



Chapter 1

Introduction

> I have discovered such wonderful things that I was amazed... Out of nothing I have created a strange new universe.
>
> **Jànos Bolyai** (Hungarian Mathematician; 1802 - 1860)

1.1 The Many Faces of Geometry

In general, much of what one learns in high school geometry does not transcend what was known over 2000 years ago. The high school canon includes important changes in language and approach, specifically trigonometry and the use of algebra, but little of true geometric substance moves beyond the ancient knowledge. It is as if nothing much had happened in our understanding and use of geometry since **Euclid of Alexandria** (Greek mathematician; circa 325 BC - 275 BC). This is a misleading perception and deprives us from learning about the wonderful areas in which geometry continues to flourish and to profoundly influence our world.

Our goal in this book is to help you to (re-)discover and explore some of the fundamental revolutions that have shaped geometry. In each of these revolutions the traditional way of thinking, the *dominant paradigm*, has been overturned by the insights of a few people who were able to break free of the prevailing views long enough to glimpse the possibilities of "strange new universes," like Bolyai who invented/discovered *non-Euclidean geometry*. Once surprising and perhaps counterintuitive, these new views have become powerful tools that are central to a contemporary understanding of geometry.

This book allows you to investigate these rich views of our world.

Imagine. You can discover whole new worlds simply by opening your mind. This is a very powerful lesson, one that we hope becomes part of your way of thinking and learning - both within and beyond your mathematical explorations.

Speaking of learning, another shortcoming of the typical high-school geometry experience is the way in which the subject is traditionally taught. As described in Section 7.3 of this book's final chapter, geometry was the area in which structured logical reasoning and rigor were first introduced into the world of mathematics in an explicit way. Our long-held efforts to use high school geometry as "the arena in which students will finally get to engage in true mathematical reasoning" have generally not been successful. Instead, as Paul Lockhart describes, it has resulted in a course which is a "virus" which "attacks mathematics at its heart, destroying the very essence of creative rational argument, poisoning the students' enjoyment of this fascinating and beautiful subject, and permanently disabling them from thinking about math in a natural and intuitive way."[1]

Certainly you will have to think, to reason, and to use logic - one cannot do mathematics without working in this way - but we hope that your explorations here will seem more natural, intuitive, and relevant than two column proofs, the formal statements of the Propositions of Euclid's Elements, and the wholly impractical presentation of geometry as a fixed, formal, and ancient system.

1.2 Dimension

The sculptures in Figure 1.1 by **Helaman Ferguson** (American sculptor and a digital artist; 1940 -) and **George Hart** (American geometer and artist; 1955 -) are these sculptors' rendition of complex mathematical objects. They are just two of the many complex mathematical objects these artists have so beautifully rendered in sculpture.[2]

Like all sculptors, Ferguson and Hart use height, breadth and depth to convey the intricate relationships, connections, and attributes of various components of the work being modeled. The spatial freedom of the sculptor contrasts sharply to that of the painter, illustrator or photographer who are limited to the fields of their canvases which consist of height and breadth but no depth. In mathematical terms, the canvases of the painter, illustrator, and photographer are *two-dimensional* while the "spatial canvas" of the sculptor is *three-dimensional*.

Figure 1.1: The sculptures "Compass Points" by George Hart and "Torus Cross Cap" by Helaman Ferguson.

[1] From A Mathematician's Lament, by Paul Lockhart, p. 67.

[2] For more on Ferguson's work see his book Mathematics and Stone and Bronze. For more on Hart's work see his Internet site at `http://www.georgehart.com/`. For more on the growing field of mathematical art in general, see `http://www.bridgesmathart.org/` and `http://virtualmathmuseum.org/mathart/MathematicalArt.html`.

The objects of our everyday existence are *all* three-dimensional. Even the sheets of paper that make up this book have length, width, and thickness. If they were two-dimensional, with no thickness, this book, made up of several dozen pages would have no thickness either. Yet much of our art is portrayed two-dimensionally - as paintings, as photographs, on computer monitors, in books, on televisions, etc. Isn't this a bit strange?

Stranger still is the analogous situation in many contemporary geometry classrooms.

1. Take a few minutes to think about your school experiences with geometry. Write down the main themes and ideas. Then write down as many of the geometric objects you studied as you can remember.

2. The geometric objects that you considered in your school geometry experience, are they generally two-dimensional or three-dimensional?

3. Have your experiences with school geometry helped enrich the way you interact with the three-dimensional world in which you live in important ways? Explain.

In "Notes to the Explorer," this book's preface, we described this book as "a route of trail markers along a path through part of the world of mathematics." The paths into the world of geometry that are taken by most as part of their school math experiences are generally pedestrian. Shapes are named: circles, ellipses, squares, triangles, rhombi, hexagons, octagons, etc. Even an occasional three-dimensional solid is named: the sphere, cone, pyramid, cube and cylinder. Some of their basic properties are considered and some of the objects are measured, with mysterious formulae seemingly concocted by the gods of mathematics.

We shall soon meet the hero of the novella Flatland by **Edwin Abbott Abbott** (English educator and author; 1838 - 1926). Named A Square, the hero decried the state of ignorant bliss in which his fellows citizens lived, saying:

> Yet I exist in the hope that these memoirs. . . may find their way to the minds of humanity in Some Dimension, and may stir up a race of rebels who shall refuse to be confined to limited Dimensionality.

Answer his call! It's your turn to break out of the confines of the narrow, well-worn, largely two-dimensional path of school geometry.

Not far along the new paths laid out here you will meet up with all sorts of people exploring the vast, rich world of geometry: painters, architects, carpenters, graphic artists, woodworkers, medical professionals, video gamers, scientists, dancers, geographers, potters and other craftspeople, etc. For all of them geometry plays a fundamental role in their constructions, creations, tools, and analyses.

The geometry one finds along these rich paths are fundamentally tied to the refusal to be tied to limited dimensionality. Cubist painters are inspired by the fourth dimension. CAT scans and other medical imaging help navigate between two and three dimensions to allow radiography to capture three-dimensional images of three-dimensional humans. Graphic artists use perspective drawing to mimic three-dimensions. Even in nature we find the call to be liberated from "limited dimensionality." As **Benoit Mandelbrot** (French mathematician and computer scientist; 1924 - 2010) said:

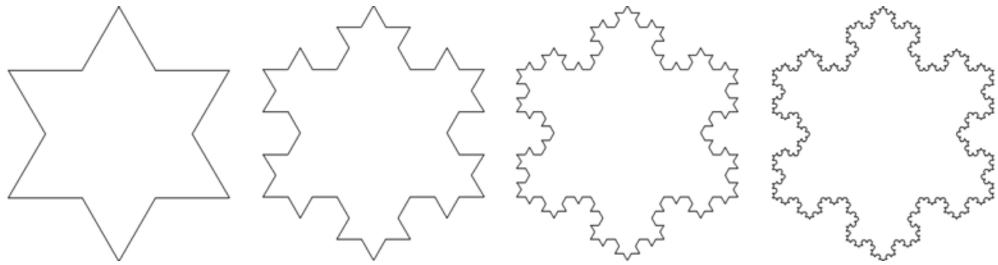


Figure 1.2: The Koch snowflake, named after its inventor **Niels Fabian Helge von Koch** (Swedish mathematician; 1870 - 1924).

> Why is geometry often described as cold and dry? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

When you explore Chapter 6 you will find that these natural objects can be described geometrically through their *self-similarity*. As such these natural objects, like the ***Koch snowflake*** in Figure 6.11, are ***fractals*** - objects whose dimensions are not even whole numbers!

So break free and follow these paths into the many beautiful dimensions of geometry.

1.3 Geometry Around Us

> Where there is matter there is geometry.
>
> **Johannes Kepler** (German mathematician and astronomer; 1571 - 1630)

4. Do you agree with Kepler? Explain.

5. In grade school teachers often have geometry scavenger hunts; they ask students to find as many triangles, rectangles, and circles as they can. Spend a few minutes doing this. Write down all of the basic shapes that you find.

Let's see what happens if we are not confined to thinking of geometry as something motivated by basic shapes. How about a snowflake?

6. Most of us have made paper snowflakes by folding paper and then cutting it with scissors. Do so.

Johannes Kepler was a fundamental figure in the *scientific revolution* in 17^{th} century Europe. Near the end of 1610, while Kepler was serving as the Imperial Mathematician to **Rudolph II** (Holy Roman Emperor; 1576 - 1612), he was preoccupied trying to find a New Year's gift for his friend **Wacker von Wackenfels** (German diplomat and scholar; 1550 - 1619). During an evening walk a snowflake that fell on his coat aroused such curiosity in Kepler that he spent months exploring the geometry of snowflakes. The result, and the gift for his friend, was the remarkable booklet The Six-Cornered Snow Flake. This book is a beautiful illustration of the type of magical curiosity and wonder that drives mathematical and scientific explorations. The booklet is also eloquently written in a witty style that makes it a novella worth reading for its literary value alone. Of greatest importance is that it uses the snowflake, "something that, while being next to Nothing, [which] would yet allow for subtle refection,"[3] as a starting place to ask essential geometric questions that have had a long-term impact on mathematics and science.

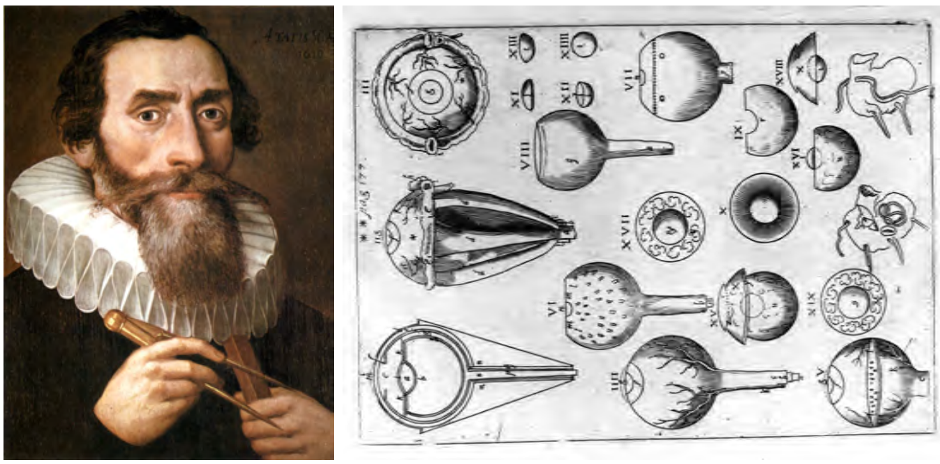


Figure 1.3: Portrait of Johannes Kepler (left) and Kepler's illustrations of human optics (right). The latter came from his Astronomiae Pars Optica which is considered to constitute the founding of the field of *geometrical optics*.

Kepler asks:

> Who shaped the little head before it fell, giving it six frozen horns? What cause establishes on that surface, at the very moment it condenses, six points for six radii to be connected all around? . . . As I consider this, I begin to wonder: Why are the radii not arranged spherically in every direction? Why, if internal heat is responsible for this, does it operate only on a plane surface? For heat distributes itself uniformly in all places, and is not present only on a flat surface of vapor.[4]

Isn't this an interesting connection to school experiences like those you considered in Investigation **2**?

7. The snowflake you made earlier, is it six-cornered and does it exhibit *six-fold symmetry*, like the snowflakes that fall from the sky? If not, try again, being more thoughtful about your approach so you can make a six-cornered, symmetric snowflake.

8. What difficulties did you encounter in trying to create a six-cornered snowflake? Explain.

[3] From p. 31 of the 2010 Paul Dry Books edition of The Six-Cornered Snow Flake.
[4] From p. 73 of the 2010 Paul Dry Books edition of The Six-Cornered Snow Flake.

Figure 1.4: Photographs of snowflakes by **Wilson Bentley** (American farmer; 1865 - 1931) who was the first known photographer to successfully photograph snowflakes.

9. Many snowflakes that one finds hanging in elementary schools during the winter are not six-cornered. What do you think about this?

Since snowflakes are crystals, Kepler began a detailed examination of the geometry of crystals in The Six-Cornered Snow Flake. He considered not only the arrangement of the "horns" on the snowflakes, but many types of crystal arrangements. In fact, this book is considered one of the pioneering works in *crystallography.*

One geometric arrangement Kepler considered was that of spheres. To investigate, you'll need a bunch of spheres of the same size - marbles, beads, tennis balls, etc.

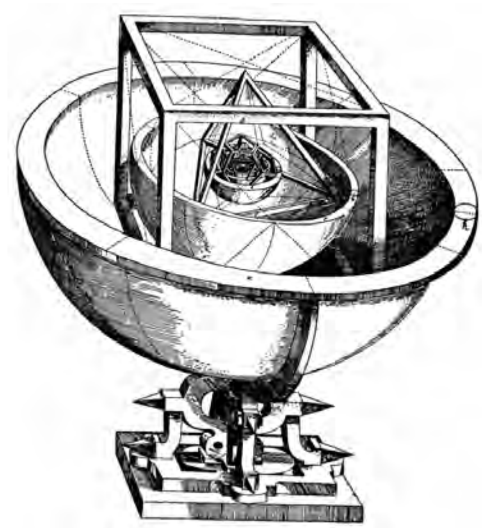


Figure 1.5: Kepler's model of the solar system based on the Platonic solids. For more on this model, see Section 1.6.2.

10. Take several of your spheres. Lay them on a flat surface. How can you pack them so they take up the least space? Can you explain why you think this packing is the most efficient?

11. The goal now is to stack the spheres so that they rise vertically more than one layer. Add several more spheres to your collection and see if you can determine how to stack them so they take up the least space. Describe your stacking. Can you explain why you think this packing is the most efficient?

12. Name some spherical objects that you have seen stacked. Is this stacking essentially the same as the one that you found? Explain.

As part of his playful investigations in The Six-Cornered Snowflake, Kepler suggested that the packing you most likely found in Investigation **11** and Investigation **12**, called ***face-centered packing***, is the most efficient possible. His claim became known as:
Kepler's conjecture: The most efficient[5] packing of spheres is the face-centered packing.

You might wonder why this is given such a fancy name, isn't it intuitively obvious? Perhaps, but a proof eluded mathematicians for almost 400 years! And mathematicians are not content until they have absolute proof. Mathematics is not an empirical science, but rather a deductive science where conclusions must be arrived at by irrefutable logic. For more than two millennia mathematicians struggled to prove that *Euclid's parallel postulate* was a consequence of four other, more intuitively obvious axioms for Euclidean geometry. Without proof they labored on. And their hesitancy was legitimate and eventually rewarded. In the middle of the nineteenth century geometry was liberated by the stranglehold of Euclidean geometry when Boylai and others discovered non-Euclidean geometries - geometries we have come to consider as natural and consistent as that of Euclid.[6]

So what is the status of Kepler's conjecture? In the 1990's, **Thomas Hales** (American mathematician; 1958 -) announced remarkable progress on this problem by reducing it to a problem that is theoretically within reach of computational methods. Computers have been used to check "all" of the necessary solutions and the 100-page "non-computer part" of proof was published in 2005 by the important *Annals of Mathematics*. Some still question whether this theorem has been deductively established, asking whether a computer can really be trusted to complete a proof. Hales continued work refining the result and on August 10, 2014, the *Flyspeck project* announced that a complete, *formal proof* had been completed.

Geometry is alive and well. Here's another example of geometric packing. In 2009 *Time* named an algorithm developed by researchers at The University of Mainz as one of the "50 Best Inventions of 2009." The invention? A dramatically improved algorithm for packing a given collection of discs into the smallest circle possible. While this may not seem like an earth-shattering invention, it is quite important for packing, storage, and many other practical *optimization problems.* It is a notoriously hard problem and for which we can only find approximate solutions - at least so far.[7]

13. Let's continue to try to break free of the confines of basic shape geometry. Spend some time searching for things in the world around you that do not contain basic shapes, but rather

[5] Measured by *average density.*

[6] The book A Certain Ambiguity by Gaurav Suri and Hartosh Singh Bal is a beautiful novel about the exploration of certainty, absolute truth, and ambiguity told through the eyes of Ravi and his grandfather Vijay who are forced to confront certainty in very different, but important, settings decades apart.

[7] A *New Scientist* article on this discovery is available online at `http://www.newscientist.com/article/dn16716-recordbreaking-algorithm-really-packs-them-in.html`.

more complicated, interesting geometric shapes, especially if these things do not have exact geometric names. Keep a list of the interesting things you have found.

14. Compare your new list with your earlier basic shapes list. What do you notice? Contrast what you have found with the Mandelbrot quote above.

15. Return to your list in Investigation **13**. Choose a few objects on your list that are interesting. Despite not containing basic geometric shapes, are there shapes (e.g. curves, arcs, contours, faces, portrusions, angles) that are interesting? What are some of the features of the surface of your objects? Describe the relative sizes and scales of your objects and/or various components of your objects.

16. Return to Investigation **4**. Have your investigations enriched your view of Kepler's quote? Explain.

17. **Classroom Discussion: Geometry Show and Tell** Bring in a solid, three-dimensional object, or a likeness of it, that you find particularly interesting. Each person should share their object with the rest of the group. As a group, each person should describe some of the geometric features (like those mentioned in Investigation **15** of the object that make it interesting. (Do not be concerned with names or basic shapes. Reach deeper. There is geometry in rainbows[8], in their shape, the configuration of colors, in the way light is defracted from the shapes of the water droplets, and many other areas. There is geometry in pineapples where spirals of *Fibonacci numbers* eminate from the base, where the nobs that form the surface fit together as a *tessellation*, in the surface of the leaves, and so on.)

1.4 Geometric Construction

It [the Larkin Building in Buffalo, New York] was an essay in the third dimension.[9]

Frank Llyod Wright (American Architect; -)

For all of the attention the naming of geometric shapes gets early on in the standard geometry curriculum, measurement becomes an enormous focus later on. What is missed is the profound role that construction has always played in geometry.

Pictured in Figure 1.7 are Google SketchUp (GSU) models which were constructed by students in our general education course at Westfield State University. None of these students had prior experience with GSU.[10] It's not just our students who are making models. Amateurs all over the world have joined with architects, regional planners, and modelers to help populate the repository of models called Google 3D Warehouse. When you use Google Earth or Google Maps to go to *any* major location in the world where there is human architecture, much of this architecture is available to you as virtual 3D models that you can explore! 3D Warehouse houses all of these models and hundreds of thousands of other models, including: cars, furniture, natural objects, proposed renovation plans, woodworking models, aircraft, and imaginary creatures of all sorts.

[8] Which was studied in detail by **Rene Descartes** (French mathematician, philosopher and author; 1596 - 1650).

[9] From the Ken Burns documentary Frank Lloyd Wright.

[10] Recently Google has sold its rights to SketchUp. The program is now just called SketchUp and is marketed and distributed by Trimble. SketchUp Make remains free.

Figure 1.6: Exterior and interior of the Larkin Building, designed by Frank Lloyd Wright. Built in Buffalo, New York in 1906 it was unfortunately demolished in 1950.

18. Find 10 structures, buildings, or monuments that have been modeled as part of the Google 3D Warehouse that you think are particularly beautiful, important, or artistic. Use the Earth mini-plugin on maps.google.com to Show 3D Images of an interesting view of each of the objects you have chosen. For each, print out your view, give the name and location, and explain why you find it compelling. (Note: You may need to use screen captures to extract the images.)

Until two decades ago, if one wanted to make a model of a building or sculpture or new geometric creation, they would have to physically construct it by hand. Now powerful ***computer-aided design programs*** (CAD) make it possible for all of us to make sophisticated models of three-dimensional objects. These models can be moved and rotated dynamically on computer screens, imported into computer generated imagery to be parts of movies, or they can be sent directly to a 3D printer so a scale, 3D model can actually be printed!

So, it is your turn to construct.

19. Envision, plan, and then construct a fabulous three-dimensional object of your own creation using SketchUp. You should be as creative as possible and use a diverse range of the tools available for construction in SketchUp.[11]

[11] There are extensive resources for SketchUp available online. This includes `http://www.3dvinci.net` and the wonderful SketchUp books by Bonnie Roskes. A short, concise introduction to SketchUp and its relation to three-dimensional explorations in mathematics is "Exploring Three-Dimensional Worlds Using Google SketchUp" by Julian F. Fleron and Jenny Livingstone, *The Mathematics Teacher*, Vol. 105, No. 6, 2012, pp. 469-473.

Figure 1.7: Original Google SketchUp models by Aaron Butler, Matt Pegorari, and Chris Fredette.

1.5 Conclusion

Our goal here has been to give you a warm-up, to get you thinking more broadly about geometry before you embark on a liberating exploration of geometry and dimensions. We hope that this has begun to open your eyes a bit to the broader role of geometry around you. As you explore going forward, please keep your eyes open for geometry in the world around you. Take a few minutes each day to ask yourself "Is there geometry here?" We believe that as you do your journey will be enriched and you will begin to agree with Kepler - "Where there is matter there is geometry." And you might even begin to break free of the confines of limited dimensionality.

1.6 Further Investigations

1.6.1 Kepler's Laws of Planetary Motion

One of Kepler's most notable contributions was ***Kepler's laws of planetary motion*** which precisely, and correctly, describe the orbits of the planets:

1. Each orbit is an *ellipse* with the sun as one of the two *foci*,

2. A line segment between a planet and the sun sweeps out equal areas in equal time intervals, and,

3. the square of the ***period*** of the planet's orbit, i.e. the planet's year, is proportional to the cube of the length of the orbit's *semi-major axis.*

F1. Compasses are used to construct circles. How can you construct an ellipse? Find out how ellipses can be constructed (there are many interesting ways) and construct one.

F2. Illustrate Kepler's second law with a diagram.

F3. Kepler's quote in Section 1.3 asked why snowflakes' growth was limited only to two dimensions when heat is distributed uniformly in three dimensions. Why are planets' orbits restricted to two-dimensional planes when gravity acts equally in all directions in three dimensions?

1.6.2 Platonic Solids and Kepler's Solar System Model

A brilliant scientist, Kepler was also somewhat of a mystic and often jumped to accepted fanciful "explanations" of scientific principles. Prior to his discovery of the three laws of planetary motion that bear his name, in his Mysterium Cosmographicum of 1596, he proposed a model of the solar system constructed based on the fact that there were exactly five *Platonic solids.* His illustration of this model is in Figure 1.5.

F4. Explain why in Kepler's model the five Platonic solids yield precisely six spherical orbits, one for each of the known planets at this time.

A **Platonic solid** is a ***polyhedron***, a three-dimensional analogue of a *polygon*, which has the following properties:

- all of its faces are *congruent*, *regular polygons*, and,
- the same number of faces meet at every *vertex* in the polyhedron.

F5. Using *mathematical manipulatives* like Polydrons, Zome, or other geometry building materials[12], try to construct as many different Platonic solids as you can. How many could you create? Describe them precisely. Then describe any useful techniques you discovered or any particular challenges you faced.

[12]If these materials are not available to you, make and cut out copies of congruent, regular polygons out of paper and tape them together edge to edge

F6. Is it surprising that only five such solids exist?

F7. After working to construct the Platonic solids, do you have any intuitive idea why only five such solids exist?

F8. List all five of the Platonic solids by name and provide an image of each.

F9. Do you think it would be hard to *prove* that there are only five Platonic solids and no others can exist?

Later in this book, in Section 5.4 of Chapter 5, ***Euler's formula for polyhedra*** is considered. It is a wonderful formula which says the number of *faces*, f, the number of *edges*, e, and the number of *vertices*, v, in any *simply connected polyhedra* are related by:

$$v - e + f = 2.$$

F10. Show that Euler's formula holds for each of the Platonic solids.

Euler's formula can be used to prove that there are precisely five Platonic solids. First, suppose you have a Platonic solid. It's faces are all the same regular polygon. Let n denote the number of *sides* each of these regular polygonal faces have. When these polygonal faces meet at a vertex, the same number have to meet at every vertex. Let m denote the number of these regular polygons that meet at each vertex of the polyhedron.

F11. What are the values of n and m for a cube? For an octahedron? For a icosahedron? For the rest of the Platonic solids?

F12. If we are attempting to make a polyhedron, what is the smallest value n can have? Explain.

F13. Similarly, what is the smallest value m can have?

F14. For a specific Platonic solid, if you add up the total number of sides on all of the individual polygonal faces, how is this total related to the number of edges of the polyhedron? Express your result as an equation in terms of n, f and e.

F15. Solve your equation so it expresses f as a function of n and e.

F16. For a specific Platonic solid, if you add up the total number of edges that meet at all vertices of the entire polyhedron, how is this total related to the number of edges of the polyhedron? Express your result as a equation involving m, v and e.

F17. Solve your equation so it expresses v as a function of m and e.

F18. Using your equations in Investigation **15** and Investigation **17**, replace f and v in Euler's formula to obtain an equation involving only e, m and n.

F19. Simplify this equation, possibly by dividing through by e or $2e$, so only one term involves the variable e.

F20. Earlier you described the smallest values m and n can take on. This dramatically limits the number of possible solutions to your equation. In fact, you should now be able to find every possible value of e, m and n that solves this equation *and* to prove that there are no other solutions. Explain why this completes the proof that there are only five Platonic solids.

Kepler pokes fun at himself and his friend Wackenfels at the beginning of The Six-Cornered Snow Flake. Wachenfels is never disappointed when Kepler repeatedly arrives without a gift. Kepler concludes, "a gift will be the more pleasing and welcome to you the closer it comes to nothing."[13] Here is Kepler's description of his epiphany that the snowflake would be the perfect gift:

> While anxiously considering these matters, I crossed over the bridge, mortified by my incivility in having appeared before you without a New Year's gift, except perhaps (to keep playing the same chord) the one that I always bring you – namely, Nothing. Nor was I able to think of something that, while being next to Nothing, would yet allow for subtle reflection. Just then, by a happy occurrence, some of the vapor in the air was gathered into snow by the force of the cold, and a few scattered flakes fell on my coat, all six-cornered, with tufted radii. By Hercules! Here was something smaller than a drop, yet endowed with a shape. Here, indeed, was a most desirable New Year's gift for the lover of Nothing, and one worthy as well of a mathematician (who has Nothing, and receives Nothing) since it descends from the sky and bears a likeness to the stars.[14]

F21. Kepler wrote in Latin, using the word "nihil" for "nothing." What is the German word for "nothing"?

F22. What is the Latin word for snow?

F23. What was Kepler's intent with these words?

1.6.3 The Six-Cornered Snow Flake as an Archetype for Learning

In his introduction to Kepler's The Six-Cornered Snow Flake, Guillermo Bleichmar writes[15]:

> Whether or not we agree with Kepler that nature plays, undoubtedly the mind does, and it is in playing that it makes some of its most important discoveries – or finds, at least, the energy that sustains it in their pursuit. The Six-Cornered Snowflake illustrates how something can be made out of nothing, not just in the world of matter but within the sphere of reason. Throughout the text, rather than at any particular point of the argument, we are faced with the question of how ideas are born in the mind.

F24. Have your past experiences with geometry, or mathematics in general, ever felt like your mind playing? These can be school-based or not and should include experiences as far back as you can remember.

[13] From p. 25 of of the 2010 Paul Dry Books edition of The Six-Cornered Snow Flake.
[14] Pp. 31, 33 of of the 2010 Paul Dry Books edition of The Six-Cornered Snow Flake.
[15] P. 20 of of the 2010 Paul Dry Books edition of The Six-Cornered Snow Flake.

F25. Discovering the Art of Mathematics - Geometry is a book whose purpose is to help inspire ideas "born in the mind." As you explore through this book, return to this question - do you now feel like your mind is playing and that this playing has helped you make discoveries?

Chapter 2

Navigating Between Dimensions

> Yet I exist in the hope that these memoirs, in some manner, I know not how, may find their way to the minds of humanity in Some Dimensions, and may stir up a race of rebels who shall refuse to be confined to limited Dimensionality.
>
> **A Square** (Hero from the book Flatland; by Edwin Abbott Abbott - 1884)

2.1 Flatland – A Romance of Many Dimensions

Written by **Edwin Abbott Abbott** (English Teacher and Theologian; 1838 - 1926) the novella Flatland: A Romance of Many Dimensions[1] is perhaps the most well-known piece of mathematical fiction of all time.[2] Flatland is both a powerful satire of the prejudices of Victorian England and a wonderful tour through the dimensions.

The setting for Flatland is a land which is a two-dimensional plane which is populated by geometric figures. Our hero, whose name is A Square, learns of the "prejudices of his own dimension" when visited by a sphere. Sphere's arrival in Flatland mystifies our hero who sees only variable circles - the *cross sections* of Sphere as he moves through Flatland. When descriptions of the third dimension do not convince A Square of a greater reality, Sphere literally lifts our hero out of Flatland and into the third dimension, providing enlightenment.

Abbott was a dedicated teacher, an important scholar of theology, and an outspoken critic of dogmatic beliefs. The prejudices and enlightenment that motivated Abbott to write Flatland went beyond the mathematical and metaphysical. As one of his important biographers **Thomas F. Banchoff** (American Mathematician; 1938 -) tells us:

> Abbott was a social reformer who criticized a great many aspects of the limitations of Victorian society. He was a firm believer in equality of educational opportunity, across social classes and in particular for women.[3] All references below refer to this edition.

[1] The book Flatland is available online for free via Project Gutenberg at `http://www.gutenberg.org/ebooks/201` and at many other sites such as `http://www.ibiblio.org/eldritch/eaa/FL.HTM`.

[2] Mathematical fiction is not a small genre. See the Mathematical Fiction site at the URL `http://kasmana.people.cofc.edu/MATHFICT/` for information on almost 1,000 other works of mathematical fiction.

[3] From the "New Introduction" in the 1991 Princeton University Press edition of Flatland.

His ultimate goal? To liberate us from prejudice and have our minds "opened to higher views of things."[4]

1. **Classroom Discussion:** Read §1 - §3 of Flatland. Talk about what life must be like in Flatland.

2. In §2: Concerning the Inhabitants of Flatland, A Square describes in detail the different social class system in Flatland:

   > Our Soldiers and Lowest Class of Workmen are Triangles with two equal sides... Our Middle Class consists of Equilateral or Equal-Sided Triangles... Our Professional Men and Gentlemen are Squares (to which class I myself belong) and Five-Sided Figures or Pentagons... Next above these come the Nobility, of whom there are several degrees, beginning at Six-Sided Figures, or Hexagons, and from thence rising in the number of their sides till they receive the honorable title of Polygonal, or many-sided. Finally, when the number of the sides become so numerous, and the sides themselves so small, that the figure cannot be distinguished from a circle, he is include in the Circular or Priestly order; and this is the highest class of all.[5].

   Why would Abbott choose these geometric attributes as the basis of Flatland's class structure?

3. In §7: Concerning Irregular Figures, A Square talks about irregular citizens of Flatland. He calls the "monsters" and tells us:

   > The irregular... is either destroyed, if he is found to exceed the fixed margin of deviation, or else immured in a Government Office as a clerk of the seventh class; prevented from marriage; forced to drudge at an uninteresting occupation for a miserable stipend; obliged to live and board at the office...[6]

   Why would Abbott have his characters portray the irregulars so dramatically?

4. Women in Flatland are straight lines. What is the dimension of a line? What is the dimension of the polygons which comprise the men of Flatland? Why would Abbott choose to portray the different genders in this way?

5. A Square describes women as "the Frail Sex" and says that because they have no angle "they are consequently wholly devoid of brain-power, and have neither reflection, judgement nor forethough, an hardly any memory."[7] If Abbott was really a social reformer interested in equality for women, why would he create a world where women were so inequitably treated?

Flatland, as we shall see in Chapter 5, is also notable in its vision of the importance of the understanding of the higher dimensions. This book predated Einstein's Theory of Relativity, CAT scans, 3D computer graphics, and many other profound advances that rest on an understanding of the higher dimensions. You'll get to explore several of these areas where interplay between the dimensions is so important in the next chapter.

[4] Flatland, p. 4
[5] Flatland, pp. 8-9
[6] Flatland, p. 30
[7] §4.

Figure 2.1: Three-dimensional objects constructed from two-dimensional components: airplane wing, Carboard Safari trophies, and the first author demonstrating the strength of *torsion boxes* with a little help from some of his friends.

6. Abbott uses a penny to help illustrate the trouble of visualization in Flatland. In your own words, with your own example, describe the visual challenge of recognizing objects in Flatland.

7. Suppose you were a Flatlander. If object in front of you was A Square, what would he look like? Would it matter how the square was "standing"?

8. Can you think of any easy we to distinguish different men in Flatland - other than moving back into three-dimensional space so you can see them from above?

9. If the object in front of you was a woman, what would she look like? Would it depend on how she was "standing"?

10. Capable of "invisibly... inflicting instantaneous death... no female is suffered to stand in any public place without swaying her back from right to left... [This is] in every respectable female, a natural instict. The rhythimical and, if I may so say, well-modulated undulation of the back in our ladies of Circular rank is envied and imitated by the wife of a common Equilateral."[8] What are the social aspects of this passage? Describe the geometrical

[8] §4.

implications.

11. Watch the trailer of the Flatland movie at `http://www.flatlandthemovie.com/`. Describe a few aspects of this trailer that you found useful, surprising, comical, or otherwise noteworthy.

In §5: Of Our Methods of Recognizing One Another of Flatland Abbott tells us:

> YOU, WHO are blessed with shade as well as light, you, who are gifted with two eyes, endowed with a knowledge of perspective, and charmed with the enjoyment of various colours, you, who can actually see an angle, and contemplate the complete circumference of a Circle in the happy region of the Three Dimensions - how shall I make clear to you the extreme difficulty which we in Flatland experience in recognizing one another's configuration?
> Recall what I told you above. All beings in Flatland, animate or inanimate, no matter what their form, present to our view the same, or nearly the same, appearance, viz. that of a straight Line. How then can one be distinguished from another, where all appear the same?

Let us experiment a bit with the difficulties one faces in visualizing in a flat world like this.

Pictured in Figure 2.2 is a construction created using the six different blocks that make up **pattern blocks**, a set of manipulatives often used in elementary mathematics classrooms. The blocks are made out of thin pieces of wood, foam, or plastic. Notice that the edges are all the same length, except the one edge of the trapezoid which is twice the length of all other edges. You can discern certain characteristics of the pattern blocks from this figure.

For the following investigations it will be helpful to have a set of pattern blocks to use. If you do not have access, a template is included in the appendix which will help you make your own set out of cardboard, foamcore, or other material.

Let us suppose we live in a flat world where the inhabitants are shaped identically with the pattern blocks, but are, as in Flatland, perfectly two-dimensional and devoid of color.

12. Suppose you came upon citizen shaped like the tan rhombus. Describe your view of this citizen as you move around him in a circular path.

13. Suppose now you came upon two citizens, one shaped like the tan rhombus the other shaped like the blue rhombus. Could you distinguish which is which? Explain.

14. Could you distinguish a citizen shaped like a hexagon from a citizen shaped like a triangle? Explain.

15. Could you distinguish a citizen shaped like a triangle from a citizen shaped like a square?

16. Can you distinguish all citizens with different shapes (of these six) or are there some that cannot be distinguished. Explain in *rigorous* detail.

17. Create a new, unique shape that is indistinguishable from a hexagon no matter what (Flatland) angle you view it from. (Hint: There are infinitely many.)

18. If the shape you created in Investigation **17** was not *symmetric*, make one that is symmetric. If it was symmetric, make one that is not symmetric.

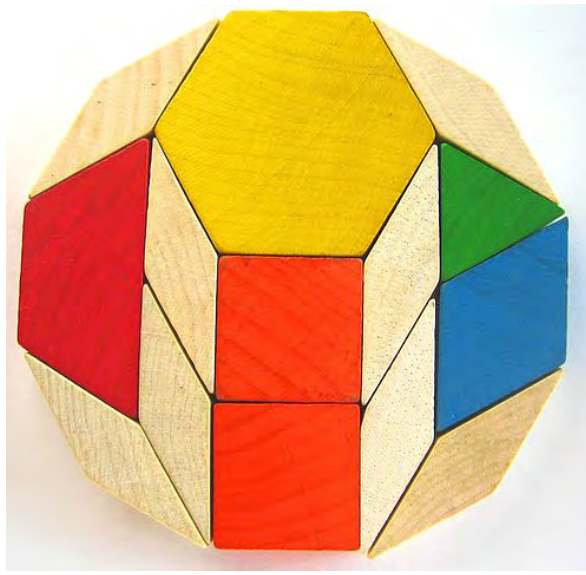


Figure 2.2: A ***regular dodecagon*** (12-sided polygon with equal sides and angles) made of wooden pattern blocks. Shown are the six different pattern blocks: squares, a regular hexagon, thin rhombi, a fat rhombus, an equilateral triangle and a trapezoid.

19. Repeat, if possible Investigation **17** and Investigation **18** for the square.

20. Could you repeat Investigation **17** and Investigation **18** for all of the pattern block shapes? Explain.

21. Choose a standard shape or create one of your own. Find a way to describe *all* shapes that would be indistinguishable from your shape no matter what angle you view it from.

22. Is there a way we can imagine what our three-dimensional world would be like if we lost our ability to see in perspective, see color, and have depth perception? Explain.

23. By naming the hero of Flatland A Square, Abbott has used a mathematical and *self-referential* pun. Explain.

24. What is the literary value of this pun and how does it fit with Abbott's theme?

2.2 Cross Sections

Pictured in Figure 2.3 and Figure 2.4 are several ***cross sections***.[9] The word cross section can have somewhat different meanings in different contexts. When mathematicians use the phrase

[9]A formal definition uses the notion of sets; a cross section of an object is the intersection of the object with a given plane.

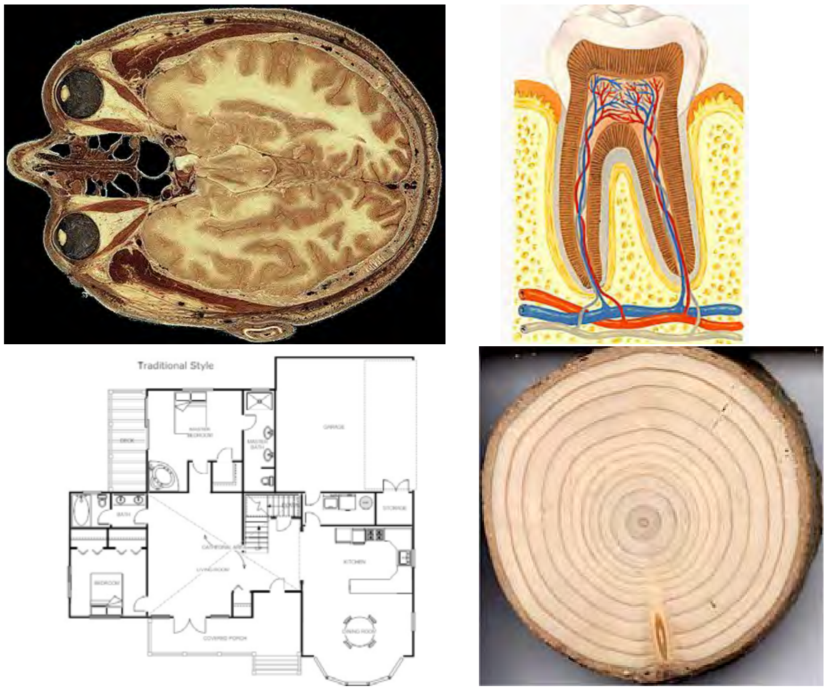


Figure 2.3: Cross sections of a human skull, a tooth, a house, and a tree trunk.

they generally use it to refer to the two-dimensional figures that are formed when a solid is sliced by a knife travelling along a fix plane. So when we say that Figure 2.5 shows cross sections of an apple, we are thinking of these slices as infinitely thin. Alternatively, the cross sections are what you would get if you dipped each slice in ink and stamped it to get a two-dimensional image.

Activity Preparation Collect together several dense, dry objects that can be easily cut by a knife or thin piece of wire. (E.g. apples, zuchinni, bagles, sponge cake, Play-Doh, cheese,...) Line a table with a large sheet of paper. Using a knife or piece of wire, you are going to slice your objects into thin, parallel, and equally spaced slices which mimic cross sections of these solids, as pictured in Figure 2.5.

25. Choose an object to slice. Choose a direction to slice this object - vertically, horizontally, or along some fixed angle. You will slice this object using equally spaced, parallel slices. Before you do any slicing, think about what the resulting cross sections of your solid will look like. In your notebook, draw a sketch of the different cross sections - lined up one after another - that you think will arise from the slicing.

26. Now find the actual cross sections by making a number of equally spaced, parallel slices through your object. Arrange the slices one after another and sketch them in your notebook.

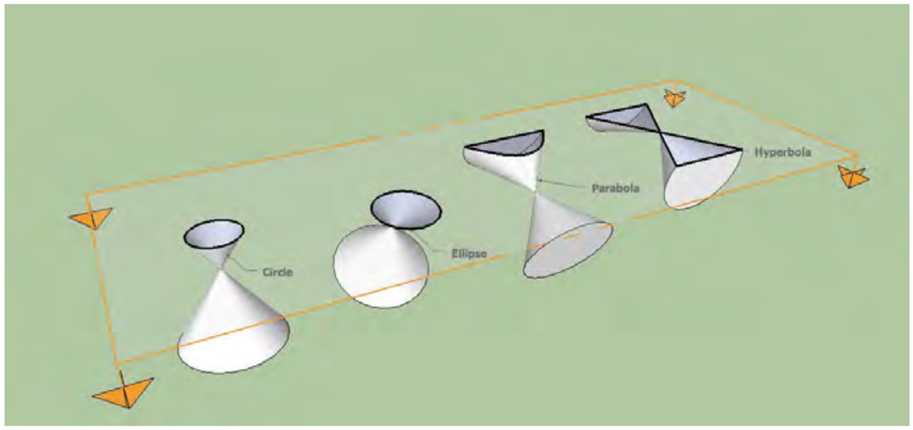


Figure 2.4: Cross sections of cones. These cross sections give rise to the important mathematical curves called, naturally, the **conic sections**: *circle*, *ellipse*, *parabola*, and *hyperbola*.

27. How do the actual cross sections compare to what you predicted? Is there anything that was surprising?

28. How do the cross sections change as you move through the object? Describe these changes in detail.

29. Take another copy of the same solid object and choose a different direction to slice in. Now repeat Investigation **25** - Investigation **28**, slicing in this new direction.

30. Choose a different, second object and repeat Investigation **25** - Investigation **29**.

31. Choose a different, third object and repeat Investigation **25** - Investigation **29**.

32. What are all of the different shapes cross sections of a cube can take?

33. What are all of the different shapes cross sections of a soda bottle can take?

34. Find, draw, or describe a solid object whose cross sections include a triangle, a circle and a square.

The climax of Flatland begins when A Square is visited by Sphere. Sphere's physical appearance is quite disturbing to A Square. Sphere tries to reason with A Square, explaining that there is a third dimension and A Square's world is simply a cross section of this much richer world. A Square proclaims Sphere a "Monster." He shrieks, "be though juggler, enchanter, dream or devil, no more will I endure thy mockeries. Either thou or I must perish." A Square collides with Sphere, trying to pierce and kill him. Instead, Sphere rises safely up out of harm's way. Exasperated he says to A Square, "Why will you refuse to listen to reason? I had hoped to find in you. . . a fit apostle for the Gospel of the Three Dimensions. . . Deeds, and not words, shall proclaim the truth."[10] With

[10] Flatland, p. 77.

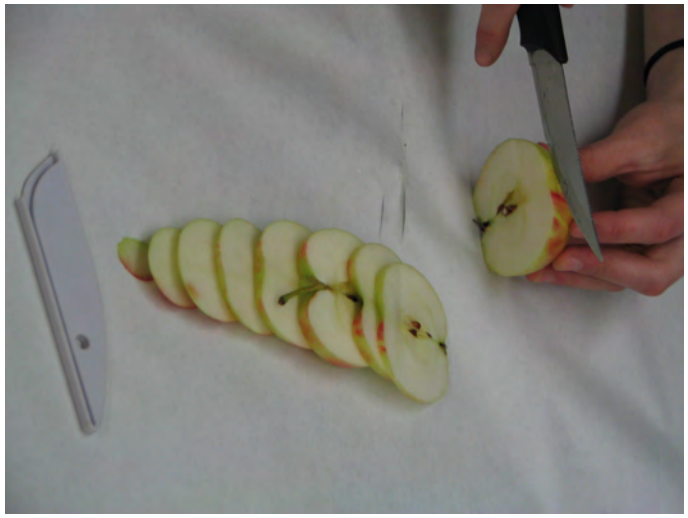


Figure 2.5: Parallel cross sections of an apple.

that Sphere latches onto A Square and pulls him up into third dimension so he sees the "plains of Flatland" below him.

35. Describe what A Square sees as the sphere moves up and down through Flatland.

36. Given A Square's view of Sphere, what kind of Flatlander would A Square think he is being visited by? Explain

37. Does it matter how Sphere was oriented as he passed through Flatland? Explain.

38. Are there any (or any other, as the case may be) objects who would appear identical to Flatlanders no matter what orientation they passed through Flatland?

39. Find several objects whose cross sections in one orientation are all identical. How are these objects similar?

40. Find several objects whose cross sections in one orientation are all identical to each other *and* are also identical to those in another orientation.

41. Can you find an object whose cross sections in one orientation are all identical to each other *and* the cross sections in a different orientation are all identical to each other but are *not* equal to the cross sections in the original orientation? Either describe these objects or explain why none can exist.

42. Is it possible for *all* of the cross sections of two objects to be the same even though the objects are different? Justify your answer.

2.3 The Flatland Game

Sphere's visit to A Square in Flatland began a journey of enlightenment. We would like to put ourself in A Square's position and see how we can be enlightened. We'll do so via the **Flatland game**.

Flatland Game Goal Determine the identity of a solid object from a series of parallel cross sections taken at regular intervals.

43. Can you guess what secret solid makes the cross sections shown in the first series of clues in Figure 2.6 as it passes through Flatland? If so, explain what the solid is and how you know its identity. If not, describe what you can ascertain about the solid from its cross sections.

44. Can you guess what secret solid makes the cross sections shown in the second series of clues in Figure 2.6 as it passes through Flatland? If so, explain what the solid is and how you know its identity. If not, describe what you can ascertain about the solid from its cross sections.

45. Can you guess what secret solid makes the cross sections shown in the third series of clues in Figure 2.6 as it passes through Flatland? If so, explain what the solid is and how you know its identity. If not, describe what you can ascertain about the solid from its cross sections.

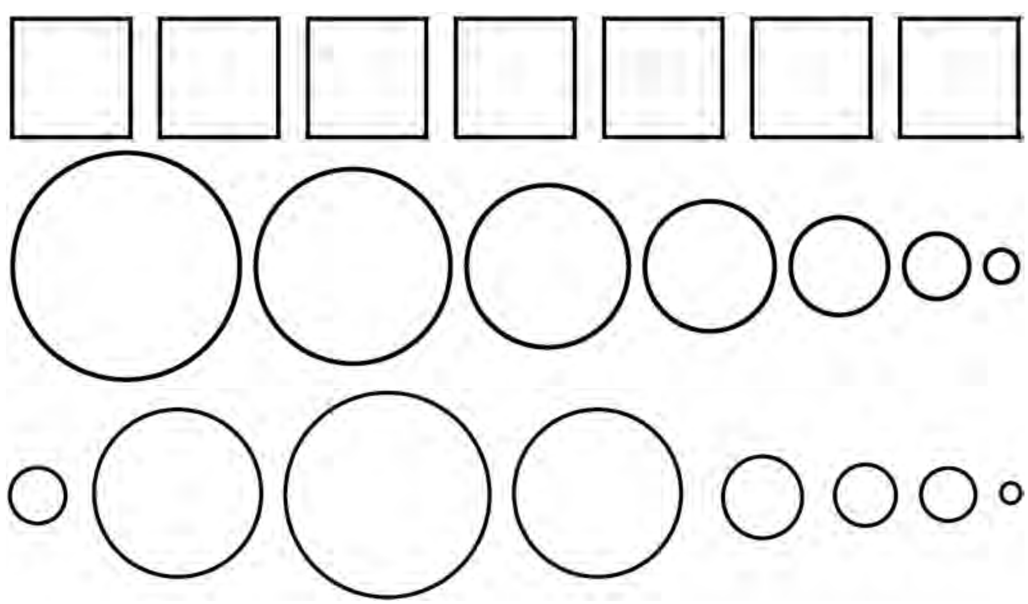


Figure 2.6: Three sets of clues for Flatland Games.

Determining what a solid is from a sequence of clues is the *inverse problem* of finding the cross sections of a solid. Both are important problems. The Flatland game will give you practice with both - allowing you to move from Spaceland to Flatland and back.

Rules and Roles for the Flatland Game

1. Choose a team of **radiographers**. This can be a single person or a small team where an illustrator has been elected.

2. Choose teams of **builders**. Teams can consist of a single person if necessary; it is best if a few small teams compete.

3. The game starts with the radiographers secretly determining a solid object from Spaceland whose identity will be the focus of the game.

4. The illustrator for the radiographers then begins play by drawing a single cross section, as viewed from above, of the secret solid.

5. The builders attempt to guess the identity of secret solid.

6. The illustrator for the radiographers then draws another cross section of the secret solid. This cross section must be parallel to earlier cross sections and the cross sections must be revealed consecutively as they would if the object actually passed through Flatland.

7. Steps 5 and 6 are repeated until:

- **a.** a team of builders correctly guesses the identity of the secret solid, in which case they are declared the winner, or,
- **b.** there are no more cross sections to draw and the radiographers are declared the winner.

46. Play the Flatland game. Draw the clues in your notebook. Once the identity of the mystery solid is discovered, record it and describe how the game went, noting any interesting geometrical issues that arose.

47. Play the Flatland game again with the same teams, recording the clues, the identity of the mystery solid and any interesting observations in your notebook.

48. Now switch roles. Let one of the teams of builders become the radiologists. Play two more games, recording the clues, the identity of the mystery solid and any observations in your notebook.

49. Play a few more Flatland games, letting each team have a chance to be the radiographers. Each time record the clues, the identity of the mystery solid and any observations you have in your notebook.

50. Compare your roles as radiographer and builder. Were there similar skills you needed? Were there similar challenges? In which did you learn the most about geometry?

51. Hopefully the radiographers weren't too tough. Name some objects whose identity would be really, really hard to determine in the Flatland game. Explain why they would be so hard.

2.4 Making Your Own Flatland Movie

As Sphere passed through Flatland A Square saw him growing and shrinking continuously, not just as a few discrete cross sections. You can mimic what it would look like as an object moves through Flatland by making a **Flatland flip book**.

52. INDEPENDENT INVESTIGATION: You will need approximately two dozen small pieces of rectangular cardstock. Anything between a business card and a 3" by 5" index card will work, but it is essential that they are all exactly the same size.

- Choose a solid object that you would like to see pass through Flatland as a movie.
- Draw successive parallel cross sections (like the clues in the Flatland game) of the solid, one on each piece of cardstock. Leave some blank space on the left (this is where you'll hold the flip book) and be sure to draw the images in consistent locations on each page.
- Assemble the pictures into a flip book, binding it with a binder clip or heavy duty staple.
- Holding it on the left, thumb through the individual pictures and you will have a movie of your object passing through Flatland!

2.5 Sliceforms

Box dividers are found in many packaging situations, protecting bottles, glasses, ornaments or other breakables, as shown in Figure 2.7. If you have not seen box dividers, see if you can locate one and figure out how it works. Mechanically they are very interesting. Simply by making regular slots in lengths of cardboard and then lacing them together to form a grid, one arrives at a very inexpensive and sturdy way to safely package a variety of objects.

In the 1870's **Olaus Henrici** (Danish Mathematician; 1840 - 1918) discovered these same mechanical principles could be used to make beautiful, dynamic mathematical models. We say dynamic because while box dividers fold nicely for storage, Olaus' *sliceform* models deform gracefully to show varied mathematical properties of the oject - in addition to the natural interest in having three-dimensional mathematical "solids" that can be folded flat to be carried in a pocket.

Sliceforms were resqued from relative obscurity[11] by **John Sharp** (English Teacher; -) who published a wonderful book of templates called Sliceforms: Mathematical Models from Paper Sections in 1995. Subsequently he published full-lenght book on the topic: Surfaces: Explorations with Sliceforms in 2004.

Subsequently, sliceforms have seen quite a resurgence as many of the figures below help illustrate.

Sliceforms are related in fundamental ways to the geometric ideas that you investigated via the Flatland game. In this section you will explore sliceforms. The ultimate goal is for you to design and build your own original sliceform.

[11] They do make a brief appearance on p. 19 of Geometry and the Imagination by **David Hilbert** (; -) and

Figure 2.7: Cardboard box dividers.

How do we go about designing and building a sliceform of a non-trivial three-dimensional object? Well, like most things you will need to explore, experiment, think, and plan. Your notebook is a perfect place to do this. The investigations below should provide some starting points.

53. Look at the sliceforms pictured in Figure 2.8, Figure 2.9, and Figure 2.11. Imagine viewing them from the top. How are the slices positioned? What shape is formed in the openings between any four slices that meet at intersecting pairs?

54. Sliceforms are constructed so they easily fold down to be flat. As they are folded, how will the shape you described in Investigation **53** change? Explain.

55. Is there any other layout of slices that can be used to make a sliceform where the slices are all "straight" and the resulting sliceform can fold flat? I.e. will any shape other than those considered in Investigation **53** - Investigation **54** make a working sliceform?

56. In the materials list graph paper is listed. Based on Investigation **53** - Investigation **55**, describe how this graph paper can be used to help build an original sliceform.

> **Sliceform Materials** Medium to heavy-weight cardstock. Sharp scissors. Ruler and possibly other measuring instruments such as a contour or profile gauge. Graph paper.

57. Cut out two small squares, say 3" on each side, from cardstock. In each cut a slit from the center of an edge directly to the center of the square. Insert one slit into the other. Is it easy to make the squares cross at right angles without holding them in place? Do the slits

Cohn-Vossen (; -).

Figure 2.8: Sliceforms as art; the sculpture of Richard Sweeney and the award-winning greeting card by Up With Paper.

act as a smooth hinge so the object can be flattened down, easily opened back up, and then flattened down the other way?

58. Cut out two more small squares. Instead of cutting slits from the edge to the center, cut ***slots*** where a small amount of cardstock is actually removed by making two parallel cuts very close to each other. Now assemble the two pieces by mating their slots. Is it easier to make the squares cross at right angles without holding them in place? Do the slots act as a better hinge?

In Figure 2.10 are templates for a stock sliceform. Larger versions of these templates are included in the appendix.

59. Can you guess what object this sliceform models when it is completed? Explain.

60. The individual pieces are labelled. What do you think the labels tell you? How were the names for these labels chosen?

61. Independent Investigation: Cut out the templates. Cut slots along each of the indicated lines. Then assemble the pieces into a completed sliceform. Describe the artistic, physical, and mechanical aspects of your sliceform.

62. Look at your sliceform and those pictured in the figures in this chapter. When vertical and horizontal slices instersect - and are in fact joined at - a slot, how are the heights of the two slices related?

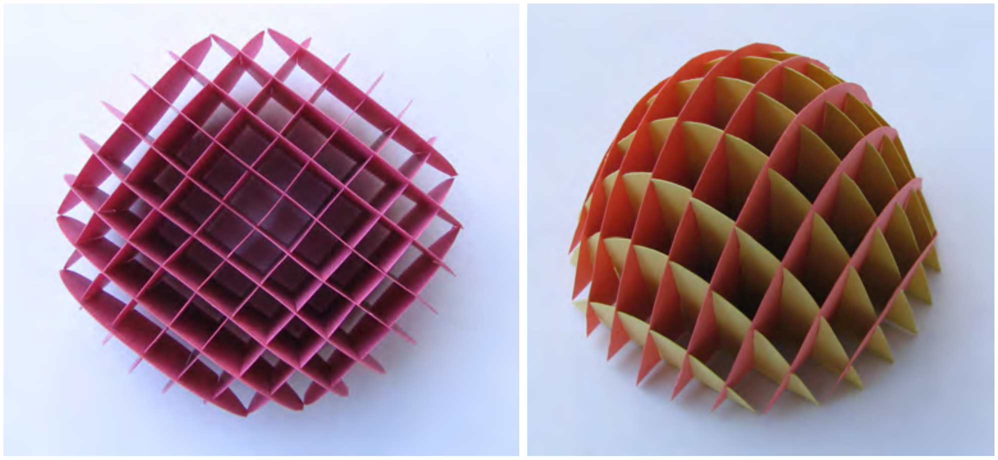


Figure 2.9: Three sets of clues for Flatland Games.

63. When you make your original sliceform you will have to figure out how deep to make each slot. Does Investigation **62** provide a good rule of thumb for doing this? Explain.

64. Suppose you loose a single slice of your model. How could you use your model to recreate this missing piece?

65. Does this give you an idea how you can create horizontal slices once you have all of the veritcal slices of your sliceform created appopriately? Explain.

66. INDEPENDENT INVESTIGATION: Design and build your own original sliceform.

This process may take you several hours over a few sittings. The design phase is quite important. As you design you should consider what you learned in the Flatland game and supplement it with tools like Play-Doh to make mock models, measuring tools like profile or contour gauges (see Figure **??**), profiles cast by shadows of light, actual cross sectioning of the object, or the use of Computer Aided Design software like the free Google SketchUp.

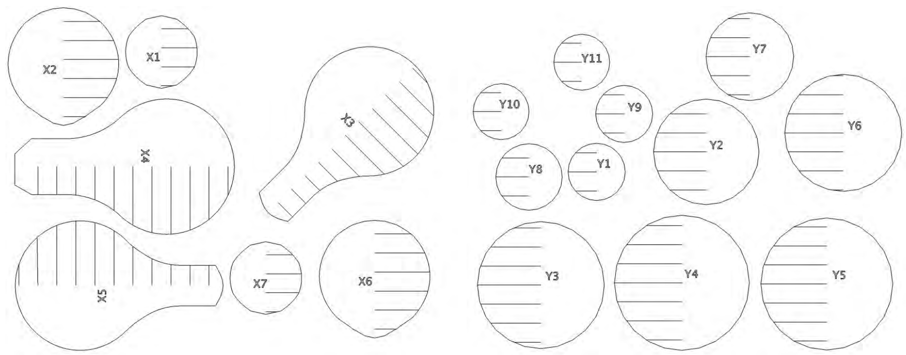


Figure 2.10: Templates for a sliceform model.

Figure 2.11: Original student sliceforms; “Guitar” by Katherine Cota, “Cactus” by Sharon Kubik-Boucher, and “The Face” the Lydia Lucia.

2.6 More General Cross Sections

End with Biesty books, general cross section, body worlds, etc. Now the possibilities are limitless. How do you want to explore our 3D world and 3D objects therein?

Body Worlds intro. There have been ethical issues related to this and similar exhibits. There are also religous and personal decisions that individuals must make. But as this exhibit travels the world thousands of people see the human body in entirely different ways - learning to see this miraculous machine in entirely new ways. Each time the authors have visited the exhibit the audience has included professionals - including doctors, nurses, athletic trainers, physical therapists - seeing new things. It is regular to hear things like “Wow, now I really see how the position of the —- impacts the motion of the —-.”

Figure 2.12: A profile gauge; used regularly by carpenters and machinists.

Figure 2.13: Images from the Body Worlds exhibit.

2.7 Alternate Approach: Projections

2.8 Connections

Needs to be something about Plato's "Cave" giving a context where we might think about a two-dimensional world. The context is quite different - but both are getting at our ability to think about the limits of our perception providing notable limits to knowledge.

Needs to have something about Google SketchUp and its ability to do cross sections. Mention Google 3D Warehouse and let them know that they can take cross sections of any of the models that are available in here. This might also be good for near the conclusion.

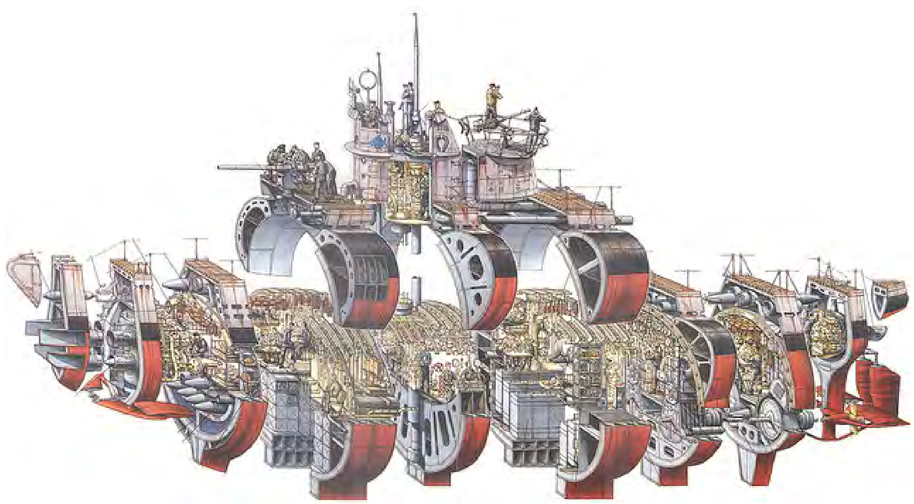
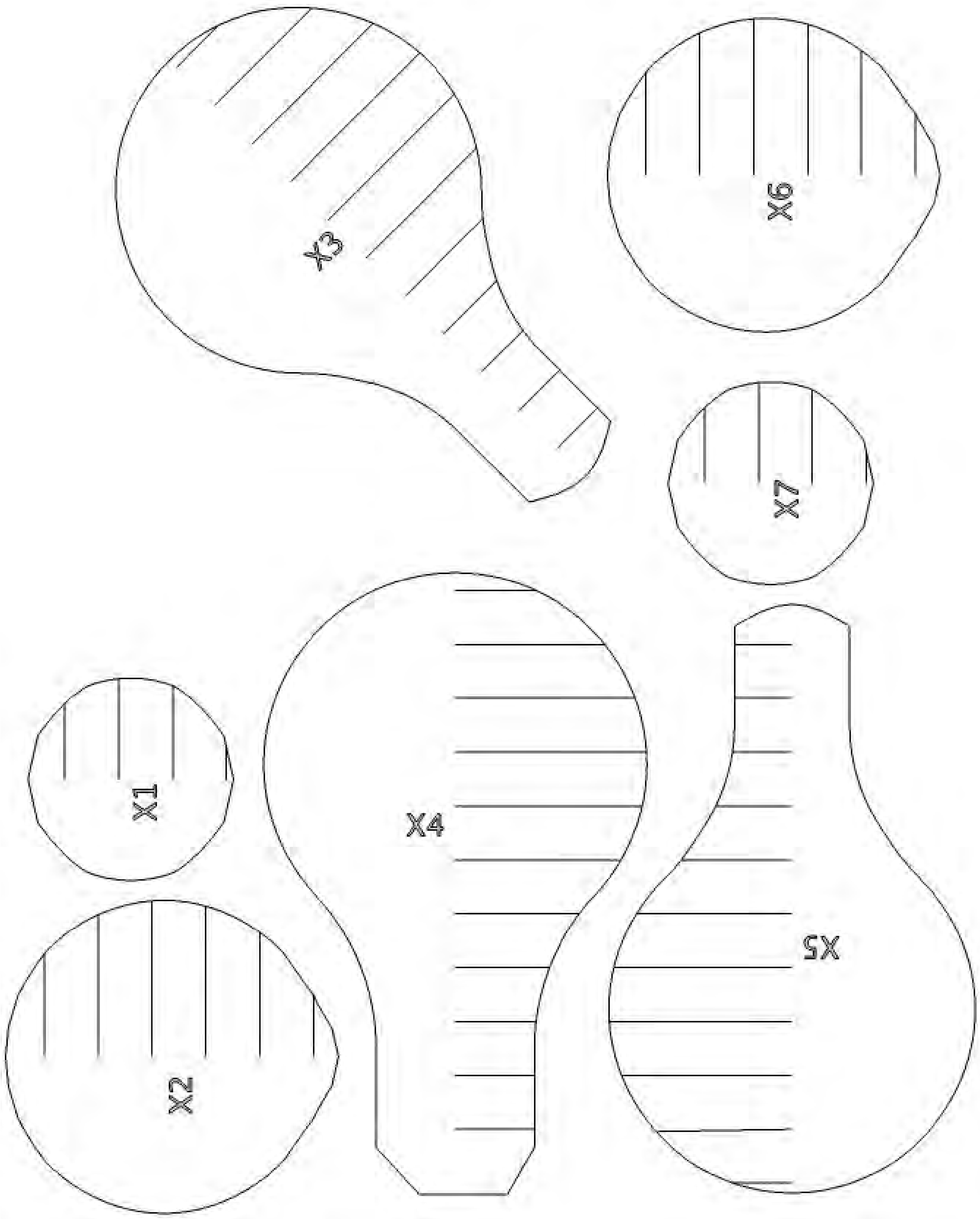
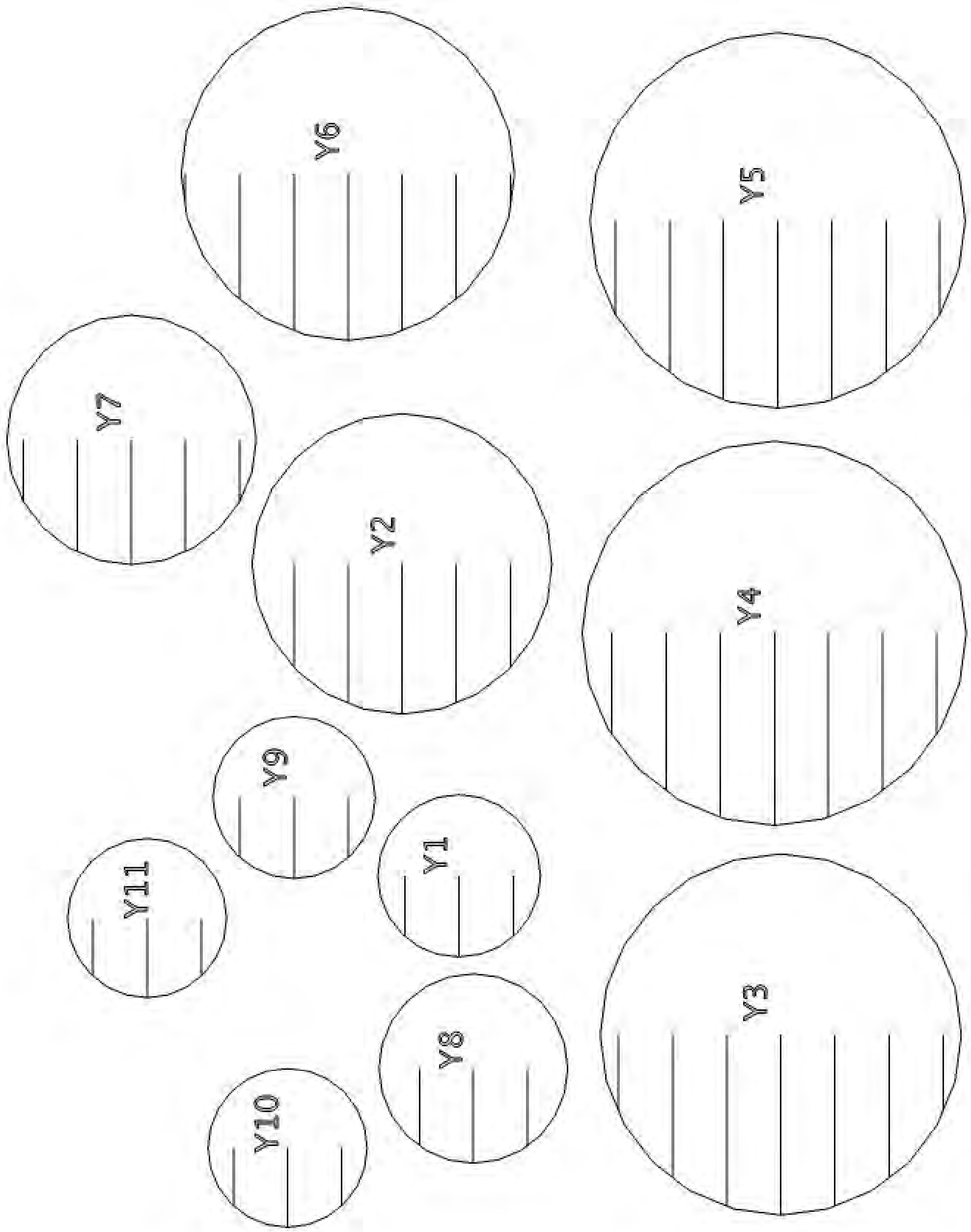


Figure 2.14: UBoat Cross Section by Stephen Biesty.


X3
X6
X7
X1
X4
X5
X2


Y6
Y5
Y7
Y2
Y4
Y9
Y1
Y11
Y3
Y8
Y10

Chapter 3

Dimensional Interplay in Other Fields

> In 1953 I realized that the straight line leads to the downfall of mankind. But the straight line has become an absolute tyranny. The straight line is something cowardly drawn with a rule, without thought or feeling; it is the line which does not exist in nature... Any design undertaken with the straight line will be stillborn. Today we are witnessing the triumph of rationalist knowhow and yet, at the same time, we find ourselves confronted with emptiness. An esthetic void, desert of uniformity, criminal sterility, loss of creative power. Even creativity is prefabricated. We have become impotent. We are no longer able to create. That is our real illiteracy.
>
> **Friedensreich Regentag Dunkelbunt Hundertwasser** (Austrian Artist and Architect; 1928 - 2000)

> The whole science of geometry may be said to owe its being to the exorbitant interest which the human mind takes in lines. We cut up space in every direction in order to manufacture them.
>
> **William James** (American Psychologist; 1842 - 1910)

3.1 Blueprints, building, and architecture

3.1.1 Computer Aided Design (CAD)

Architects, designers, and engineers routinely use Computer Aided Design (CAD) software to flexibly navigate the dimensional ladder. With this software customers can take a virtual walk through designs of their new house, the architect can explore how the lighting inside the building might change in the course of the day, and a builder can determine the total amount of materials needed.

Google SketchUp is a free CAD tool you can use to create, view, and modify three-dimensional designs yourself. Download the most recent version from `http://sketchup.google.com/` and explore some of its basic tools. This software was designed to allow users to provide a three-dimensional rendering of their house for inclusion in Google Earth. It is sophisticated enough to create realistic models of houses; see Figures 3.1-3.2.

Figure 3.1: Wire diagram of a house. Source: SketchUp Gallery at `http://picasaweb.google.com/gallery.sketchup`.

Basic SketchUp operations can lift us from two into three dimensions. After drawing a two-dimensional figure (such as a circle, a rectangle, or a polygon), we can pull the two-dimensional figure into the third dimension: starting with a circle we could create a cylinder, for example.

Similarly, we can also step *down* the dimension ladder: placing a "section plane" into our design will cut it open and reveal a particular cross-section; moving the plane will reveal a radiographers movie. We can produce x-ray views by making surfaces opaque; shadows provide realistic views. A particular fly-through can be captured as a slide show.

1. SketchUp Introduction: Use the short series of self-paced tutorials included in SketchUp to learn about basic tools (Intro tutorial). Follow the basic steps in creating a simple house (look for three short tutorials called "Start A Drawing," downloaded through the Help menu).

2. SketchUp Building: Open SketchUp and create a closed, planar shape using Rectangle, Circle, or Polygon. Select this shape and then activate the Push/Pull tool. You have created a right prism. Every horizontal cross section of this solid is congruent. In other words, if it passed vertically through Flatland, a Flatlander would not see the shape changing at

Figure 3.2: Town model created by Eighth Grader Andrew, using SketchUp. Source: SketchUp Gallery at `http://picasaweb.google.com/gallery.sketchup`.

all.

3. SketchUp Shaping: Create another right prism as in Investigation **1**. Choose the Rotate tool and rotate the top surface of the prism. Choose the Orbit tool to closely inspect the object you created. What would a Flatlander observe? Experiment with rotating by different angles.

4. SketchUp Radiography: Create a solid in SketchUp, as in Investigation **1** or using GetModel to get a more sophisticated model such as the Statue of Liberty from the Internet. Activate Section Plane and create a section plane. Select this plane, activate Move, and then move the section plane by clicking and dragging on the plane. As this plane moves you will various cross sections of your SketchUp solid.

3.2 CAT scans, MRI, PET scans, and other medical imaging.

In 1895, Wilhelm Conrad Röntgen created the first plates of X-ray photographs, taking shadowy images of his wifes hand; see Figure 3.3. This profound medical advance allowed us for the first

Figure 3.3: Photographs by Wilhelm Röntgen: Berta Röntgen's hand with wedding band. Courtesy General Electric Co.

time to see internal body structures in a non-invasive manner. Röntgen's discovery earned him the very first Nobel Prize (for Physics in 1901), but also contributed to his death in 1923 from Leukemia due to X-ray exposure. On November 8 2010, Google celebrated the 115-year anniversary of this discovery by showing the logo in Figure 3.4 on their web page.

Figure 3.4: Google celebrates the 115-year anniversary of Röntgen's discovery of X-rays.

Important as X-Rays are, they offer only a single planar image of even the most complex anatomical structures. Extending imaging techniques to capture 3-dimensional structures in their entirety via tools like CAT scans, MRI, and PET scans accomplished over the last 35 years or so has revolutionized modern medicine by allowing us to visualize every bodily organ with tremendous resolution as a real 3-dimensional object; see Figure 3.5.

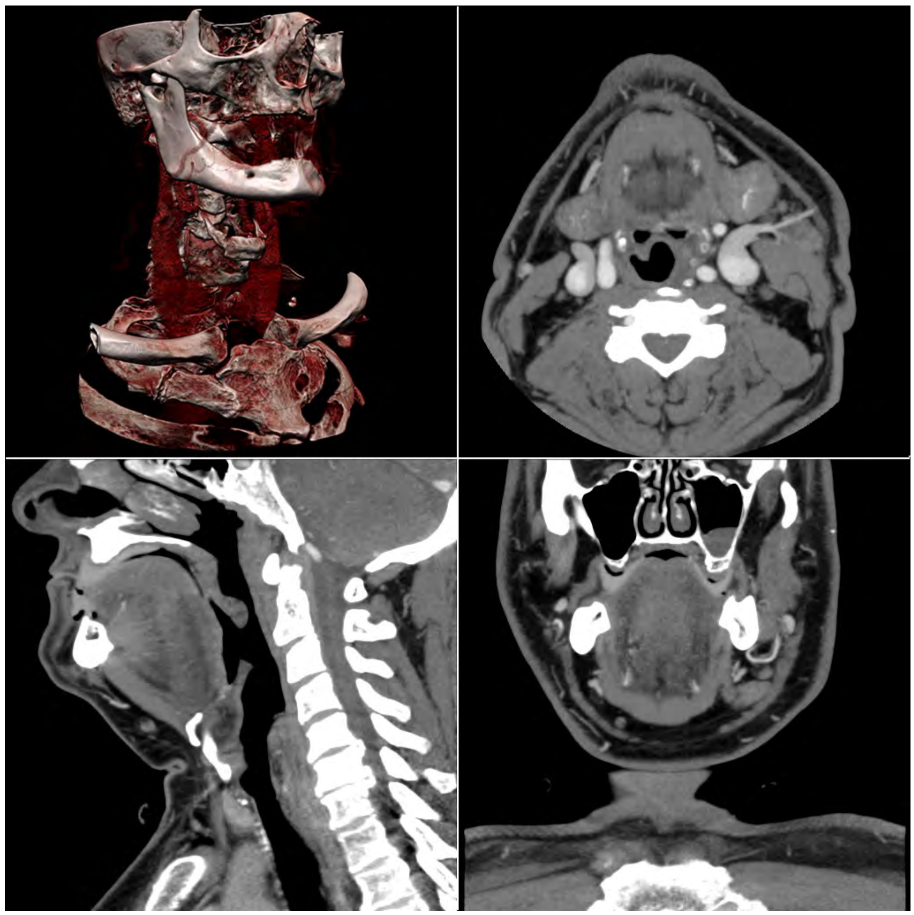


Figure 3.5: Typical screen layout of workstation software used for reviewing multi-detector CT studies. Clockwise from top-left: Volume rendering overview, axial slices, coronal slices, sagittal slices. Source: Wikimedia.

CAT scans–like the other imaging technology–work by little more than a combination of the slicing and projection techniques we have considered so far. Figure 3.6 shows a modern (2006) CT scanner with the cover removed, demonstrating the principle of operation. The X-ray tube (labeled “T”) and the detectors (labeled “D”) are mounted on a ring shaped gantry. The patient lies in the center of the gantry while the gantry rotates around him as a broad fan-shaped X-ray beam (labeled “X”) passes through the body.

Using this circular array of X-Rays and detectors, numerous images from different perspectives

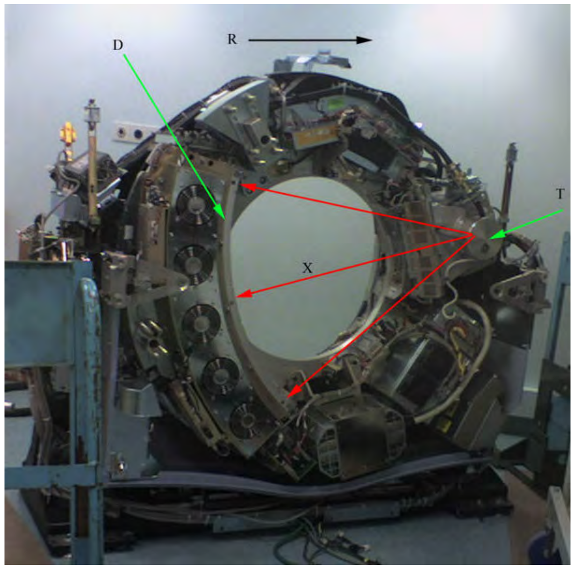


Figure 3.6: A modern (2006) CT scanner with the cover removed, demonstrating the principle of operation. Source: Wikimedia.

along an axial slice (the A in CAT) are relayed to a computer (the C in CAT). Using mathematical algorithms known as tomographic reconstruction (the T in CAT), the computer obtains a high-resolution image of an axial slice. This process is repeated to obtain hundreds of slices and the computer integrates these much like we did in constructing sliceforms to provide a true 3-dimensional map of the solid being analyzed. The solid can be seen using interactive computer graphics or even through stereolithographic models like those that are made for hip replacement surgery. Those who make this possible are Flatland game experts with real life rewards. For accessible further information, see Sochurek's book "Medicines New Vision" [**?**] or the Physics 2000 Internet site materials on X-rays and CAT scans [**?**].

5. Working in small groups, draw axial cross sections of different regions of your bodies. (This is the view one would see as if they were passing through Flatland standing up.) Check your accuracy using one of the many available Visible Human Program viewers available online (e.g. the NPAC/OLDA Visible Human Viewer at `http://www.dhpc.adelaide.edu.au/projects/vishuman2/VisibleHuman.html` or the viewer at the Center for Human Simulation at `http://www.uchsc.edu/sm/chs/browse/browse.htm`.)

6. Working in small groups, play the Flatland game using human body parts as the mystery objects.

7. Working in small groups, play the Flatland game using a variety of different animals.

3.3 Geography, maps, and topography

3.3.1 Topographic Maps:

8. Consider the five landscape photographs in Figure 3.7. For each of these, draw a series of Flatland Game clues.

9. What features of the landscape do you see reflected in your clues?

(a) Landscape 1

(b) Landscape 2

(c) Landscape 3

(d) Landscape 4

Figure 3.7: Investigation **8**: Draw a series of Flatland clues for these landscapes.

10. Consider the topographic map shown in Figure 3.8 and focus on the contour lines. What relationship do you notice between these contour lines and the clues you drew for the Flatland Game in Investigation **8**? Be specifc.

11. Which of the images in Figure 3.7 do you think corresponds to the landscape shown in the topographic map in Figure 3.8? Explain your reasoning.

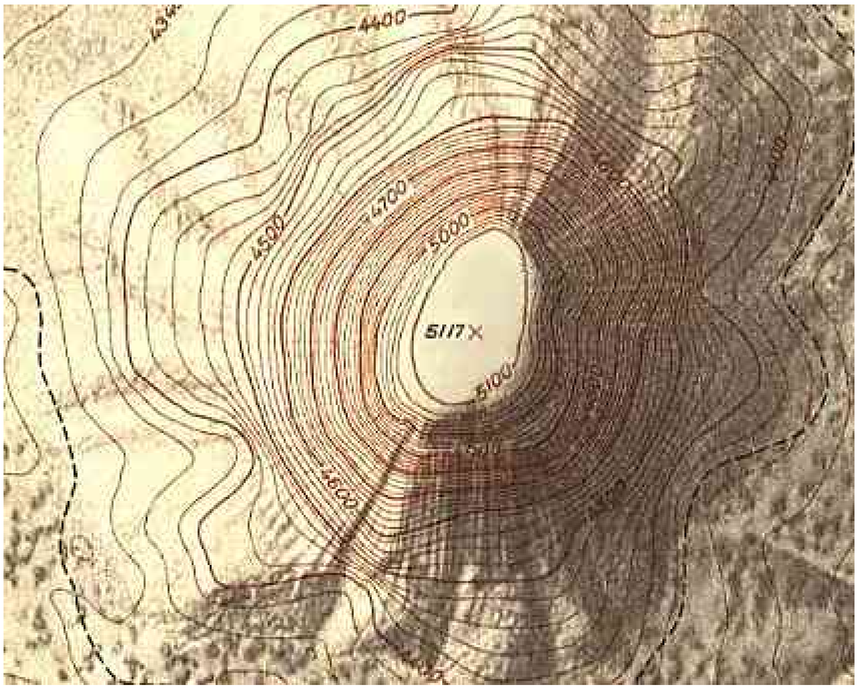


Figure 3.8: Source: Compass Dude.

12. Find a topographic map of a hilly hiking area, perhaps close to where you live. The US Geological Survey is a good source. Lower resolution versions of some of their maps are available for free on the web via services like `MyTopo.com` or `http://www.digital-topo-maps.com/`. Figure 3.9 was obtained in this manner; it shows the area around Mount Tom, the Whiting Street Reservoir, and the Connecticut River north of Holyoke, MA.

13. In what ways do the contour lines on the map provide information about landscape features? How can you tell where is it flat; where to find gullys or canyons; where to find steep cliffs, etc.?

14. Can you find a location on the map where different contour lines touch or cross? What would that mean in the landscape?

15. Using the map, design hiking trails for people with varying abilities: a mostly flat trail for wheelchair access, a gentle walk, a trail with strenuous climbs, or even a rock-climbing path?

16. Build a model of the area using Play-Doh.

3.3.2 Maps and Sliceforms:

Find a contour map of a mountain with longitude and latitude grid overlay. The map shown in Figure 3.9 shows an example of this.

17. Draw a straight line through the top of the mountain at whatever angle you like. Imagine you cut the mountain along that line. Create a drawing of what the surface along the cut would look. You can do this by laying a piece of paper along the line, marking its intersection with each of the contour lines, drawing perpendicular lines of appropriate height above these intersections, and then connecting the tops of the lines. Geographers call this a ***topographical profile***.

18. Make a sliceform model of the mountain whose contour map is give above. You will need to make topographical profiles along each longitude and latitude first. Then you cut slots at the intersection of the longitude and latitude lines, cutting the slots down on the topographical profiles in one direction and up in the other direction, splitting the difference each time. (For more details see Sharp [**?**], pp. 54 - 60.)

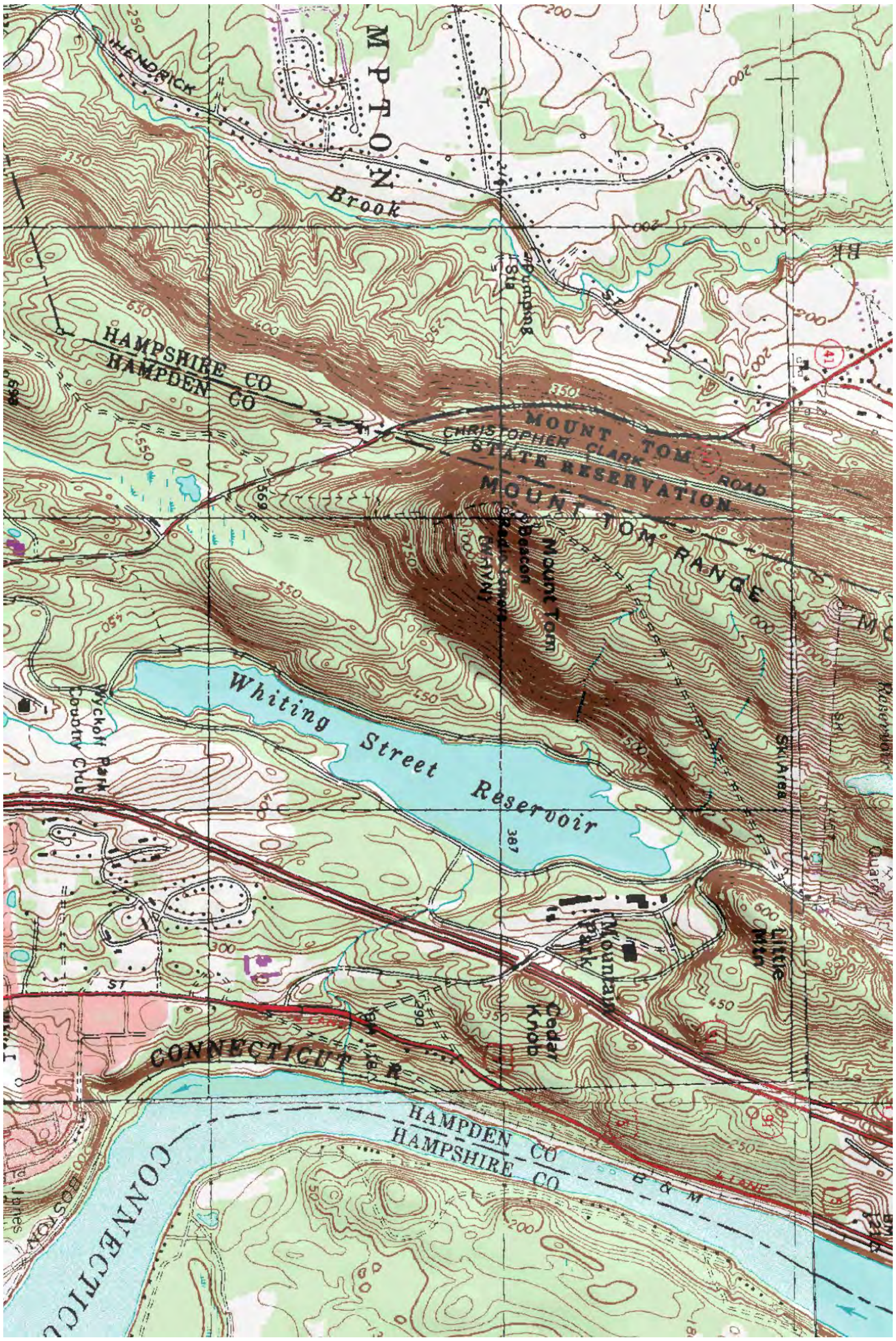


Figure 3.9: Mount Tom and Connecticut River in Western Massachusetts, north of Holyoke. Source: `MyTopo.com`.

3.4 3D Laser Scanning and 3D Printing

We have grown accustomed to scanning and printing photographs and images using paper documents. Imagine if we could actually "scan" entire three-dimensional objects, such as a sculpture, a piece of furniture, or a new hip joint? In fact, the technology already exists, and will likely find wider use as the price of the instruments will come down. As an example, consider the "Digital Michelangelo" project at Stanford University which had as its goal to create a 3D scan of the famous statue "David," created by the artist **Michelangelo** (Italian Renaissance painter, artist, sculptor; 6 March 1475 - 18 February 1564); see Figure 3.10.

3.4.1 Three-dimensional laser scanning and stereolithographic building

Researchers from Stanford University have used laser rangefinders to scan the surface of famous sculptures, such as Michelangelos David, and create high resolution digital models; see [**?**]. This is done using a carefully calibrated laser which takes detailed measurements of the location of points in one horizontal slice after another, capturing a radiographers outline at each stage; see Figure 3.10(a). A computer collects all the data to build a digital model of the statue; see Figure **??**. This digital model can be used, for example, to determine the volume and weight of the statue, or to determine the center of gravity for a structural analysis (As we might expect by now, this slicing process can be reversed. Using the digital models, builders can recreate actual three-dimensional replicas; see Figure 3.10(b). Important not only to artists and historians, this process is critical in medical, manufacturing, moviemaking, and video game industries. Many different kinds of precision machining tools can be used to make these replicas.

One example is called ***stereolithography***. This widely used ***rapid prototyping*** technology. Digital models created by CAD programs drive an ultraviolet laser which solidifies liquid plastic as it moves along a two dimensional cross section. Once a layer is complete, a platform holding the model drops a the specified layer thickness, covering the top layer again with liquid plastic for the next layer to be built. Fine layer by fine layer, the machine proceeds – just like our builders – until a complete 3D model is completed.

3.4.2 3D Laser Scanning:

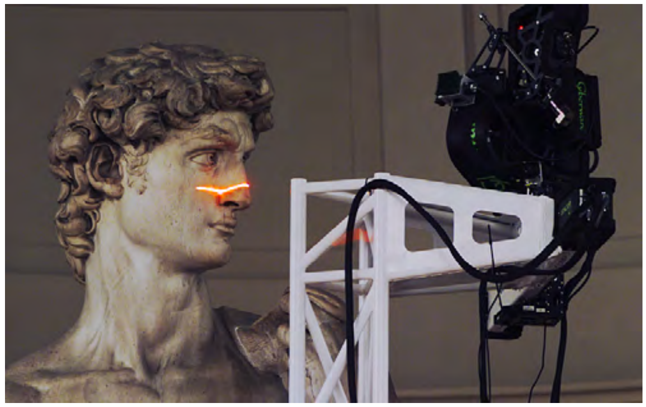
In small groups, you will use a laser level to collect scan data for a modestly complicated but reasonably sized solid object.

19. On a sheet of paper, draw a straight line whose length is the width of the solid and is divided into about a dozen equally spaced intervals.

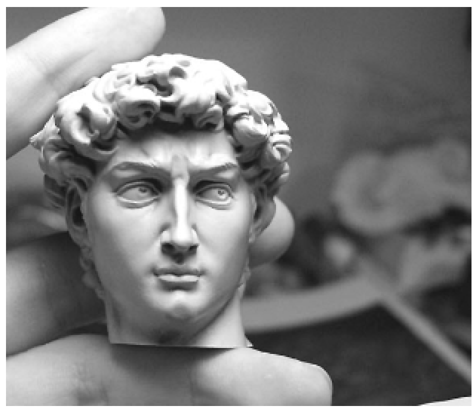
20. Set the object on the paper so it does not obscure the axis drawn in Investigation **19**.

21. Set a laser level to "horizontal" and position it in front of the solid. The laser now provides the boundary of a cross sectional slice. (If you do not have a laser level handy, you can also use a carpenter's gauge. See Investigation **24** and Figure 3.12(a).)

22. As shown in Figure 3.11 (STILL NEED THIS), have one group member align a ruler perpendicular to the axis with its tip touching the object where the laser meets it. Have another group member measure the distance from the axis to the solid.



(a) Laser scanning Michelangelo's sculpture of "David."



(b) 3D replica of "David" sculpture created from the scanned data.

Figure 3.10: © Stanford University's "Digital Michelangelo" Project. Source: `http://graphics.stanford.edu/projects/mich/`. Notice: The images of Michelangelo's statues that appear on this web page are the property of the Digital Michelangelo Project and the Soprintendenza ai beni artistici e storici per le province di Firenze, Pistoia, e Prato. They may not be copied, downloaded and stored, forwarded, or reproduced in any form, including electronic forms such as email or the web, by any persons, regardless of purpose, without express written permission from the project director Marc Levoy. Any commerical use also requires written permission from the Soprintendenza.

23. When all measurements are taken, make a cross section much as you did with the topographical profiles in Investigation **18**.

24. Alternatively, each group should use a carpenters profile gauge (Figure 3.12(a)) to quickly copy the boundary of different cross sections of the solid.

25. Taking horizontal and vertical cross sections at regular intervals, make precise cross sections

Need a picture here for doing a laser scan by hand:

with a horizontal laser range finder illuminating one slice,

one person holds a ruler to a certain line, while another person

measures the length/depth. See investigation \ref{inv:measure}
s

Figure 3.11: Investigation **22**: One group member aligns a ruler perpendicular to the axis with its tip touching the object where the laser meets it. Another group member measures the distance from the axis to the solid.

to build a sliceform such as the human face in Figure 3.12(b).

3.4.3 3D Laser Building:

While creating sliceforms gives us a rough skeleton of the three-dimensional object, say a mountain, the following investigations aim at creating a **solid** copy of this object, using layering techniques similar to those used in stereolithography.

26. Find an interesting contour map (available in Gazetteers in your library) that has a range of at least 20 contour lines separating the highest and lowest elevation.

27. Prominently mark two points on a copy of the map: the highest elevation and another point a significant distance away, choosing a point of highest local elevation if possible.

28. Make as many copies of the marked map as there are contour lines.

29. Each class member is responsible for a specific contour line. Cut along this contour to obtain a horizontal cross section of the regions topography.

30. Affix or trace the outline of each cross section to a piece of 1/4" foam-core or double thickness of corrugated cardboard. Then cut out each cross section, carefully labeling it, and poking two holes at the marked locations of the map.

31. Beginning with the lowest elevation cross section, successively stack up the cross sections, using pencils or short pieces of dowel to thread through the holes to keep the cross sections aligned.

Your completed object is a raised relief map of the region constructed in much the same way stereolithography makes prototypes.

(a) Carpenter's Gauge Tool

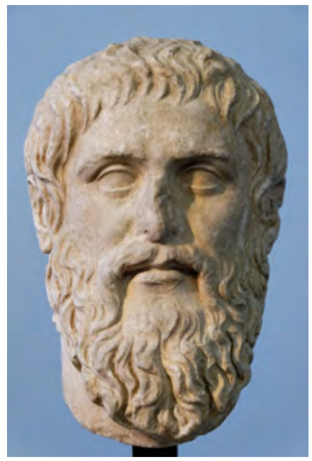
(b) Sliceform of a face, by Lydia Lucia, Westfield State University.

Figure 3.12: 3D Laser Scanning and Building.

3.5 CAVES and Virtual Reality Theaters

In his "Allegory of the Cave," **Plato** (Classical Greek Philosopher; 428/427 BC - 348/347 BC) (Figure 3.13(a)) explores the ideas of perception, reality, and illusion by using the analogy of a person with no three-dimensional experiences; their experiences are limited to two-dimensional shadows projected on the back of a cave.

Many of us have experienced the thrill of feeling completely immersed watching a movie in the cave of an IMAX theater. In similar ways, virtual reality games, computer-generated animation, and flight simulators draw us deep into their reality – with sophisticated use of geometry.



(a) Copy of a portrait bust showing Plato created by Silanion.

(b) CAVE Automatic Virtual Environment

Figure 3.13: Plato and a Modern CAVE.

Now, what would it be like to freely explore a three-dimensional world with a similarly rich sense of immersion? Imagine meeting Michelangelos David face-to-face, or peeking around in a CAT scan model of your own body? The CAVE, a virtual reality theater, is one step towards realizing this vision. "CAVE," the name selected for the virtual reality theater, is both a recursive acronym (CAVE Automatic Virtual Environment) and a reference to honor Platos “Allegory of the Cave.” It consists of a multi-person, room-sized, high-resolution, 3D video and audio environment. Computer-generated graphics are projected in stereo onto three walls and viewed with 3D stereo glasses. A viewer wearing a position sensor moves freely within its display boundaries. Instantly, the correct perspective and stereo projections of the environment are updated by a powerful computer: the images move with and surround the viewer. To the viewer with 3D stereo glasses the projection screens become seemingly transparent: three-dimensional objects such as tables and chairs appear to be present both inside and outside this projection-room. To a viewer these objects are really there until they try to touch them or walk beyond the boundaries of the projection-room. There are many rips and tears on projections screens where viewers have forgotten to be careful when walking within these invisible boundaries.

32. Search online for videos of people moving inside a CAVE.

3.5.1 CAVE Game Activity

This game is a variant of the Flatland Game, where the goal is now to recognize an object from the shape of one or more of its shadows. Hide an object behind a blind in such a way that an overhead projector casts its shadow onto a wall for all to see.

33. Break into groups: one group of radiographers and several groups of builders. The radiographers secretly select a solid mystery object.

34. The radiographers project an image of their object and teams of builders try to guess the object.

35. The radiographers then project a different image of their object and the builders guess again.

36. Step 3 continues until

- **a.** a team of builders correctly guesses the identity of the mystery object, in which case they are declared the winner, or,
- **b.** after a fixed number of projections has been surpassed the radiographers are declared the winners and the object is revealed.

37. The game can be made easier by showing the projections as the object is turned (using a lazy susan) or harder by simply showing the final projections.

3.6 Blueprints, building, and architecture

Like contour maps, traditional blue-prints, represent an amalgamation of information based on slices of the object along major directions. A builder synthesizes the information from such a collection of two-dimensional prints to create a fully-featured three-dimensional object, be it a porch, a roof, or an entire house. These are the “builders” of the Flatland game in real life.

Consider the set of blue-prints in Figure **??**.

38. What is this? What information can you read off the blueprint?

39. Build a model using manila envelopes and tape. It is easiest if your model is the same size as the blue-print.

Chapter 4

Visualizing Between Dimensions

The eyes are not responsible when the mind does the seeing.

Publilius Syrus (Syrian writer; 46 BC - 29 BC)

The real voyage of discovery consists not in seeking new landscapes but in having new eyes.

Marcel Proust (French author; 1871 - 1922)

Artists who work in two dimensions - those who paint on canvas, those who draw on paper, those who do graphic design on computers, photographers - are faced with the challenge of artistically representing the three-dimensional world within the confines of two dimensions.

4.1 Perspectives on Art

Every child is an artist. The problem is how to remain an artist after he(she) grows up.

Pablo Picasso (French Artist; -)

1. Think back to your childhood years prior to middle school. What types of art did you create? Please think broadly - a Lego building is a sculpture as much as Micheangelo's David is.

2. The art that you created - was it mostly two-dimensional or mostly three-dimensional? What are some possible reasons why the balance of your art was this way.

3. Find - in your memory, with the help of family, or by finding the physical objects - actual examples of your early artwork. Are there ways - as a child - that you tried to represent the three-dimensional world in two-dimensions? Explain how.

4. At some point - generally in the middle grades - most of us are taught how to draw a cube. Carefully draw a cube. Explain how you have drawn it. What important geometric features does your representation of a cube have? Be as specific as possible. Which features are comparable to those in an actual cube? What features are different than those in a cube?

5. How has your involvement with art - again, broadly defined - changed since you started middle school? Do you do more or less art? Is the art of different types than previously? What has happened to the balance between two-dimensional and three-dimensional art that you use? Describe possible reasons for this change and whether Picasso's quote has any meaning for you.

4.2 Perspective Drawing

As early as 25 B.C. **Marcus Vitruvius** (Roman author, architect and civil engineer; circa 75 BC - circa 15 BC) wrote "perspective is the method of sketching a front with sides withdrawing into the background."[1] Evidence of perspective drawings have been found in murals and frescoes in the Roman city of Pompei - preserved only because the entire city was buried by the volcanic eruption of Mount Vesuvius in 79 A.D. There is also evidence of perspective drawing in ninth century Chinese art.[2]

After the long darkness of the Middle Ages, the more modern and permanent (re)discovery of perspective drawing is widely attributed to **Fillippo Brunelleschi** (Italian architect and artist; 1377 - 1446) at the height of the Rennaissance; his first drawing said to be of the cathedral in Piazza del Duomo in Florence, Italy. In a time when most of us have high resolution, digital cameras built into our cell phones, creating a life-like image may not seem to be important. But the development of perspective drawing fundamentally changed the world of art and, as a result, the way we view the world.[3]

4.2.1 One-Point Perspective

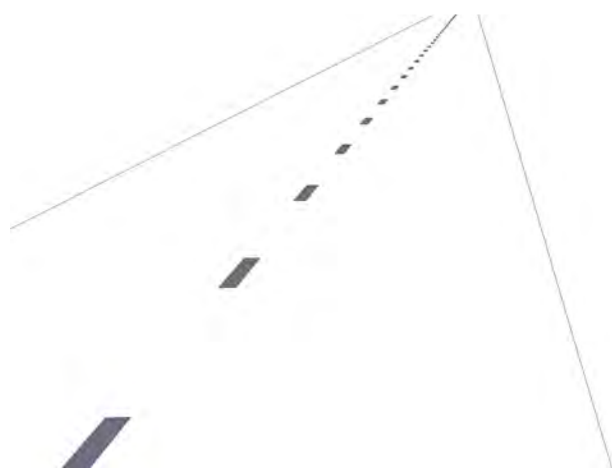


Figure 4.1: A portion of a one point perspective drawing of a road.

[1]Quoted in Squaring the Circle: Geometry in Art and Architecture by Paul A. Calter, p. 373.

[2]See e.g. p. 3 of Perspective Drawing: Freehand and Mechanical by Joseph William Hull for more on these early cultures' use of perspective.

[3]For much more on this fascinating topic, at a level that is similar to the book you are currently working through, see the wonderful book Mathematical Perspective and Fractal Drawing by **Annalisa Crannell** (American Mathematician; -) and **Marc Franz** (American Mathematician; -).

6. Figure 4.1 shows a portion of a *one-point perspective* drawing of a rudimentary road. Find the exact location of the *vanishing point*, also known as the *point at infinity*, used to create this drawing. Precisely locate the *horizon line* for this drawing.

7. Draw your own one-point perspective drawing of railroad tracks vanishing into the horizon. Line the railroad tracks with a row of telephone polls on one side and a path/road on the other. Briefly explain how you created your drawing and any difficulties you faced.

8. Explain - geometrically - how you knew how to orient the railroad ties, telephone polls, and other important features in your drawing.

9. On the upper-left of Figure 4.2 is a square and a point. Using the larger version of Figure 4.2 in the Appendix, draw lines from each of the vertices of the square to the point.

10. Repeat Investigation **9** for the figures in the upper-right, lower-left, and lower-right of Figure 4.2.

11. Can these images that you have drawn represent finite geometric solids that you can name or describe? Either do so or explain why not.

12. Now consider each of your images as perspective drawings, with the point through which your lines run the point at infinity. As perspective drawings, the geometric objects represented here are infinite geometric objects. What objects are they?

13. As your images can represent both finite and infinite geometric objects depending on their interpretation, they are ambiguous. In your drawings in Investigation **7** there are many visual clues to the reader that the image is a perspective drawing - trees, light poles, sidewalk sections, etc. Find a way to decorate the "floors", "walls", and "ceilings" of your one-point perspective drawings in Investigation **12** using appropriate motifs so it is visually clear that your drawing is a perspective drawing. Use the same motif for each of the drawings.

We would now like to draw cubes in one-point perspective.

14. Using the larger version of Figure 4.2 in the Appendix, draw the one-point perspective drawing of a cube whose front face and point at infinity are the square and the point in the figure in the upper-left. Do you know exactly how long to make the side edges? Explain.

15. Repeat Investigation **14** with the square and point at infinity for the figure in the upper-right of the larger version of Figure 4.2.

16. Repeat Investigation **14** with the square and point at infinity for the figure in the lower-left of the larger version of Figure 4.2.

17. Repeat Investigation **14** with the square and point at infinity for the figure in the lower-right of the larger version of Figure 4.2. The figure you have drawn is called the **foreshortened cube**.

18. Can you find a way to hold a cube so that your view of the cube is similar to that shown in each of your one-point perspective drawings? Explain.

19. Carefully compare all of these perspective drawings to the cube you drew in Investigation **4**. Are any of these drawings similar to the cube you drew? If not, how do they differ? Can you move the point at infinity to a location so the one-point perspective drawing will exactly correspond to your cube? Either do so or *prove* why it cannot be done.

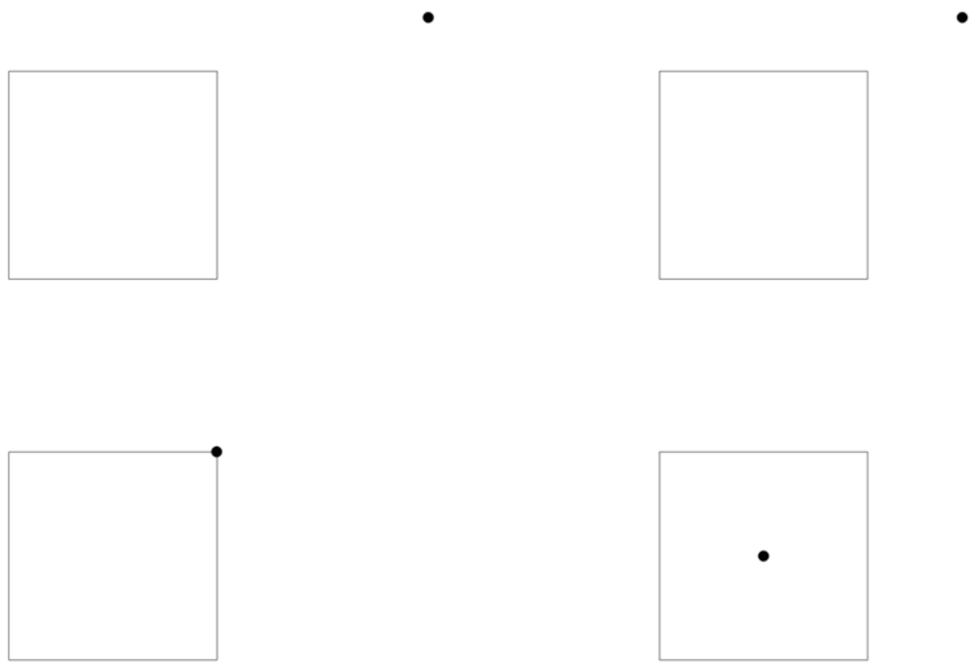


Figure 4.2: Set-ups for creating different one point perspective drawings with square front faces.

Figure 4.3 and Figure 4.4 show images by **Albrecht Dürer** (German artist; 1471 - 1528). An important Renaissance artist, Dürer was the first to write systematically about the use of perspective in art - his instructions first appearing in The Painter's Manual.

20. The images in Figure 4.3 and Figure 4.4 show *perspective machines* that could be used to make realistic perspective drawings. Study these drawings and describe how these machines seem to work.

21. In both of these images by Dürer, how is the *picture plane* or *viewing plane* oriented in relation to the table on which the subject being rendered is located? Artistically, why do you think Dürer chose such an orientation? How would the image differ if this was not the orientation?

22. If you were one of the artists in these images, how could you find a vanishing point for these drawings?

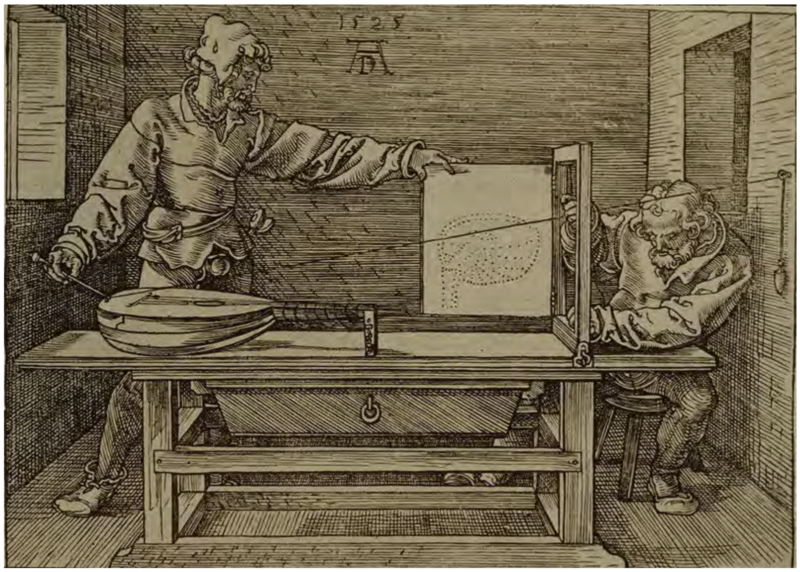


Figure 4.3: Dürer's "Man Drawing a Lute" from 1525.

4.2.2 Two-Point Perspective

Figure 4.5 shows students using tape and window panes to create perspective drawings, like that in Figure 4.6, of nearby buildings. The student on the far left and far right are sighting the image, being careful not to move and looking through one eye, and instructing the "drawers" where to locate the tape. (See the wonderful book Viewpoints: Mathematical Perspective and Fractal Geometry in Art[4] for more on this activity and many other related to the topics in this chapter.)

23. There is a vanishing point somewhere to the left of the "drawing" in Figure 4.6. Explain precisely how this vanishing point can be located.

24. None of the faces of Courtney Hall, the building pictured, are parallel to the plane of the window pane. Explain why this means there must be a vanishing point to the right as well. How hard will it be to accurately locate this vanishing point? Explain.

The Flatiron Building in New York City, pictured in Figure 4.7, is remarkable building. As it is pictured here you can clearly see the two streets that frame it, Fifth Avenue and Broadway, receding away from it.

25. Explain why a perspective drawing of the Flatiron building would naturally have two vanishing points. How could you determine the location of these two vanishing points?

26. A building whose shape is a rectangular prism is located at an intersection to two perpendicular streets. Draw a ***two-point perspective*** drawing of such a building as you would see

[4] By Marc Frantz and Annalisa Crannell, Princeton University Press, 2011.

Figure 4.4: Dürer's "Draughtsman Making a Perspective of a Woman" from 1525.

it if you were located on the sidewalk diagonally across from the building. Carefully explain how you have located the two vanishing points and used them to draw key elements like curbs, windows, doors, and rooflines.

27. Use the set-up in Figure 4.8 to draw a cube in two-point perspective. (A larger version of this image is included in the Appendix.) Do you know how long to make the edges? Do you know where the rear, vertical edge should (approximately) be? If you need to, redraw your drawing until it really looks like a cube.

28. Now compare your cube drawn in two-point perspective to those that you drew in one-point perspective and the one that you drew in Investigation **4**. Are any of these identical, up to their physical scale? Explain in detail how you know certain drawings are identical or what exactly it is that distinguishes them.

29. Consider the two-point perspective drawing of a building that you made in Figure 26. If the building in front of you was *very* tall, as you looked up at it, what would you notice about the sides of the building?

30. Explain how you can now locate a third vanishing point.

31. For the image of the Flatiron building in Figure 4.7, the taped images of Courtney Hall in Figure 4.5 and Figure 4.6, and other images of buildings shaped like rectangular prisms which are viewed from vantage points where the picture plane is not parallel to one of the building's walls, how many vanishing points are there in the image? Why are they not always all obvious?

32. Use the set-up in Figure 4.9 to draw a cube in three-point perspective.

Figure 4.5: Creating a two-point perspective model.

4.3 The Necker Cube

As you have seen, there are many different ways to draw a cube. We'll see even more below.

One of the most common is created by drawing a square, drawing a congruent square partially superimposed over the first, and then connecting corresponding corners, as shown in Figure 4.10. Perhaps this is the type of cube you drew in Investigation **4**. The object that many recognize as a drawing of a cube in Figure 4.10 is called a **Necker cube** after **Louis Albert Necker** (Swiss Geologist; 1786 - 1861). Necker mentions this object explicitly in a 1832 letter to **David Brewster** (Scottish Physicist, Writer, and Inventor; 1781 - 1868) - inventor of the kaleidoscope.

33. What important geometric features does the Necker cube have? Be as specific as possible. Which features are comparable to those in an actual cube? What features are different than those in a cube? (You may want to refer back to 4 if you drew a Necker-like cube there.)

34. Is the Necker cube a cube rendered in one-point perspective? Either create the one-point perspective drawing or *prove* that it cannot be done.

35. Is the Necker cube a cube rendered in two-point perspective? Either create the two-point perspective drawing or *prove* that it cannot be done.

36. Is the Necker cube a cube rendered in three-point perspective? Either create the three-point perspective drawing or *prove* that it cannot be done.

See the Connections Section for connections of the Necker cube to perception and psychology as well as its key role in the novel Factoring Humanity.

Figure 4.6: A finished two-point perspective taping.

4.4 Projection

> Since I found that one could make a case shadow from a three-dimensional thing, any object whatsoever – just as the projecting of the sun on the earth makes two dimensions – I thought that by simple intellectual analogy, the fourth dimension could project an object of three dimensions, or, to put it another way, any three-dimensional object, which we see dispassionately, is a projection of something four-dimensional, something we are not familiar with.[5]
>
> **Marcel Duchamp** (French Artist; 1887 - 1968)

Below we will consider Duchamp's challenge more directly - the fourth dimension. But first, let us think about shadows. In the previous chapter you considered the difference between slicing and projection; CAT scans and XRays; life in Flatland versus life in Plato's cave. (JF Notes - Did Volker do this? If not, do a bit of it here. Make the distinction.)

37. Build two ***wire frame cubes***, cubes with solid vertices and edges but empty faces. The edges of one should be 8 - 16 inches. The edges for the other should be less than three inches. Suggestions for materials include: Zome; stiff wire; marshmallows/gumdrops/Play Doh and skewers/toothpicks; straws with paperclips to join them at vertices.

We would like to consider the different types of shadows that are cast by this cube.

4.4.1 Parallel Projections

The first shadows you should make are those called ***parallel projections***. These are made by a light source that is so far away that the rays are essentially parallel. The sun works very nicely

[5]Quoted in The Fourth Dimension and non-Euclidean Geometry in Modern Art by Linda Dalrymple Henderson.

Figure 4.7: The Flatiron Building.

for this. If you need to use artificial light, just about any type of light will do. A flashlight or bare lightbuld works fine as long as the distance between the light source and the cube is quite significantly greater than the distance from the cube to the surface it is projected on.

The relative location of the light source and cube is also important. To begin, we would like to insure that the line from the light source to the cube is perpendicular to the surface the shadow is projected onto. If you are using a lightbulb or flashlight as a source, this is easy to arrange. As the sun is generally not directly overhead, if you use the sun you will have to bring a large piece of cardboard with you so that it can be tilted to be perpendicular to the rays of the sun.

38. Do you think parallel projections of the cube will differ from the cube drawing and perspective drawings that you created previously? If so, how. If not, why not?

39. Now create a parallel projection of the cube. You should be as exact as possible in duplicating the projection. If possible, trace over the actual shadow as it is projected onto a large sheet of

Figure 4.8: Set-up for creating a cube in two-point perspective.

paper, cardboard, chalkboard, or whiteboard. Then copy a scale version into your notebook.

40. Now rotate the cube to see different parallel projections. When you find a projection you particularly like, stop rotating and carefully draw it. Do this five times so you have completed Investigation **48** a total of six times.

41. Carefully compare all of these parallel projections to the cube you drew in Investigation **4**. Are any of these drawings the same as your cube? If not, how do they differ? Can you rotate the cube so the parallel projection will correspond to your cube? Either do so or explain why it cannot be done.

The shadows created will be quite different if the line from the light source through the cube is *not* perpendicular to the surface the shadow is projected onto. Projections of this type are called ***oblique parallel projections***.

42. Do you think oblique parallel projections of the cube will differ from the cube drawing, perspective drawings, and parallel projections that you have created previously? If so, how. If not, why not?

43. Now create an oblique parallel projection of the cube. You should be as exact as possible in duplicating the projection. If possible, trace over the actual shadow as it is projected onto a large sheet of paper, cardboard, chalkboard, or whiteboard. Then copy a scale version into your notebook.

44. As above, rotate the cube to see different oblique parallel projections. And perhaps change the angle at which the light source hits the surface of projection. When you find a projection you particularly like, stop rotating and carefully draw it. Do this five times so you have completed Investigation **48** a total of six times.

45. Carefully compare all of these oblique parallel projections to the cube you drew in Investigation **4**. Are any of these drawings the same as your cube? If not, how do they differ? Can you rotate the cube so the parallel projection will correspond to your cube? Either do so or explain why it cannot be done.

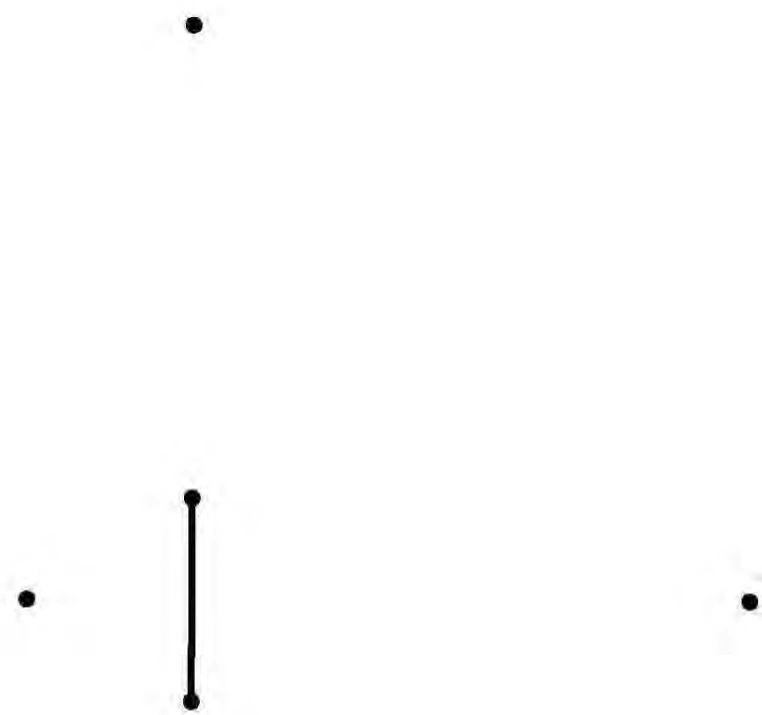


Figure 4.9: Set-up for creating a cube in three-point perspective.

4.4.2 Point Projections

Now we are going to change the type of light source. Now you need a ***point light source*** where light is coming from (essentially) a single point in space with no focussing, dispersion, etc. Regular lightbulbs are designed to cast lots of ambient light; they do not work here. Most flashlights are designed with reflective surfaces that help direct a beam that is wide enough to be useful without losing its power to illuminate. These don't work here. What is best is a small, incandescent flashlight bulbs. Sometimes the lens and reflective surface of a flashlight can be removed to leave the working bare bulb. If not, such bulbs can often be found in school Science Departments or are inexpensive to purchase from Radio Shack.

Point projections are created using point light sources which are quite significantly closer to the object being projected than the object is from the surface it is being projected onto. As above, it is generally assumed that the line from the source to the object is perpendicular to the surface. If this is *not* the case then the projection is called an ***oblique point projection***.

46. Explain why the projections of your cube will change as you rotate the cube.

47. Carefully draw what you think of few of the projections will look like.

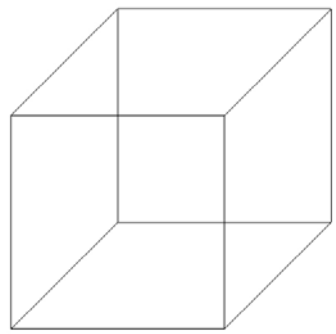


Figure 4.10: The Necker cube.

48. Now actually create a number of different projections of the cube as you rotate. When you find a projection you particularly like, carefully draw it - perhaps tracing over the actual shadow on your piece of paper.

49. Repeat Investigation **48** five more times to find five more interesting projections of the cube.

50. Carefully compare all of these projections to the cube you drew in Investigation **4**. Are any of these drawings the same as your cube? If not, how do they differ? Can you rotate the cube so the projection will correspond to your cube? Either do so or explain why it cannot be done.

51. Create three different parallel projections where the cube is oriented in the same direction relative to the sun as it was to the point-source of line in three of your projections above. How do the parallel projections differ from the projections?

4.5 3D Drawing

Above you considered a wealth of different ways to draw cubes. Each had its own benefits and drawbacks - as well as its many applications. We consider one last one here that is widely used in technical drawings, in engineering drawings, in computer graphics, and to help children as a first experience in drawing three-dimensional representations. It is called ***isometric drawing*** and is a special type of parallel projection. The main tools in creating isometric drawings are ***isometric dot paper***, shown in Figure **??**, and shading.

The word isometric is from the Greek roots iso, for same, and metron, for measure. In isometric drawings all lengths along the three coordinate axes are preserved. For a cube this means that the edge lengths are all preserved.

52. Draw a cube - whose edge lengths are equal to the length between adjacent dots - on isometric dot paper. Compare your cube to those of others. (Extra copies are included in the appendix.)

53. Experiment with shading some of the faces of your cube to see if you can make it appear more "cubish."

54. Figure 4.12 shows an absurd attempt to draw a cube using isometric dot paper. What is with using isometric dot paper to make this representation of a cube?[6]

55. Draw another cube on your dot paper. Now draw another cube adjacent to the first.

56. Continue by drawing a whole row of adjacent cubes. Is the isometric dot paper helpful?

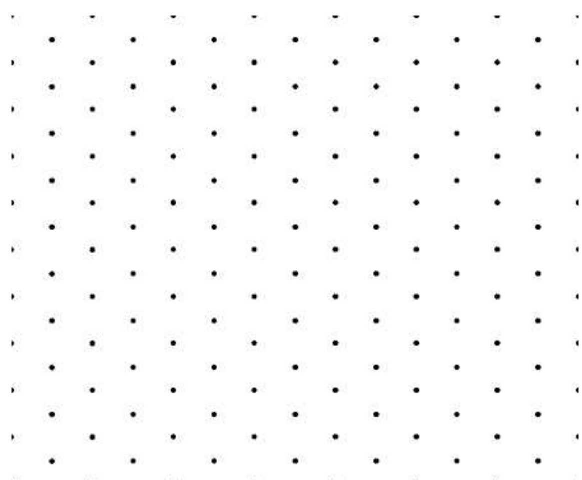


Figure 4.11: Isometric dot paper.

Figure 4.13 shows a block structure rendered isometrically. Figure **??** shows another such block structure as well as its top, front, and right views.

57. Figure 4.15 shows top, front and right views of a *block building* that has been created from cubes. In other words, you are looking at a blueprint of this building. Using actual blocks if you wish, determine how this building was constructed. Draw the result on isometric dot paper.

58. Compare your result with others. Is it the same?

59. Figure 4.16 shows top, front and right views of a *block building* that has been created from cubes. In other words, you are looking at a blueprint of this building. Using actual blocks if you wish, determine how this building was constructed. Draw the result on isometric dot paper.

60. Compare your result with others. Is it the same?

[6]It is quite sad that this was from an "educational" wwwebsite.

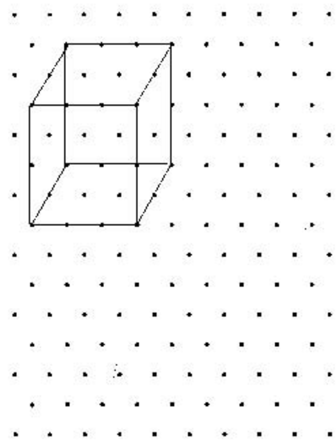


Figure 4.12: A failing attempt at drawing a cube using isometric dot paper.

Figure 4.13: A block structure rendered on isomoetric graph paper; with and without grid lines.

61. Figure 4.17 shows top, front and right views of a different *block building*. Determine how this building was constructed. Draw the result on isometric dot paper as if your view was from the right, front corner.

62. Compare your result with others. Is it the same? Does this make sense?

63. Figure 4.17 shows top, front and right views of a third building *block building*. Determine how this building was constructed.

64. Draw the result on isometric dot paper as if your viewpoint was from the left, front corner.

65. Compare your result with others. Is it the same? Does this make sense?

66. Now draw the result on isometric dot paper as if your viewpoint was from the right front corner. Does this make sense?

67. Suppose you showed your projection drawing in Investigation **66** to somebody who had not determined how to build this object first. What would they think?

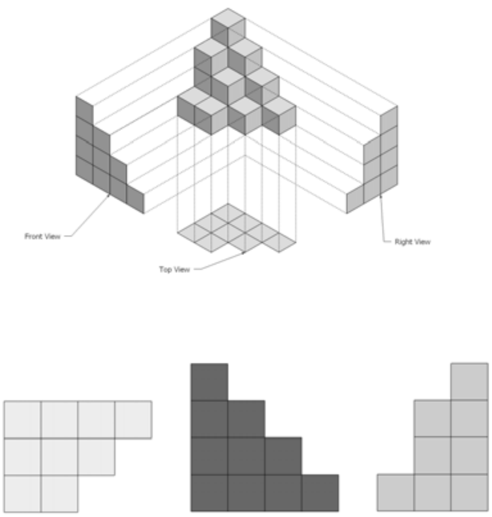


Figure 4.14: A block structure rendered isometrically (left) with top, front and right views (right). Note orientation of the views; this will be used consistently.

68. Figure 4.18 shows a famous engraving by **M.C. Escher** (?? artist; **-**). Does the example in Investigation **66** help you understand how Escher made this engraving?

69. If drawing with isometric dot paper only creates optical illusions, how can one actually make "Ascending and Descending" out of Legos?

70. Find some other optical illusions that deal with perspective and projection.[7]

Two wonderful pieces of art by **Shigeo Fukuda** (; 1932 -) are shown in Figure 4.19.

You might try your hand at some optical illusion artwork. We also highly suggest the game ***Rumis*** (aka 3D Blockus) which is a mutliplayer game where players must place three-dimensional Tetris-like pieces in arrangements that maximize the number of pieces that they can use.

[7] Masters of Deception: Escher, Dali & Artists of the Optical Illusion by **Al Steckel** (; -) is a wonderful book of optical illusions.

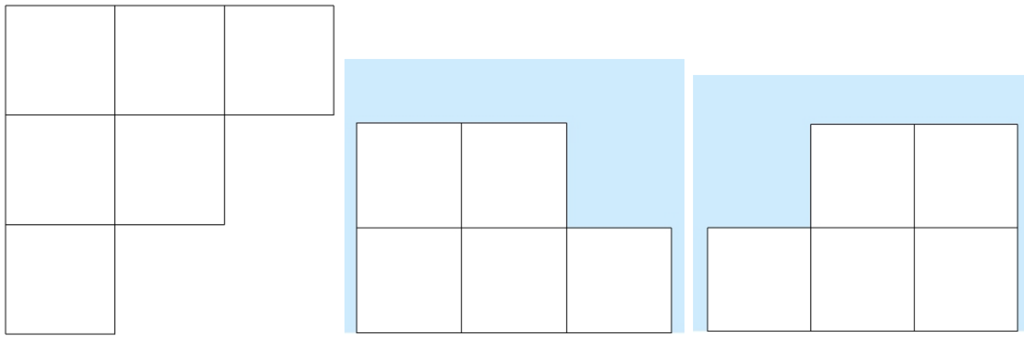


Figure 4.15: Top, front, and right views of a block building.

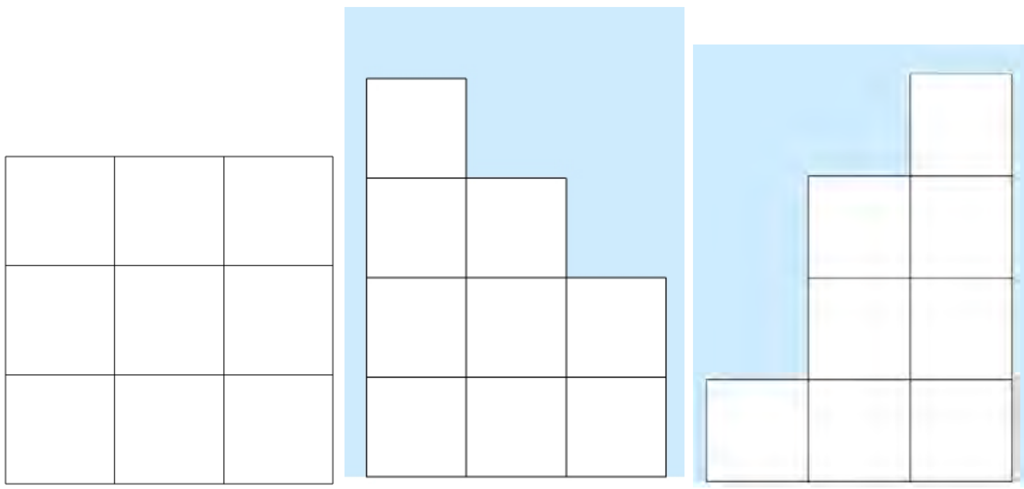


Figure 4.16: Top, front, and right views of a block building.

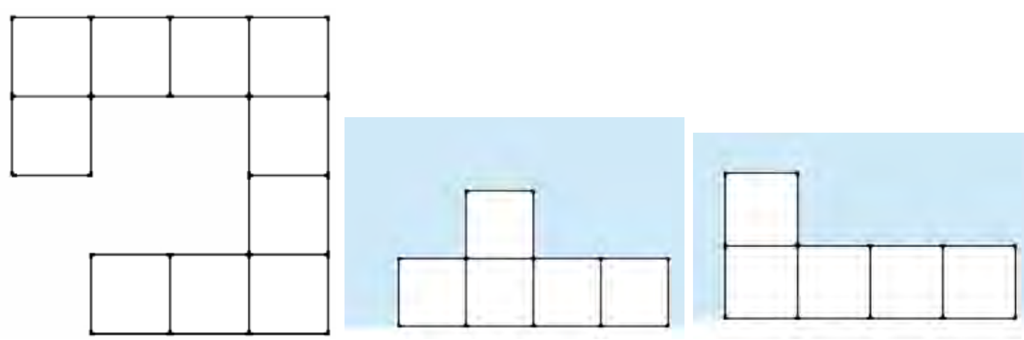


Figure 4.17: Top, front, and right views of a different block building.

Figure 4.18: M.C. Escher's "Ascending and Descending," The Strokes album cover artwork from "Angles" and "Ascending and Descending in Lego."

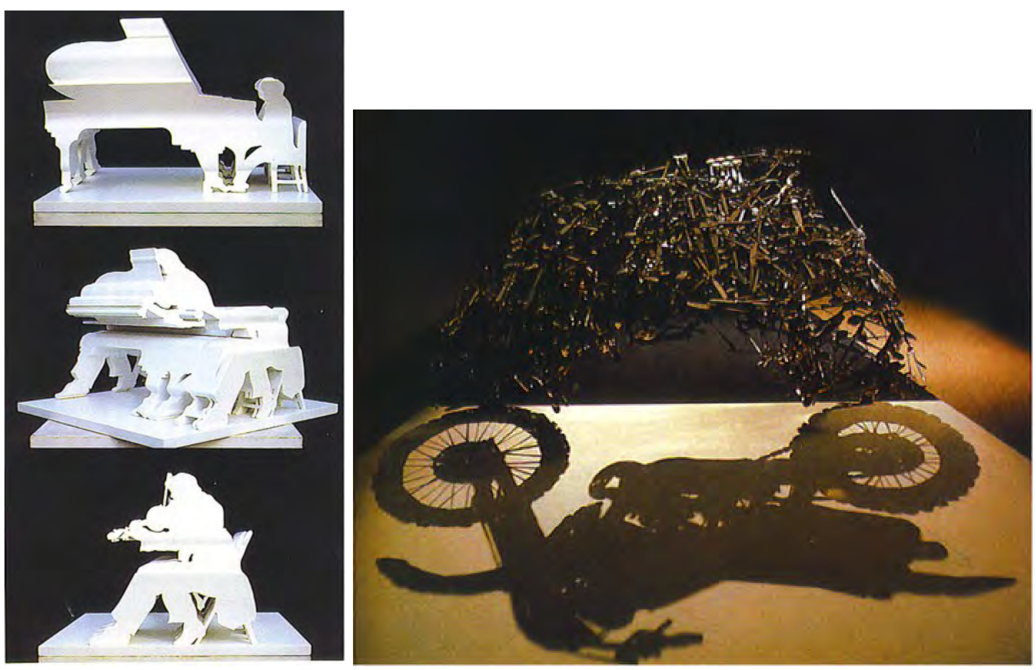


Figure 4.19: Encore and Lunch with a Helmut On by Shigeo Fukuda. The former is from wood, the latter from 848 steel spoons, knives and forks!

Chapter 5

Climbing a Ladder to the Higher Dimensions

> This is because you think of space only in three dimensions... We travel in the fifth dimension. This is something you can understand, Meg. Don't be afraid to try. Was your mother able to explain a tesseract to you?[1]
>
> **Madeleine L'Engle** (American Writer; 1918 - 2007)

.

> Today the major reason for our interest in Flatland is that for the first time we can achieve some of the dreams of our ancestors a century ago and obtain direct visual experience of phenomena in a dimension higher than our own.[2]
>
> **Thomas Banchoff** (American Mathematician; -)

5.1 The Dimensional Ladder

The schematic in Figure 5.1 is commonly referred to as the ***dimensional ladder***. The focus of this book is playing around on the dimensional ladder. In slicing and projecting you were moving from the three-dimensional world to the two-dimensional world. When you built sliceforms you created models mimicking three-dimensional solids - much like CAT and MRI scans do. Using perspective drawing, isometric drawing and topographical maps you tried to model three-dimensional objects using two-dimensional images. These are interesting topics with deep connections to many other areas - art, perception, computer graphics, etc.

We have focussed on moving between dimensions two and three because we are three-dimensional beings with two-dimensional visual fields. The ideas of slicing and projections and perspective are important to lower-dimensional "beings" as we learn from the characters in Flatland.

It would be remarkable if we could move further up the ladder. Is there a fourth dimension that our methods can help us discover?

[1] Spoken by the fictional character Mrs. Whatsit from A Wrinkle in Time.
[2] From the Introduction to the 1991 Princeton University Press edition of Flatland.

After the hero A Square of Flatland was lifted out of his flat world and into the three-dimensional world by Sphere he saw this only as a beginning to a multitude of different worlds, begging Sphere to take him higher:

> But my Lord has shewn me the intestines of all my countrymen in the Land of Two Dimensions by taking me with him into the Land of Three. What therefore more easy than now to take his servant on a second journey into the blessed region of the Fourth Dimension, where I shall look down with him once more upon this land of Three Dimensions, and see the inside of every three-dimensional house, the secrets of the solid earth, the treasures of the mines in Spaceland, and the intestines of every solid living creature.[3]

Indeed, the whole point of Abbott's Flatland is to implore us not to "be confined to limited Dimensionality." So let us see if we can climb higher.

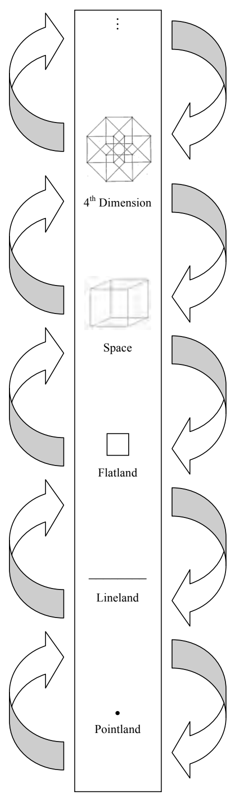


Figure 5.1: The dimensional ladder.

You will find you are not alone as you climb higher, authors (see Section 5.8) and the artists are already playing around higher up on the ladder.

[3]Flatland, p. 88.

5.2 The Fourth Dimension in Art

> M. Poincet [sic] read Henri Poincaré in the text. M. Princet has studied at length non-Euclidean geometry and the theorems of Riemann, of which Gleizes and Metzinger speak rather carelessly. Now then, M. Princet one day met M. Max Jacob and confided him one or two of his discoveries relating to the fourth dimension. M. Jacob informed the ingenious M. Picasso of it, and M. Picasso saw there a possibility of new ornamental schemes. M. Picasso explained his intentions to M. Apollinaire, who hastened to write them up in formularies and codify them. The thing spread and propagated... Cubism, the child of M. Princet, was born.[4]
>
> **Loius Vauxcelles** (; -)

Vauxcelles coined the term *cubism* - one of the major movements in the history of art. Figure 5.2 shows one of the more iconic cubist paintings, "Nude Descending Staircase" by **Marcel Duchamp** (French Artist; 1887 - 1968). As described in Discovering the Art of Mathematics - Patterns this painting was inspired not just by the cubist movement, but also the ***chronophotography*** begun by **Étienne-Jules Marey** (French scientist and photographer; 1830 - 1904) and **Eadweard J. Muybridge** (English photographer; 1830 - 1904) in the late 1800's.

Marey and Muybridge used trip wires to trigger multiple cameras to take a sequence of pictures one after another. Put together next to each other, you then saw a sequence of pictures that showed stages in the motion. So we saw, for the first time, a horse running "frame by frame" as we might now call it. These were precursors to moving pictures. One particularly well-known chronophotograph is of a nude woman descending a staircase.

Figure 5.2: Nude Descending Staircase by Marcel Duchamp.

[4]Quoted in The Fourth Dimension and non-Euclidean Geometry in Modern Art by **Linda Dalrymple Henderson** (; -)

Figure 5.3: Nude Descending Staircase by Gjon Mili.

Famous *Life Magazine* photographer **Gjon Mili** (Albanian photographer; 1904 - 1984) was one of the first artists to prominently explore the use of *stroboscopic light* to show motion in still photographs. Inspired in turn by Duchamp, Mili created his own "Nude Descending Staircase," shown in Figure 5.3.

Mili's most well-known photograph is of Picasso and is particularly relevant here as we talk about the fourth and higher dimensions. Mili told Picasso, "I would like you to draw in space while I photograph you." Picasso replied, "That would ammuse me.[5]" The result is the photograph in Figure 5.4.

5.3 Drawing Shadows of the Higher Dimensions

> The point is the result of the initial collision of the tool with the material plane. Paper, wood, canvas, stucco, metal - may serve as this basic plane. The tool may be pencil, burin, brush, pen, etching-point, etc. The basic plane is impregnated by this first collision... The geometric line is an invisible thing. It is the track made by the moving point; that is, its product. It is created by movement - specifically through the destruction of the intense self-contained repose of the point. Here, the leap out of the static into the dynamic occurs.[6]
>
> **Wassily Kandinsky** (Russian Artist; 1866 - 1944)

Notice that the name of Kandisky's book, Point and Line to Plane, is an homage to the dimensional ladder. At this point we should note that we have not yet provided a formal, mathematical definition of dimension. We have used the terms *two-dimensional* and *three-dimensional* throughout relying on everyday usage and only a brief mention in the Section 1 as the canvases of the painter and the solids of the sculptor. We expect that the use of a line as an archetype for a

[5]Quoted in the 60^{th} anniversary issue of *Life*, October, 1996, p. 64.
[6]From Point and Line to Plane, pp. 28, 57

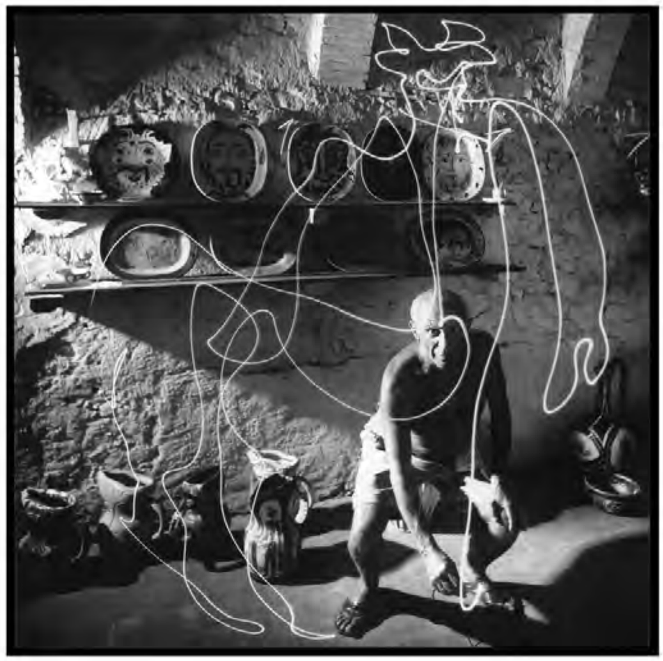


Figure 5.4: Pablo Picasso "painting" in time; Photograph by Gjon Mili, January 30, 1950.

one-dimensional object and a point as an archetype for a *zero-dimensional* object seemed unobjectionable enough. You shall uncover a precise mathematical definition shortly, but for now we investigate informally, freely using these terms.

1. With a red pen draw a point. Imagine this point moving directly to the right one unit. Draw another red point which marks where the point ends its movement. Then connect these two points by a blue or black line segment which represents the path of the moving point - the red points Kandinsky's "initial collision of the tool with the material plane" and then it's "repose" and the blue or black line the "leap out of the static into the dynamic."

2. What are the dimensions of the red components? The dimension of the entire object created in Investigation **1**? How do these dimensions compare?

3. Draw a red line segment of unit length. Imagine this line segment moving one unit perpendicular to its length - your "tool" now like a brush dynamically creating a square. Draw another red line segment which marks where the line ends its movement. Then connect corresponding endpoints of the two red line segments by blue or black lines.

4. What are the dimensions of the red components? The dimension of the entire object created in Investigation **3**? How do these dimensions compare?

Figure 5.5: Wassily Kandinsky's Unbroken Line from 1923 and Yellow-Red-Blue from 1925.

5. Mimic Investigation **1** and Investigation **3** by drawing the two squares in red and connecting corresponding edges in blue or black to form a Necker cube.

6. What are the dimensions of the red components? The dimension of the entire object created in Investigation **5**? How do these dimensions compare?

7. Use Google SketchUp to illustrate the dynamic process in Investigation **5**. Namely, use the Rectangle Tool to draw a square. Then use the Push/Pull Tool to transform it into a cube - *extruding* your square into a three-dimensional cube. Describe relationships between this computer generated approach, your hands-on approach above, and the process described above by the artist Kandinsky.

8. Now extend this drawing process to the *fourth dimension*. That is, draw a cube in red and then another also in red. Now connect corresponding vertices of the cubes in blue or black.

The object you have just created is a projection of a *rectangular* 4-*parallelopiped*; i.e. a 4-dimensional box. What type of projection you have drawn - as you may have guessed - depends on the types of projections you used to draw your cubes *and* the way you have located these cubes. Viewed from an appropriate location yours is the projection of a 4-dimensional cube, which is generally called a **tesseract**, a **4-hypercube**, or simply **hypercube**.

9. Following Investigation **8**, draw four more, different, projections of a 4-hypercube. Experiment with a number of different locations and projection types.

10. Compare your figure in Investigation **8** and Investigation **9** with those of other peers. What similarities and differences do they share? Are you surprised that there are so many figures that are so very different? Explain.

11. There are many different online scripts where you can interact with and animate projections of 4-hypercubes in an attempt to visualize them better. Go to one of these sites and play.[7] Do you see projections that look like some of those you drew in Investigation **9**? Explain.

[7]At this printing two recommended sites are `http://dogfeathers.com/java/hyprcube.html` and `http://darkwing.uoregon.edu/$\sim$koch/java/FourD.html`

12. Figure 5.6 shows four projections of 4-hypercubes. For each use a marker to highlight twelve edges that make up the projection of a (three-dimensional) cube. Can you see another, identical projected cube in the projected hypercube? Can you see how the vertices of these cubes are connected? I.e. can you see how to reverse the drawing process used to create the 4-hypercubes?

13. Explain whether these dynamic scripts help you get a sense of what the 4-hypercube looks like and how it behaves under rotations.

14. Some of these scripts employ 3-D glasses to help you see the 4-hypecube as if it were three dimensional. Does this help? What might some implications of this technology be?

15. Describe *how* you would make a 5-hypercube. Could you also make a 6-hypercube? Can this process continue indefinitely? Explain.

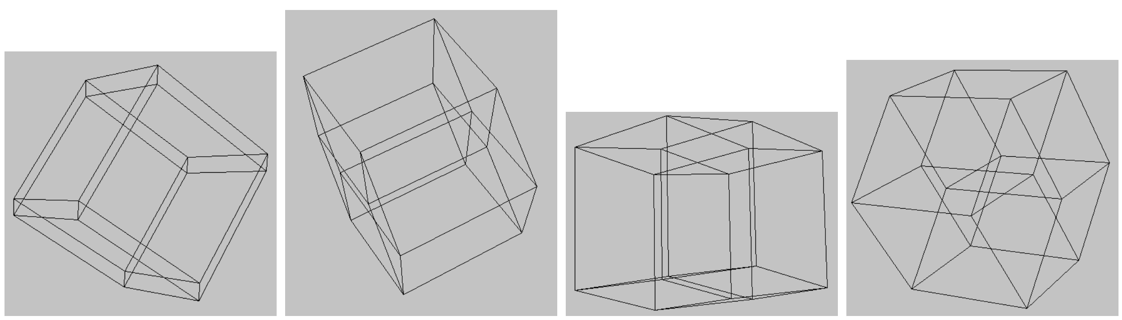


Figure 5.6: 4-hypercube projections.

5.4 Patterns: Gear to Help Climb the Dimensional Ladder

At this point you might be somewhat dubious about the shadows of the hypercube that we've asked you to create. Sure, the drawing trick is an interesting process. But we hear you thinking, "How do you know this says anything about the existence of objects in higher dimensions?" This is a legitimate criticism.

In this section you will explore some additional evidence for the appropriateness of our claim that these really are projections of objects from the higher dimensions.

5.4.1 Point, Line, Square, Cube and Hypercube Components

The table below contains information about some of the geometric components that make up our ***family of hypercubes***: point as a 0-hypercube, line segment as a 1-hypercube, square as a 2-hypercube, etc.

Dimension	Object	#Vertices	#Edges	#Faces	#Cubes	#4HCubes	Boundary
0	Point	1	0	0	0	0	Empty
1	Line	2					
2	Square						4 Lines
3	Cube						
4	4HCube						
5	5HCube						
6	6HCube						
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
n	nHCube						

16. Make a copy of this table in your notebook. In your table, fill in the number of vertices in a square and in a cube. Do you see a pattern in the data of the vertices? If so, what does this pattern predict for the number of vertices a 4-hypercube should have? A 5-hypercube? A 6-hypercube?

17. Count the number of vertices in a 4-hypercube. Does this number agree with your prediction in Investigation **19**?

18. Above you were asked how you would draw a 5-hypercube and a 6-hypercube. Based on your instructions above, will the number of vertices in these hypercubes agree with your predictions in Investigation **19**? Explain. Will your prediction hold for dimensions after this? Explain.

19. Determine a formula which expresses the number of vertices as a function of the dimension n.

20. Return to the table and fill in the number of edges in a square and a cube. Do you see a pattern in the data for the number of edges? If so, what does this pattern predict for the number of edges a 4-hypercube should have? Can you use this pattern to easily determine the number of edges in a 6-hypercube? A 17-hypercube?

21. We'd like to determine a formula for the number of edges in an n-hypercube as a function of the dimension n. Concentrate on the situation at a single vertex. How many edges emanate from each vertex in a square, a cube, a 4-hypercube? Is there a clear pattern? You've already determined the number of vertices in an n-hypercube, use it to determine a formula for the number of edges in an n-hypercube. Check that your pattern agrees with the data you have collected.

22. In Figure 5.7 are projections of the cube and the hypercube. There are multiple copies of these images in the appendix/online. Using a marker or highlighter, highlight all of the different faces of the cube - organizing your collection by using several different images to record your findings.

23. Repeat Investigation **22** to highlight all of the faces in a hypercube.

24. Do you see a clear pattern that allows you to predict how many faces there are in an n-hypercube?

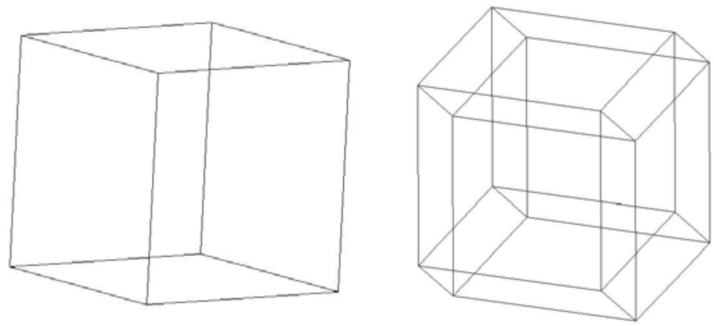


Figure 5.7: Orthogonal parallel projections of the cube and 4-hypercube. Notice that the front and back faces of the cube are not square in the projection.

25. To help uncover a pattern for the number of faces in an n-hypercube, consider the situation at each vertex, as you did in Investigation **21**. How many faces emanate from each vertex in a cube? How many faces emanate from each vertex in a 4-hypercube? Explain how you can use this to determine the total number of faces in a 4-hypercube.

To extend your reasoning in Investigation **25** you need to determine the number faces at each vertex of an n-hypercube. How many edges emanate from each vertex? How many edges need to be specified to determine an individual face? Determine how many faces there will be at each vertex in an n-hypercube. Use this to find a formula for the number of faces in an n-hypercube as a function of the dimensions, n, filling in appropriate data in the table.

26. Repeat Investigation **22** to highlight all of the cubes in a 4-hypercube. Fill your data in the table.

27. Following Investigation **21** and Investigation **25**, determine the total number of cubes in an n-hypercube, explaining carefully how you have made this determination.

28. Could you extend your reasoning to determine how many 4-hypercubes there were in an n-hypercube? How many 5-hypercubes in an n-hypercube? Etc.? Explain.

29. Consider what we find when we find the sum of the individual rows in the table above:

a. Line: 2 0-hypercubes + 1 1-hypercube = 3 total hypercubes

b. Square: 4 0-hypercubes + 4 1-hypercubes + 1 2-hypercube = 9 total hypercubes

Extend these calculations to a standard cube (i.e. 3-hypercube) and then to a 4-hypercube. What pattern do you see?

30. Does the consistency of all of these patterns help address some of your concerns about the legitimacy of our projection of the 4-hypercube and beyond? Explain.

5.4.2 Euler's Formula for Polyhedra

When one is considering geometric objects that are made up of points, lines, and planes/polygonal faces it is natural to ask if the number of components of each time are related in any way.

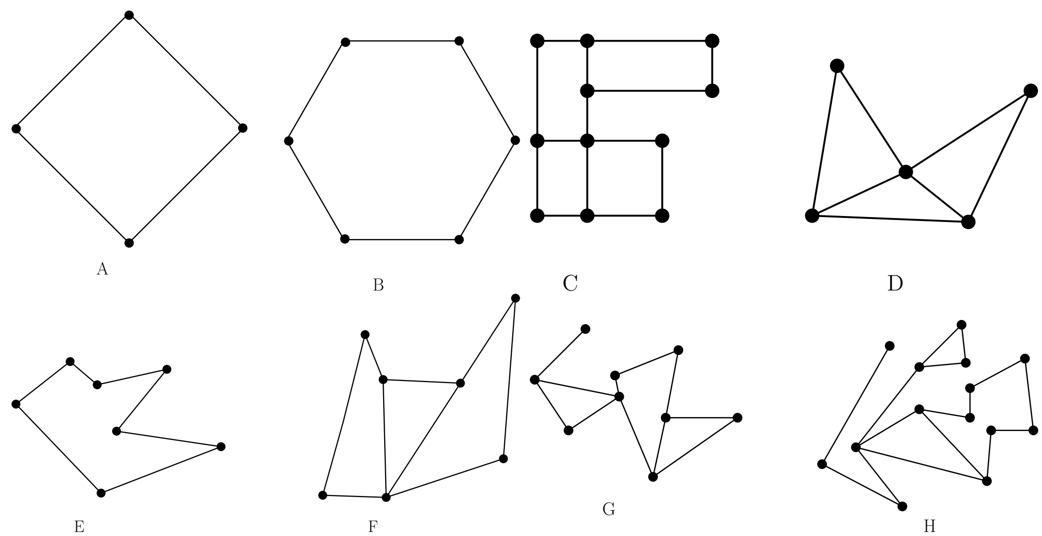


Figure 5.8: Planar figures.

31. In Figure 5.8 are several planar figures. For each figure, count the number of vertices, v; edges, e, and faces, f. Then compute $v - e$ for each. Record your data in the following table:

Figure	v	e	f	$v - e$
A				
B				
C				
D				
E				
F				
G				
H				

32. You should notice an interesting pattern in your data. Describe this pattern in a formula that relates v, e and f.

33. Draw several planar figures of your own, determine the number of vertices, edges and faces in each, and determine whether the formula from Investigation **32** holds for your planar figures. Do you think that this formula holds for all planar figures like those that we've been considering?

34. In Figure 5.9 are several 3-dimensional figures. For each, record the number of vertices, edges, faces and solids (cubes in the example), s. Can you adapt the formula in Investigation **32** so it holds for these 3-dimensional solids?

35. Make, draw or locate several other solids which are polyhedra or polyhedra joined together. Does your new formula hold for these solids?

The formulas arising above were first discovered by **Leonhard Euler** (Swiss mathematician; 1707 - 1783). Because Euler made this discovery studying of polyhedra (which have $s = 1$ as they are each comprised of a single solid) the classical formula is called ***Euler's formula for polyhedra***.[8]

36. Based on your versions of Euler's formula for 2- and 3-dimensional objects, suggest a formula that relates the number of vertices, edges, faces, solids and ***hyper solids*** (e.g. 4-dimensional "solids"), h.

37. Does this new formula hold for the 4-hypercube?

38. Does this give the 4-hypercube more legitimacy in your mind? Explain.

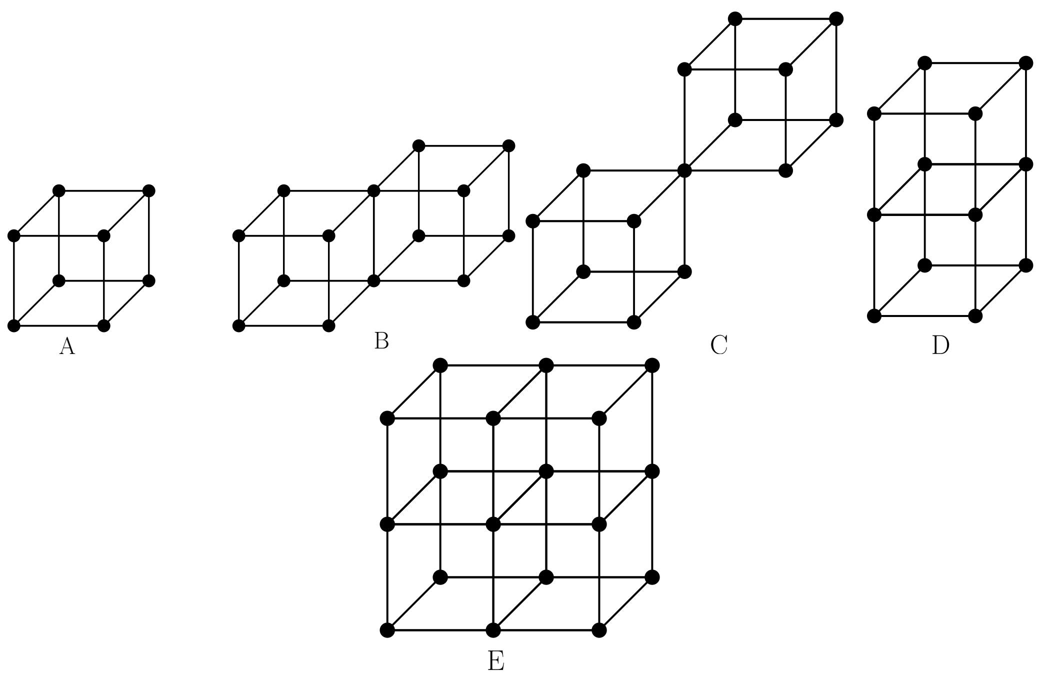


Figure 5.9: Solid figures.

[8]See the "Further Investigations" of the "Introduction" of this book for the application of Euler's formula to prove that there are only five Platonic solids.

5.5 Coordinate Geometry

One of the revolutionary changes to geometry over its whole history is something that you likely saw in high school geometry. Introduced most notably by **Rene Descartes** (; -) in his essay "La Geometrie" of 1637, the use of coordinate systems provided a revolutionary new way to do geometry. We use it here to provide another way to navigate the dimensional ladder.

On the left of Figure 5.10 is a unit line segment - like those that make up the edges of the squares, cubes and hypercubes we've been drawing. Notice that the vertices need only one coordinate to be located on the number line: 0 and 1.

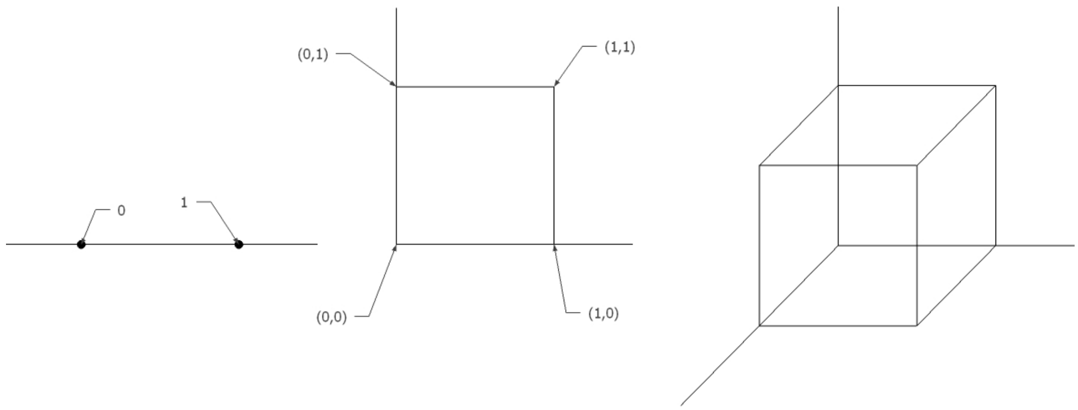


Figure 5.10: Line, plane, and space - the dimensional ladder.

39. In the center of Figure 5.10 we have drawn a unit square. Describe how the coordinates of the vertices are named.

40. Similarly, on the right of Figure 5.10 we have drawn in a unit cube and the three coordinate axes: positive x pointing to the right, positive y pointing up, and positive z pointing "out of the plane." Find the coordinates of each vertex of the cube.

41. You should see a pattern forming in the way the vertices of the line, square and cube are formed. Describe it in detail.

42. Use Investigation **41** to write down the coordinates of all of the vertices of a standard, unit 4-hypercube. Show that your answer agrees with your results above.

43. Explain what form the vertices of a standard, unit 5-hypercube would take. A 6-hypercube? How high can you go in describing, precisely, all of the vertices of a standard, unit n-hypercube?

44. If we can describe the vertices then we can begin to analytically describe the edges using something like the standard formula for a line (i.e. $y = mx + b$) and then the faces and then... So, if we can analytically describe all of these components of an n-hypercube, is there any question that such n dimensional objects exist - even when $n \geq 4$? Explain.

5.6 Defining Dimension

We say that space is 3-dimensional because the walls of a prison are 2-dimensional.
Hermann Weyl (German mathematician and physicist; 1885 - 1955)

The goal of this section is to develop a more formal definition of dimension. There are many ways to define dimension - you'll see another in the next chapter. Mathematicians have many different definitions of dimension, to be used and applied in different contexts.

We start with our intuitive notion of dimension that has been implicit throughout. This is most easily seen via the use of Cartesian coordinates as in the last section. A line is one-dimensional because there is one *degree of freedom* - from a fixed origin you can move only left and right and the distance from this fixed origin is specified by a single variable x which determines the location of any point on the line. A plane is two-dimensional because you have two degrees of freedom. You can move left-right and up-down and from a fixed origin we can specify any point in the plane using two coordinates, written (x, y). Similarly, points in three-dimensional Euclidean space are specified by three coordinates, (x, y, z), as we have three degrees of freedom. Theoretically there is no limit to the number of coordinates that can be specified.

We are used to other specialized surfaces being described in similar ways. For example, we describe locations on the surface of the earth by longitude and latitude which make up a two-dimensional coordinate system for this surface. For pilots and divers this can be extended to three-dimensional space by specifying longitude, latitude, and height above/below sea level.

Mathematicians work with idealized objects which approximate real objects. As we work with the intuitive notion of dimension, let us agree to do think of real objects by their idealized versions. For example, string will be considered an infinitely thin, one-dimensional object and sheets of paper will be considered infinitely thin, two-dimensional surfaces.

45. Find a dozen everyday objects that, in their idealized versions, are one-dimensional. Explain why you believe they are one-dimensional.

46. Find a dozen everyday objects that, in their idealized versions, are two-dimensional. Explain why you believe they are two-dimensional.

47. Find a dozen everyday objects that, in their idealized versions, are three-dimensional. Explain why you believe they are three-dimensional.

Mathematicians have discovered how to piece together objects that look locally like Euclidean spaces, calling the results *manifolds*. Instead, the notion of dimension that is suggested by the investigations below is that of *small inductive dimension*, which is a topological notion of dimension.[9] This definition is less technical than the machinery necessary to define maninfolds, is both historically and mathematically important, and is fully formalizable.

We must make one caveat. Fundamental to these investigations is the notion of *boundary* which we will not define formally - to do so would require developing a number of non-trivial notions that make up the first several weeks of an introductory topology course for mathematics majors. *However*, we do use the appropriate terms below and the investigations below parallel the more formal development.[10]

[9] *Topology* is a critcal area of mathematics that arises in many books in this series, including: Discovering the Art of Mathematics - Knot Theory and Discovering the Art of Mathematics - Art and Sculpture.

[10] If you wish to see more of the precise development, Chapter 3 - Topological Dimension of the beautiful book Measure, Topology, and Fractal Geometry by Gerald A. Edgar is recommended.

Mathematicians name terms in accordance with their everyday meanings, but on more precise levels. So, in mathematics, as in everyday usage, a ***boundary*** is the periphery of a region, a bound or frontier, or a limiting extent of a region.

Using this terminology, describe the boundary of each of the mathematically idealized[11] versions of the everyday objects below.

48. . . . a country.

49. . . . a pear.

50. . . . a piece of property.

51. . . . a stop sign.

52. . . . a house.

53. . . . the surface of a drinking glass.

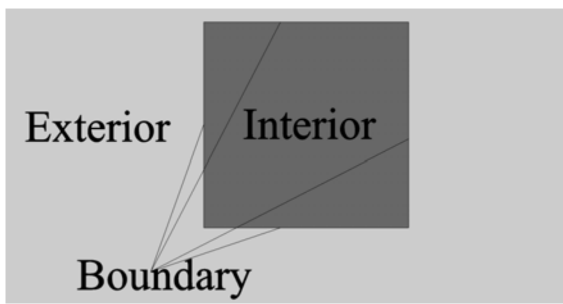


Figure 5.11: The interior (dark grey), exterior (light grey), and boundary of a square.

The boundary has several important properties. Figure 5.11 shows the boundary of a square, which ***separates*** the *interior* from the *exterior*. In constructing a stop sign, large sheet metal is stamped with a punch which sheers the metal and separates the sign from the larger sheet. The bark of a tree separates the wood which makes up the tree from its surroundings.

Conceptualized differently, if you are in a country and walk in a specific direction you will eventually get to a point where you cannot walk any further and remain in the country. This *limit point* is a *boundary point* and the entire boundary of the country can be determined by finding limit points like this. Figure 5.12 shows a drinking glass which has been given a coordinate system. As we follow each *geodesic* on the glass from the center of the bottom we see that each stops at a point on the rim, a boundary point.

54. Return to Investigation **48** - Investigation **53** and show how the notion of separation and/or limit points can be used to check that your previous answers are correct.

55. What is the boundary of a cube?

[11]For example, the drinking glass is - like an idealized mathematical plane - infinitely thin.

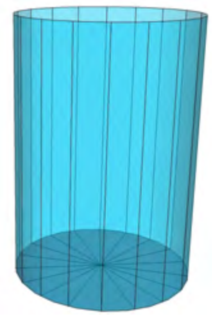


Figure 5.12: A ruled drinking glass.

56. What is the boundary of a line segment?

57. Return to the table 5.4.1 and fill in the boundaries to complete the last column.

58. As we move up the dimensional ladder point → line segment → square → cube → 4-hypercube, how does the dimension of the object and its boundary appear to be related? Explain.

59. Investigation **58** should suggest a first attempt at a definition of dimension:

A point is zero-dimensional. A topological object has dimension n precisely when its boundary $\boxed{}$.

Of course, in making a definition we need to insure that it correctly applies to those examples we hope to use it to describe. Let's check.

60. What is the dimension of a ***solid ball***, like a bowling ball without holes?

61. What is the boundary of a solid ball?

62. What is the dimension of the boundary of a solid ball?

63. What is the dimension of a ***disc***, like a flattened frisbee?

64. What is the boundary of a disc?

65. What is the dimension of the boundary of a disc?

66. What is the dimension of a solid cylinder?

67. What is the boundary of a solid cylinder?

68. What is the dimension of the boundary of a solid cylinder?

69. What is the dimension of a ***solid torus***, like a doughnut?

70. What is the boundary of a solid torus?

71. What is the dimension of the boundary of a solid torus?

72. Use your definition to show that a 1-hypercube (i.e. line segment) is 1-dimensional.

73. Use your definition to show that a 2-hypercube (i.e. square) is 2-dimensional.

74. Use your definition to show that a 3-hypercube (i.e. cube) is 3-dimensional.

75. Use the definition to show that a 4-hypercube is 4-dimensional.

76. Compare your observations thus far with Weyl's quote that opens this section.

So far, so good. This seems a useful, compelling definition.

77. What is the boundary of a sphere?

78. What is the boundary of a circle?

79. What is the boundary of a surface of a torus?

80. What happens if you try to apply your definition of dimension to a sphere, circle, or surface of a torus? Can you think of a way to circumvent this problem?

We hope you feel your definition is compelling enough that we should attempt to adjust it - allow it to be adapted to be compatible with these new examples.

A circle can naturally be decomposed into two semicircles.

81. What is the boundary of a semicirle?

82. Can you use your definition dimension to determine the dimension of a semicircle? Explain.

83. Does this help to determine the dimension of a circle? How?

84. Can you decompose a sphere so your definition of dimension can be applied to each of the components?

85. Can you decompose the surface of a torus so your definition of dimension can be applied to each of the components?

86. Is there are limit on the number of pieces that an object can be decomposed into before this process breaks down? Either describe why their is no limit or provide an example which illustrates the limit.

87. Adapt your definition of dimension in light of the examples just considered.

For typical mathematical objects[12] this is the definition of ***small inductive dimension***.

[12]E.g. *compact*, *separable*, *metric spaces*.

5.7 Implications of the Fourth Dimension

All of us are slaves to the prejudices of our own dimension.

Thomas Banchoff (American Mathematician; -)

The greatest advantage to be derived from the study of geometry of more than three dimensions is a real understanding of the great science of geometry. Our plane and solid geometries are but the beginning of this science. The four-dimensional geometry is far more extensive than the three-dimensional, and all the higher geometries are more extensive than the lower.[13]

Henry Parker Manning (American Mathematician; 1859 - 1956)

One's mind, once stretched by a new idea, never regains its original dimensions.

Oliver Wendell Holmes (; -)

Artists who are interested in four dimensional space are not motivated by a desire to illustrate new physical theories, nor by a desire to solve mathematical problems. We are motivated by a desire to complete our subjective experience by inventing new aesthetic and conceptual capabilities. We are not in the least surprised, however, to find physicists and mathematicians working simultaneously on a metaphor for space in which paradoxical three dimensional experience are resolved only by a four dimensional space. Our reading of the history of culture has shown us that in the development of new metaphors for space artists, physicists, and mathematicians are usually in step.[14]

Tony Robbin (; -)

Figure 5.13 shows what is called a *net* of a dodecahedron.

88. Cut out the net, fold along the lines, and tape edges together to create a dodecahedron.

89. Create a net of a square-based pyramid.

90. Create several different nets of a cube. Cut two of them out and check to make sure they appropriately form a cube.

91. Can you create a net for a cube that resembles a cross? Either do so or explain why it cannot be done.

92. Figure 5.14 shows the famous 1954 painting Crucifixion (Corpus Hypercubus) by **Salvidore Dali** (Spanish artist; 1904 - 1989). How is the cross related to topics we have been discussing? Explain in detail.

93. Can you think of some reasons Dali may have utilized this type of cross? I.e. what might his artistic, cultural, and/religious message have been?

[13]From Geometry of Four Dimensions.

[14]Quoted in The Fourth Dimension and non-Euclidean Geometry in Modern Art by Linda Dalrymple Henderson.

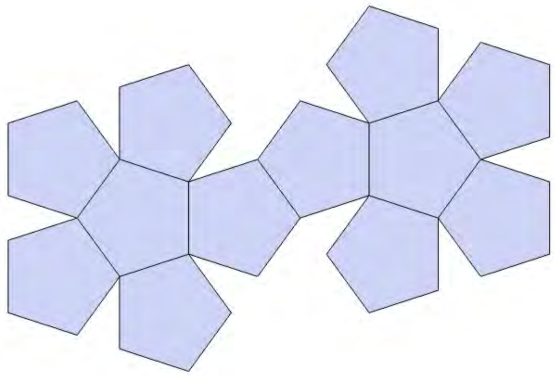


Figure 5.13: Net of a dodecahedron

When Sphere visits Flatland, the hero A Square eventually becomes exasperated and in his confusion/fear he assaults Sphere, exclaiming, “Monster, be thou juggler, enchanter, drea, or devil, no more will I endure they mockeries. Either thou or I must perish.” A Square then collides into the Sphere, seeking to destroy him. Yet he finds, “I could feel him slowly and unarrestbly slipping from my contact; no edging to the rigt nor to the left, but moving somehow out of the world and vanishing to nothing.[15]”

Frustrated that A Square “will not listen to reason,” Sphere decides to resort to deeds.

94. Sphere’s first deed is to demonstrate how he “can see from my position in Space the inside of all things that you consider closed.” He goes on to describe the contents of A Square’s cubbard, including money hidden in closed boxes. How is it that Sphere can see these things and so amaze A Square with his power to see through Flatland’s boundaries?

95. Sphere’s final deed, “as crowning proof,” is to giving A Square “a touch, just the least touch” in his stomach. One that “will not seriously injure... and the slight pain... cannot be compared with the mental benefit” which knowledge of a third dimension will impart. A Sphere reports of a “shooting pain in my inside, and a demoniacal laugh seemed to issue from within me.” How did Sphere touch A Square in his stomach? If A Square was ill and needed some sort of surgery, what advantage would Sphere have over Flatland doctors?

96. Extend the analogy now. Suppose you had cash locked in a safe. If you were visited a being from the fourth dimension, would they have access to the contents of your safe? Explain precisely.

97. If you were ill and needed surgery, would a doctor from the fourth dimension have any advantages over Spaceland doctors? Explain precisely.

All sorts of “paradoxes” arise when we think about the fourth dimension. Here is another.

[15] P. 77.

Figure 5.14: The Crucifixion (Corpus Hypercubus) by Salvidore Dali.

When you played the Flatland game, it is likely that some of your cross sections were *disconnnected*. For example, an axial cross section of a hand shows a thumb and four independent fingers. They look entirely disconnected from one another from the view of a Flatlander. Only by seeing into Spaceland can one ascertain that these independent objects are actually connected together as part of a larger whole.

98. **Carl Gustav Jung** (Swiss Psychiatrist; 1875 - 1961) wrote of the *collective unconscious*. Might this be part of a larger physical collective? E.g. might each of us be part of a larger, four-dimensional physical whole - called humanity - and that each human only seems independent because our views are limited to three dimensions and we do not realize that we are all connected? Explain.

5.8 The Fourth Dimension in Literature

The fourth dimension, and beyond, appears in literature of many different sorts:

- Time Machine by **H.G. Wells** (; -) - An 1895 science fiction novella that spurred our interest in time travel and thereby spawned hundreds of other books, stories, and movies.
- "Mimzy were the Borogoves" by **Lewis Padgett** (Pseudonym for **Henry Kuttner** (; -) and **C. L. Moore** (; -); -) - a 1943 short story which was later turned into the hit 2007 movie The Last Mimzy.
- A Wrinkle in Time by **Madeline L'Engle** (American writer; 1918 - 2007) - Award-winning young adult science fiction novel written in 1962.
- Factoring Humanity by **Robert J. Sawyer** (Canadian writer; 1960 -) - Outstanding science-fiction novel from which the analogy in Investigation **98** was borrowed.

- Slaughterhouse Five by **Kurt Vonnegut** (American writer; -) - Ranked # 19 on the Modern Library's 100 Best English Novels, this famous science fiction includes as fundamental characters the *Tralfalmadorians* who see in four dimensions and have thus seen every moment of their lives.

99. Choose two of the pieces of literature above. Find out more about these pieces of literature, focussing particulary on the role that the fourth dimension plays as a literary vehicle for the piece. Describe this role and its relationship with what you have learned about the higher dimensions here.

100. Find two examples of literature that are not included on the list above where the fourth, or higher, dimensions plays a critical role. Describe the role of the higher dimensions and its relationship with what you have learned about the higher dimensions here.

5.9 Further Investigations

5.9.1 Different Types of Hypercube Projections

In Chapter 4 we saw that the way we see and represent and project cubes can be best described by the number of vanishing points - and that $0, 1, 2$, and 3 vanishing points were possible. Different artists have decided to use the names *four point perspective*, *five point perspective* and *six point perspective* to describe different drawing methods that introduce curved perspective and flat images of 360 degree viewing spheres. These are very interesting topics, topics that are considered in detail in Discovering the Art of Mathematics: Art Sculpture and where you can learn to create amazing images like that shown in Figure 5.15, but we prefer to continue with a stricter analogy. Namely, in the fourth dimension we have four perpendicular coordinates so it makes sense to consider not only $0, 1, 2$ and 3 vanishing points for parallel lines, but also to consider 4.

F1. Draw a projection of a hypercube with no vanishing points - what one might call a Necker hypercube.

F2. Draw a projection of a hypercube that has exactly one vanishing point.

F3. Draw a projection of a hypercube that has exactly two vanishing points.

F4. Draw a projection of a hypercube that has exactly three vanishing points.

F5. Draw a projection of a hypercube that has exactly four vanishing points.

5.9.2 Extending the Pattern of Hypercube Components

In Investigation **21**, Investigation **25** and Investigation **27** you determined how to find the number of edges, faces, and cubes in n-hypercubes of all dimensions. Investigation **28** asks whether or not you can extend the approach you used to find all lower dimensional k-hypercubes that are part of an n-hypercube. This generalization is considered here.

F6. As before, consider the situation at each vertex. How many edges are needed to *span* a k-hypercube which emanates from the chosen vertex?

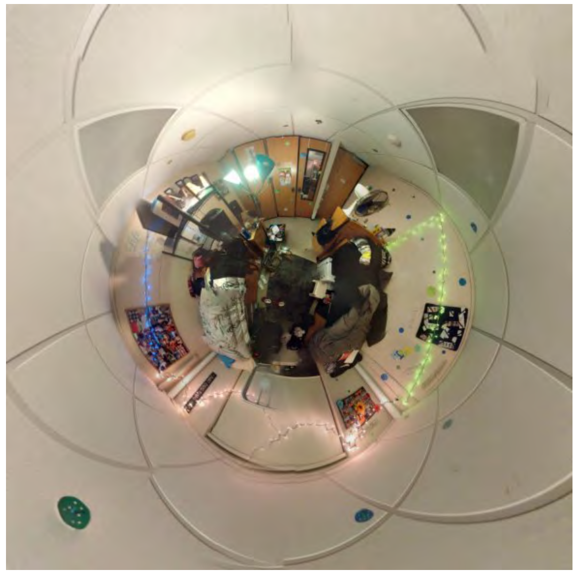


Figure 5.15: 360 view of a Westfield State University dorm room by Erin.

F7. How many edges emanate from the vertex?

F8. How many different possible ways are there to choose the required number of edges from those emanating from the vertex? Explain.

F9. The same situation exists at every vertex. If we temporarily ignore counting the same k-hypercube multiple times, how many k-hypercubes do we have?

F10. How many times is each distinct k-hypercube counting in this approach?

F11. How many k-hypercubes are there in an n-hypercube? [Hint: Check that your answer correctly predicts the correct number using some of the data you have above.]

F12. Use the *binomial theorem* to prove the result in Investigation **29**.

5.9.3 Euler's Formula

> In the "commentatio" (note presented to the Russian Academy) in which his theorem on polyhedra (on the number of faces, edges and vertices) was first published Euler gives no proof. In place of a proof, he offers an inductive argument: he verifies the relation in a variety of special cases. There is little doubt that he also discovered the theorem, as many of his other results, inductively.
>
> **George Polya** (; -)

Euler is considered one of the greatest mathematicians of all time. His collected works fill over 80 volumes. Completely blind by age 60 he continued to do an enormous amount of important mathematics through the last sixteen years of his life, using three dedicated scribes to write his work while performed remarkable calculations in his heads as his grandchildren from his 13 children played on his lap.

Mathematics is indebted to Euler for contributions in many different areas of mathematics. Despite his incredible insights, Euler's work often relied only on inductive evidence and deductive proofs were filled in by later generations of mathematicians.

Euler's formula is unique because there remains debate about whether we have a complete understanding of this result.

Euler's formula needs to be adapted to account for solids which have "holes", like that on the left in Figure 5.16. But even then there are solids for which this new formula does not hold, like those in the center and on the right in Figure 5.16. Efforts to rectify these problems have lead to some vocal debates and deep questions about the nature of mathematics' development.

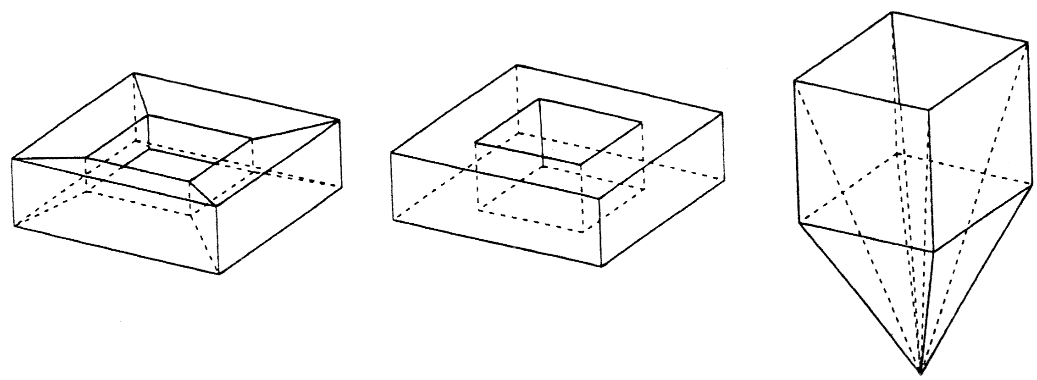


Figure 5.16: Counterexamples to Euler's formula.

For example, the important book Proofs and Refutations: The Logic of Mathematical Discovery by **Imre Lakatos** (; -) is an important book the philosophy and culture of mathematical discovery and proof. The majority of this book is a Socratic dialogue in which the characters discovery Euler's formula for polyhedra and then repeatedly find their attempts at proof to be refuted. They continue, searching for deeper understandings that will enable them to make sense of this mysterious relationship among the components of geometric objects. It is a deep, important book whose messages about mathematical discovery are fundamental to the genesis of the Discovering the Art of Mathematics series.

In 1994 well-known pattern experts **Branko Grünbaum** (; -) and **C.G. Shephard** (; -) wrote "A New Look at Euler's Theorem for Polyhedra"[16] to give "unexpectedly simple answers" to the concerns that counterexamples which arise from "complicated and unusual polyhedral sets" had introduced. Shortly thereafter two other well-known mathematicians, **Peter Hilton** (; -) and **Jean Pedersen** (; -), wrote a pointed response.[17] While they called Grünbaum and Shephard's

[16] *American Mathematical Monthly*, Vol. 101, No. 2, Feb. 1994, pp. 109-128.

[17] "Euler's Theorem for Polyhedra: A Topologist and Geometer Respond," *American Mathematical Monthly*, Vol. 101, No. 10, Dec. 1994, pp. 959-961.

paper "stimulating. . . [and] a very real contribution to polyhedral geometry" they had very strong objections:

> The paper appears. . . to both neglect and distort the contribution which the algebraic and combinatorial topologists have made to the development of the Euler characteristic. We claim that the 'problem' . . . is no real problem at all, and that topologists have completely and satisfactorily formalized the Euler characteristic. . . Moreover, we deny categorically that 'this development lead to the loss of the connection to the origins of Euler's theorem'. . . Specifically we reject the calculation of $v - e + f$ for Figure 1(b).

The journal's editors allowed the original authors to respond,[18] and their response is no less pointed:

> The "Response" by Peter Hilton and Jean Pedersen seems to us to be misleading and inappropriate. . . The rejection of that "calculation", and of the similar one for the polyhedron in Figure 2, was our explicit aim, obviously missed by Hilton and Pedersen. However, our objection to the "Response" is much deeper. Their comment. . . is true, but irrelevant. In particular, we reject the implication that the geometrical approach we propose is redundant. . . Euler himself, if he were alive today, would, we believe, feel far more comfortable with our approach to his theorem than that advocated in the "Response."

F13. Are you surprised that prominent, contemporary mathematicians are fighting over a result that is called a "theorem" and was discovered in 1752?

F14. In fact, debates in mathematics are as typical as they are in other fields - despite peoples' perceptions of mathematics as a field whose results are definitive, cut and dry. Examples of areas of debate include the proof of the *four color theorem*, the *classification of the finite simple groups*, and Hales' proof of Kepler's sphere packing conjecture which we considered earlier in this book. Look up one of these debates, or find another, and briefly describe the nature of the controversy.

F15. Create several planar shapes and several polyhedral solids which have "holes". See if you can determine how to adapt Euler's formula to work for these objects. Can you see, geometrically, why these adaptations are necessary?

[18] "Response from Grünbaum and Shephard," *American Mathematical Monthly*, Vol. 101, No. 10, Dec. 1994, pp. 961-2.

Chapter 6

Inbetween the Dimensions - Fractals

> The existence of these patterns [fractals] challenges us to study forms that Euclid leaves aside as being formless, to investigate the morphology of the amorphous. Mathematicians have disdained this challenge, however, and have increasingly chosen to flee from nature by devising theories unrelated to anything we can see or feel.
>
> **Benoit Mandelbrot** (Polish/French mathematician; 1924 - 2010)

> The real voyage of discovery consists not in seeking new landscapes but in having new eyes.
>
> **Marcel Proust** (French novelist and critic; 1871 - 1922)

Earlier in this book you discovered how to navigate between the dimensions, about visual interplay among the different dimensions and what critical roles these things play in mathematics and many other fields, including: art, architecture, geography, and medicine among them. Then you scaled the dimensional ladder, climbing right past the third dimension we call space into the fourth dimension, the fifth dimension, etc. You discovered that you can climb as high as you wants on this ladder. Entirely new worlds have been opened.

While mathematicians - and physicists and authors and artists and philosophers and the religious and others - explored and marveled at our ascent into the higher dimensions in the late nineteenth century, some imaginative travelers undertook an entirely new exploration. They searched for different geometries by inventing different conceptual lenses to see the world through. In this way they slowly discovered that between the rungs of the dimension ladder there were rich, provocative, beautiful and important geometries to be discovered.

These are the geometries of this chapter, *fractal geometries* which have dimensions are no longer whole numbers. These geometries have fractured our belief that the dimensions simply march from 1 to 2 to 3 to... Instead we realize that an immensely greater infinity of realms in between are there to be explored.

6.1 Proportion and Scale

The notion of proportion is fundamental in mathematics. As we shall see in the final chapter, **Eratosthenes of Cyrene** (Greek mathematician and Geographer; circa 276 B.C. - 195 B.C.)

used proportions to measure the circumference of the Earth to within a few percent of its actual value! This is a tremendous accomplishment.

We begin with some basic examples.

Figure 6.1 is a satellite image of the west end of the United States National Mall. At the center of the circle on the left in this image is the Jefferson Memorial. On the right side, inside of the interlocking ovals, is the Washington Monument. The *scale* of this map is 1 inch = 0.2 miles.

Figure 6.1: West end of the United States National Mall.

1. Determine the distance, in miles, from the center of the Jefferson Memorial to the center of the Washington Monument.

2. The scale of this map was given as an equality. Often it is written as a 1 inch : 0.2 miles, signifying a ratio. What is the value of the ratio $\frac{1}{0.2}$?

3. What is the value of the ratio

$$\frac{\text{Inches Between Jefferson Memorial and Washington Monument on Map}}{\text{Miles Between Jefferson Memorial and Washington Monument}}?$$

4. How do the ratios in Investigation **2** and Investigation **3** compare?

The main entrance of Westfield State College's Courtney Hall is shown in the image in Figure 6.2. (A larger version appears in the Appendix.) Direct measurement shows that the inside distance between the two brick pillars that form the entryway (roughly vertically below the space between the "O" and "U" to the center of the "A") is 110 inches.

5. Create a scale for this photo.

6. Measure the height of Courtney Hall on the photo, from the ground to the top of the *cupola*, in inches.

Figure 6.2: Westfield States Courtney Hall.

7. Determine the approximate height of Courtney Hall in inches.[1]

8. Using your data, record the following two ratios:

$$\frac{\text{Door Width in Courtney Photo}}{\text{Door Width in Courtney}} = \underline{\qquad} \text{ and } \frac{\text{Height of Courtney in Photo}}{\text{Height of Courtney}} = \underline{\qquad}$$

9. Simplify both of the ratios in Investigation **8**. What do you notice? Why does this happen?

10. Determine the width of this main section of Courtney Hall.

11. Determine the widths of the square, *3 over 3 windows* on the third floor.

12. Suppose we told you that the original scale was wrong, that the doors were really 220" apart. How would this change yours calculations?

Likely these are not new ideas to you. However, they provide a starting point that will lead to quite surprising discoveries.

[1] As we learned in the chapter "Visualizing Between the Dimensions," awareness of perspective and its implications is essential. While this picture was taken from fairly far away, there will be a vanishing point quite some distance above the top of the picture. This creates a ***keystone effect*** where vertical lines are not truly parallel. As the size of this effect is small in this picture, its impact will be small. A slightly more significant problem here is that the cupola, building face and entry face are not on the same plane. So this will slightly distort the distances we compute.

6.2 Scaling Perimeters, Areas and Volumes

Figure 6.3 shows a square drawn on square graph paper. We call this a unit square because we have used the smallest scale of our paper to dictate the size of the square. We define the lengths of its sides to be $s = 1$ and we define its area to be $A = 1$. You and your fellow students may use graph paper whose square grids are different sizes, but your graph paper will set a scale for you in the following investigation.

We will refer to the unit square by S_u.

The ***perimeter*** of the unit square is $P = 4$ because its boundary is made up of four unit line segments, each with length 1.

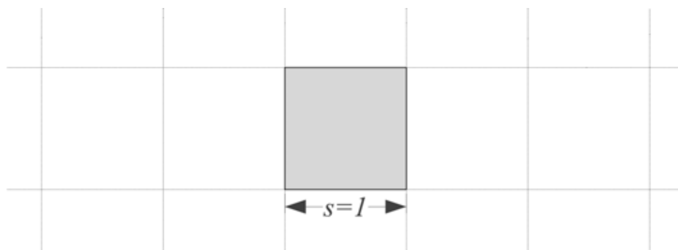


Figure 6.3: Unit square.

13. Using graph paper or dot paper, draw your unit square, labeling the unit side length and the unit area.

14. Draw a differently sized square on your graph paper, one whose sides are concurrent with the lines on your graph paper. Label it S_A.

15. Determine the side length and the perimeter of S_A.

16. What are the values of the two ratios

$$\frac{\text{Side Length of } S_A}{\text{Side Length of } S_u} \text{ and } \frac{\text{Perimeter of } S_A}{\text{Perimeter of } S_u}$$

and how do they compare?

17. Draw a differently sized square on your graph paper, one whose sides are concurrent with the lines on your graph paper. Label it S_B.

18. Determine the side length and the perimeter of S_B.

19. What are the values of the two ratios

$$\frac{\text{Side Length of } S_B}{\text{Side Length of } S_u} \text{ and } \frac{\text{Perimeter of } S_B}{\text{Perimeter of } S_u}$$

and how do they compare?

20. What are the values of the two ratios

$$\frac{\text{Side Length of } S_B}{\text{Side Length of } S_A} \text{ and } \frac{\text{Perimeter of } S_B}{\text{Perimeter of } S_A}$$

and how do they compare?

21. Find a formula which relates the two ratios

$$\frac{\text{Side Length of } S_1}{\text{Side Length of } S_2} \text{ and } \frac{\text{Perimeter of } S_1}{\text{Perimeter of } S_2}$$

for any two squares S_1 and S_2. Explain geometrically why it will always hold.

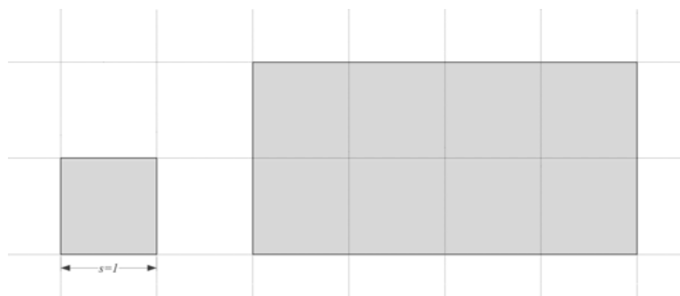


Figure 6.4: Rectangle of area $A = 8$ units.

Figure 6.4 shows a rectangle. Its ***area*** is $A = 8$ units because it can be decomposed into eight pieces, each which is an identical copy of our unit square.

22. Find the area of your square S_A.

23. What are the values of the two ratios

$$\frac{\text{Side Length of } S_A}{\text{Side Length of } S_u} \text{ and } \frac{\text{Area of } S_A}{\text{Area of } S_u}$$

and how do they compare?

24. Determine the area of S_B.

25. What are the values of the two ratios

$$\frac{\text{Side Length of } S_B}{\text{Side Length of } S_u} \text{ and } \frac{\text{Area of } S_B}{\text{Area of } S_u}$$

and how do they compare?

26. What are the values of the two ratios

$$\frac{\text{Side Length of } S_B}{\text{Side Length of } S_A} \text{ and } \frac{\text{Area of } S_B}{\text{Area of } S_A}$$

and how do they compare?

27. Find a formula which relates the two ratios

$$\frac{\text{Side Length of } S_1}{\text{Side Length of } S_2} \text{ and } \frac{\text{Area of } S_1}{\text{Area of } S_2}$$

for any two squares S_1 and S_2. Explain geometrically why it will always hold.

Let us now move up a dimension, from Flatland objects like squares to solids like cubes. To visualize and keep track of the objects you are building and exploring it will be helpful to use a collection of small, congruent cubes to build with, or to use the isometric drawing techniques from the chapter "Visualizing Between the Dimensions," or to use SketchUp or some other sort of CAD program.

The left image in Figure 6.5 shows an isometric drawing of a unit cube **unit cube**. Letting the lengths on the isometric graph paper dictate our scale, the lengths of the sides of the cube are $s = 1$ and we define its volume to be $V = 1$.

We will refer to this unit cube by C_u.

The ***surface area*** of the unit cube is $SA = 6$ because its boundary is made up of six unit squares, each with area 1.

The right image in Figure 6.5 shows a rectangular prism whose dimensions are 3 by 2 by 4. The ***volume*** of this rectangular prism is $V = 24$ because it can be decomposed into twenty-four pieces, each which is an identical copy of our unit cube.

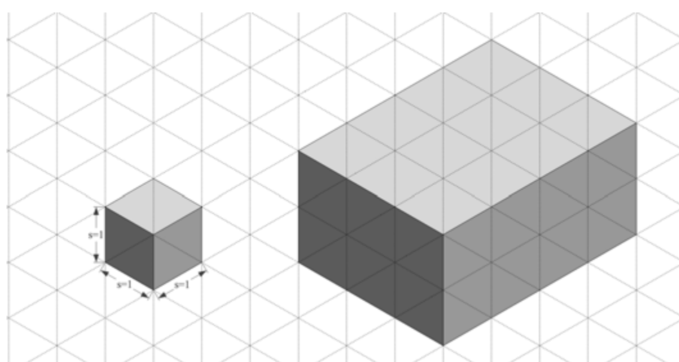


Figure 6.5: Unit cube and rectangular prism.

Using actual cubes, isometric graph paper or a CAD project, choose a unit cube for your investigations.

28. Create a differently sized cube. Call it C_A.

29. Determine the side length and volume of C_A.

30. What are the values of the two ratios

$$\frac{\text{Side Length of } C_A}{\text{Side Length of } C_u} \text{ and } \frac{\text{Volume of } C_A}{\text{Volume of } C_u}$$

and how do they compare?

31. Create a differently sized cube. Call it C_B.

32. Determine the side length and volume of C_B.

33. What are the values of the two ratios

$$\frac{\text{Side Length of } C_B}{\text{Side Length of } C_u} \text{ and } \frac{\text{Volume of } C_B}{\text{Volume of } C_u}$$

and how do they compare?

34. What are the values of the two ratios

$$\frac{\text{Side Length of } C_B}{\text{Side Length of } C_A} \text{ and } \frac{\text{Volume of } C_B}{\text{Volume of } C_A}$$

and how do they compare?

35. Find a formula which relates the two ratios

$$\frac{\text{Side Length of } C_1}{\text{Side Length of } C_2} \text{ and } \frac{\text{Volume of } C_1}{\text{Volume of } C_2}$$

for any two cubes C_1 and C_2. Explain geometrically why it will always hold.

36. Find the surface area of C_A.

37. What are the values of the two ratios

$$\frac{\text{Side Length of } C_A}{\text{Side Length of } C_u} \text{ and } \frac{\text{Surface Area of } C_A}{\text{Surface Area of } C_u}$$

and how do they compare?

38. Find the surface area of C_B.

39. What are the values of the two ratios

$$\frac{\text{Side Length of } C_B}{\text{Side Length of } C_u} \text{ and } \frac{\text{Surface Area of } C_B}{\text{Surface Area of } C_u}$$

and how do they compare?

40. What are the values of the two ratios

$$\frac{\text{Side Length of } C_B}{\text{Side Length of } C_A} \text{ and } \frac{\text{Surface Area of } C_B}{\text{Surface Area of } C_A}$$

and how do they compare?

41. Find a formula that relates the two ratios

$$\frac{\text{Side Length of } C_1}{\text{Side Length of } C_2} \text{ and } \frac{\text{Surface Area of } C_1}{\text{Surface Area of } C_2}$$

for any two cubes C_1 and C_2.

Written by **Euclid** (Greek mathematician; circa 340 B.C. - circa 280 B.C.), the thirteen volume Elements is one of the most important texts in history. It treated much of the mathematics that was known at that time, and did so in what was the first systematic, organized, and logically rigorous way. It contains results on circles and spheres that are analogous to those you have discovered above. Namely, in Book VII, Proposition 2, Euclid proves, "Circles are to one another as the squares on their radii." In other words:

$$\frac{\text{Area of } Circle_1}{\text{Area of } Circle_2} = \left(\frac{\text{Radius of } Circle_1}{\text{Radius of } Circle_2}\right)^2.$$

Later, in Proposition 18 of Book XII Euclid proves, "Spheres are to one another as the triplicate ratio of their respective radii." In other words:

$$\frac{\text{Volume of } Sphere_1}{\text{Volume of } Sphere_2} = \left(\frac{\text{Radius of } Sphere_1}{\text{Radius of } Sphere_2}\right)^3.$$

Surprisingly, Euclid said nothing about the perimeter (a.k.a circumference) of the circle and nothing about the surface area of a sphere.

42. What do you think an appropriate relationship between

$$\frac{\text{Perimeter of } Circle_1}{\text{Perimeter of } Circle_2} \text{ and } \frac{\text{Radius of } Circle_1}{\text{Radius of } Circle_2}$$

would be and any two circles $Circle_1$ and $Circle_2$? Explain.

43. What do you think an appropriate relationship between

$$\frac{\text{Surface Area of } Sphere_1}{\text{Surface Area of } Sphere_2} \text{ and } \frac{\text{Radius of } Sphere_1}{\text{Radius of } Sphere_2}$$

would be for any two spheres $Sphere_1$ and $Sphere_2$? Explain.

6.3 Power Laws and Familiar Formulas

Two variable quantities p and x are said to be in a **power law relationship** if there are constants a and d such that

$$p = a \cdot x^d$$

for all p and x in some specified range.

44. If p and x are in a power law relationship, prove

$$\frac{p_1}{p_2} = \left(\frac{x_1}{x_2}\right)^d.$$

45. Apply your formula in Investigation **21** to the unit square and a square, S_s, of side length s. Solve this formula for the Perimeter of S_s.

46. Apply your formula in Investigation **27** to the unit square and a square, S_s, of side length s. Solve this formula for the Area of S_s.

47. Apply your formula in Investigation **35** to the unit cube and a cube, C_s, of side length s. Solve this formula for the Volume of C_s.

48. Apply your formula in Investigation **41** to the unit cube and a cube, C_s, of side length s. Solve this formula for the Surface Area of C_s.

You should recognize these formulas as familiar formulas that you learned already some time ago. And you may wonder why we have taken such time considering relationships between ratios when we could have just relied in these tried-and-true formulae. One reason is that many people have not understood why these formulas hold and we hope these investigations have helped you make sense of them. But there are two other important reasons for what follows.

First, our understanding of these results developed historically via ratios. And while Investigation **44** - Investigation **48** suggest power law relationships and the types of ratios we're considering are synonymous, there is an essential distinction. If we apply Euclid's VII.2 to a unit circle, C_u and circle of radius r, denoted by $Circle_r$, we have

$$\frac{\text{Area of } Circle_r}{\text{Area of } Circle_u} = \left(\frac{r}{1}\right)^2$$

which we can rewrite as:

$$\text{Area of } Circle_r = \text{Area of } Circle_u \cdot r^2.$$

This is wonderful. *But* we have *no idea* what the area of a unit circle is. Neither Euclid nor any of his contemporaries were know to have suggested what constant may be needed here. Your observation in Investigation **42** suggests

$$\text{Perimeter of } Circle_r = \text{Perimeter of } Circle_u \cdot r.$$

While surveyors and artisans of the time were likely aware that such a result held and that the perimeter of a unit circle was about 3, nothing precise about this constant was known. Similarly, Euclid's XII.18 and your observation in Investigation **43** yield

$$\text{Volume of } Sphere_r = \text{Volume of } Sphere_u \cdot r^3$$

and

$$\text{Surface Area of } Sphere_r = \text{Surface Area of } Sphere_u \cdot r^2.$$

Again, these results are useful only if the volume of the unit sphere and surface area of the unit sphere are known.

So there are four missing constants, those for the perimeter and area of a unit circle and those for the surface area and volume of a unit sphere. For a square the analogous constants are 4 and 1 and for a cube they are 6 and 1. On the face of it, there is nothing to suggest that the constants for circles and spheres should be related in any particular way.

Around the time of the death of Euclid is born the great **Archimedes of Syracuse** (Greek mathematician, inventor, physicist and astronomer; 287 B.C. - 212 B.C). Archimedes made some of the most profound contributions in the history of mathematics - including beautiful analysis that would foreshadow the development of calculus almost two millennia later. In Measurement of a Circle and On the Sphere and Cylinder Archimedes discovered and proved that these constants are $2\pi, \pi, \frac{4\pi}{3}$, and π. I.e. he proved that:

$$\text{Perimeter of } Circle_r = 2\pi r$$
$$\text{Area of } Circle_r = \pi r^2$$
$$\text{Surface Area of } Sphere_r = \frac{4\pi}{3} r^2$$
$$\text{Volume of } Sphere_r = \pi r^3.$$

It is remarkable that each involves the same constant, π. Realizing the importance of this constant which united all of these important measures, he determined the value of *pi* correct to two decimal places.[2]

The second important reason for considering these ratios is that its is this representation of these types of results that allows us to analyze wonderful new shapes that are known as *fractals*.

6.4 Fractals

> Why is geometry often described as cold and dry? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line. . . Nature exhibits not simply a higher degree but an altogether different level of complexity.
>
> **Benoit Mandelbrot** (; -)

> Fractal geometry will make you see everything differently. There is danger in reading further. You risk the loss of your childhood vision of clouds, forests, flowers, galaxies, leaves, feathers, rocks, mountains, torrents of water, carpets, bricks, and much else besides. Never again will your interpreation of these things be quite the same.
>
> **Michael F. Barnsley** (; -)

The word *fractal* comes from the Latin fractus which means fractured or broken. The term was coined in the mid-1970's by Benoit Mandelbrot who was the first to exploit advances in modern computers and computer graphics to explore many fabulously complex objects that had been invented by famous mathematicians such as **Pierre Joseph Louis Fatou** (French mathematician; 1878 - 1929), **Gaston Maurice Julia** (French mathematician; 1893 - 1978), **Niels Fabian Helge von Koch** (Swedish mathematician; 1870 - 1924), and **Waclaw Franciszek Sierpinski** (Polish mathematician; 1882 - 1969) some 70 years earlier.

Of course, like many things, people of earlier eras had discovered objects like this well before they were re-discovered by mathematicians. Sierpinski's triangle can be found in Italian tilework 600 years earlier. Shown in Figure 6.6, "The Wave" is one of the most famous woodblock prints of **Katsushika Hokusai** (Japanese artist; 1760 - 1849) and it shows clear fractal-like structure.

The fractals we will build and analyze here and below are *self-similar fractals*.

[2]For a brief, beautiful treatment of Archimedes and these results, see Chapter 4 - Archimedes Determination of Circular Area in the book Journey Through Genius: The Great Theorems of Mathematics by William Dunham. For much more on π see the section "π - A Final Case Study" in the chapter "Establishing Truth: Certainty and Burdens of Proof" in Discovering the Art of Mathematics Truth, Reasoning, Certainty & Proof.

Figure 6.6: "The Wave" by Katsushika Hokusai.

Many pieces of online software and smartphone apps allow you to create self-similar fractals. One that is particularly robust and user-friendly is the spatially-oriented programming environment Recursive Drawing (`http://recursivedrawing.com`) developed by Toby Schachman. Two student-generated images, including one that bears a striking resemblance to the fractal structure Hokusai's work, are shown in Figure 6.7.

49. Using Recursive Drawing, or some other piece of software that enables dynamic creation of self-similar fractals, experiment with the fractal-generating process at work. Are there particular forms or patterns that remind you of natural or artistic things that you recognize?

50. Use your experience to describe what makes the images that you are creating self-similar.

51. After you have experimented, create your own, original, self-similar fractal. Describe your fractal and both the artistic and mathematical aspects you have chosen to employ in the design and creation of your fractal.

Another way to build self-similar fractals is to use a *iterative process* where the shape is built on smaller and smaller scales by the application of a single set of defining rules:[3]

- Choose an **Initiator** which is your starting shape.
- Choose a **Generator** which is a collection of scaled copies of the Initiator.
- In each stage replace every copy of the Initiator in the existing stage with an appropriately scaled copy of the Generator.
- Continue this process infinitely, arriving at a self-similar fractal in the *limit*.

[3]The approach and terminology used here are from Michael Frame's Fractal course taught for many years at Yale University.

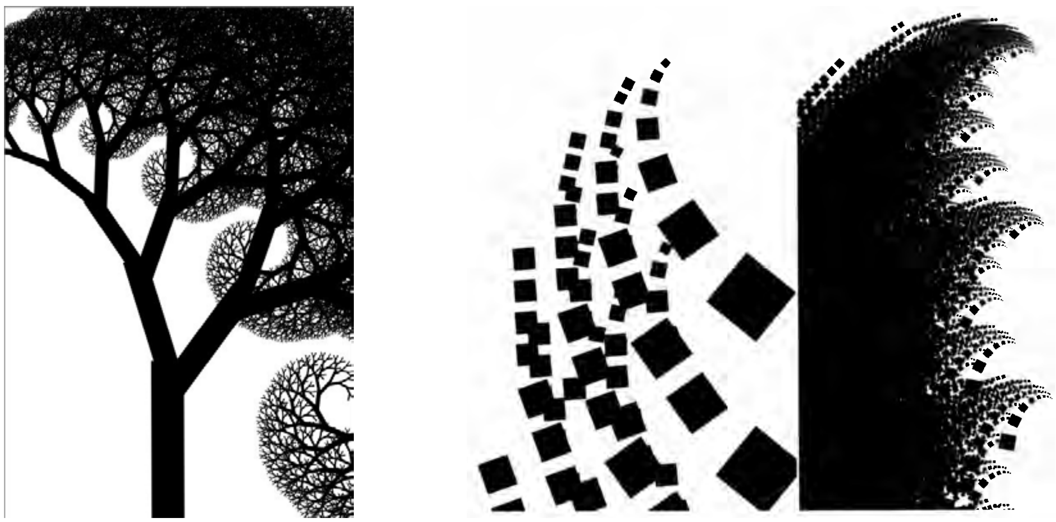


Figure 6.7: Fractal tree (by Rachel Sherman) and fractal wave (by Ethan Goldberg) created using Recursive Drawing.

When naming the stages, the initiator is generally considered stage 0 and the generator stage 1.

Pictured in Figure 6.8 is the **Menger Sponge** invented by **Karl Menger** (Austrian-American mathematician; 1902 - 1985) in 1926. The initiator is simply a unit cube, shown on the left. The generator is the figure second from the left. One way to think about the generator is that its is formed by cross-sectioning the initiator by cuts that are parallel to each face and trisect the edges, then removing all cubies on each face as well as the one in the center.

Another way to think of the generator is through the *iterative* process that defines the fractal in the limit.

52. How many copies of the Initiator are used to create the Generator?

53. How are the linear dimensions scaled as the initiator is used in creating the generator? (This is called the ***linear scaling factor***.)

54. What is the volume of each of the scaled copies of the Initiator in the Generator?

55. What is the volume of the Generator? (Hint: Here and below it will help in locating patterns if you leave your results as fractions.)

56. How many copies of the Generator are used in creating Stage 2?

57. What is the volume of each of the scaled copies of the Generator in Stage 2?

58. What is the volume of Stage 2?

59. How many copies of Stage 2 are used in creating Stage 3?

60. What is the volume of each of the scaled copies of Stage 2 in Stage 3?

61. What is the volume of Stage 3?

62. Find an expression for the volume of Stage n.

63. Make a table and/or plot data for values for n to analyze what happens to the volume in Stage n for larger and larger values of n.

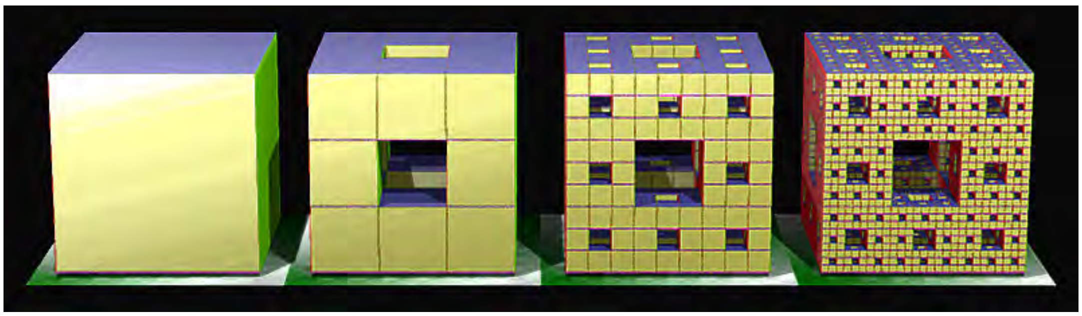


Figure 6.8: Menger Sponge: Stage 0 = Initiator, Stage 1 = Generator, Stage 2, and Stage 3 (from left to right).

The Menger Sponge is the limiting object obtained from this process. Although one starts with a cube, the process pokes an unbounded number of (square) holes through the cube. So many (square) holes are poked that the Menger Sponge has zero volume (as results in Investigation **63** should show)! There are so many holes that if feels almost as if the Menger Sponge becomes invisible. Has it slipped out of the realm of being three-dimensional?

A wonderful thing about natural and synthetic sponges is their ability to "soak up" water. How do they work? They have an excessive number of pores throughout their volume. These pores contribute enormous surface area relative to their volume and water has all of this surface area to adhere to.

If we could actually make one, the Menger Sponge would be the ultimate sponge, for it has infinite surface area! (See the Further Investigations Section below for details.)

A finite object with zero volume and yet infinite surface area? What dimension does this remarkable object naturally occupy?

6.4.1 The Sierpinski Triangle

The Sierpinski triangle was rediscovered by Sierpinski in 1915. Sierpinski was held in such high regard by his contemporaries that one of the craters on the moon is named in his honor.

Although Sierpinski was the first (known) to describe and analyze this object mathematically, similar figures were invented as early as the 13th century when it was used in the Cosmati mosaic tile work in the Cathedral of Agnani in Italy. As shown in Figure 6.9 the initiator is an equilateral triangle and the generator three equilateral triangles surrounding an empty space created by an equilateral triangle of the same size.

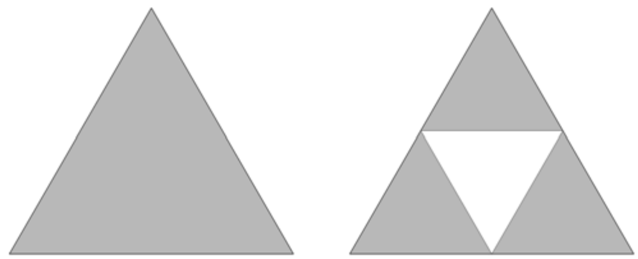


Figure 6.9: Sierpinski triangle Initiator and Generator.

In the Appendix are copies of *isometric dot paper* which is particularly useful for creating equilateral triangles and that where the Initiator has been indicated.

64. Create Stage 2 of the Sierpinski triangle.

65. On a separate sheet, create Stage 3 of the Sierpinski triangle.

66. On a separate sheet, create Stage 4 of the Sierpinski triangle.

67. On a separate sheet, create Stage 5 of the Sierpinski triangle.

68. Consider the Initiator of the Sierpinski triangle to have unit area; i.e. $A = 1$. What is the area of each copy of the Initiator used to create the Generator?

69. What is the area of the Generator?

70. How many copies of the Generator are used to create Stage 2?

71. What is the area of each of the scaled copies of the Generator used in Stage 2?

72. What is the area of Stage 2?

73. How many copies of Stage 2 are used in creating Stage 3?

74. What is the area of each of the scaled copies of Stage 2 used in Stage 3?

75. What is the area of Stage 3?

76. Find an expression for the area of Stage n.

77. Make a table and/or plot data for values of n to analyze what happens to the area in Stage n for larger and larger values of n.

78. The Sierpinski triangle started from a 2-dimensional triangle. Based on your explorations, does it seem like the Sierpinski triangle is 2-dimensional? Explain.

What about the perimeter of the Sierpinski triangle? As results from Investigation **77** should show, the area of the Sierpinski triangle is zero. It is zero no matter what the units of area measurement is. Since you have drawn the stages of the Sierpinski triangle on dot paper, use the units of this paper to measure perimeter - each side of the Initiator is 32 units long.

79. What is the perimeter of the Initiator?

80. What is the perimeter of the Generator? (Note: Since the interior triangle has been removed, their is boundary on the inside which contributes to the perimeter as perimeter is a measure of the boundary.)

81. What is the perimeter of Stage 2?

82. What is the perimeter of Stage 3?

83. What is the perimeter of Stage 4?

84. What is the perimeter of Stage 5?

85. What can you conclude about the perimeter of the Sierpinski triangle? How does this result compare with the earlier analysis of the Menger Sponge?

6.4.2 Koch Curve

The next fractal you will create is the Koch curve, invented by Koch in 1904. The **Koch curve** is the self-similar fractal formed from the Initiator and Generator shown in Figure 6.10. Notice that the Generator is formed by dividing the Initiator in thirds and replacing the middle third with two legs of an equilateral triangle whose base is the size of the removed piece. Scale copies of the Koch curve Initiator and Generator, on isometric dot paper, are included in the Appendix.

Figure 6.10: Koch curve initiator and generator.

86. Create stage 2 of the Koch curve.

87. How many line segments make up Stage 2? If we set our scale so the length of the Initiator is 1 unit, how long is each of the line segments that make up Stage 2? So what is the overall length of Stage 2 of the Koch curve?

88. Create Stage 3 of the Koch curve.

89. How many line segments make up Stage 3? How long is each line segment? So what is the overall length of Stage 3 of the Koch curve?

90. Create stage 4 of the Koch curve.

91. How many line segments make up Stage 4? How long is each line segment? So what is the overall length of Stage 4 of the Koch curve?

92. Find an algebraic expression for the length of Stage n of the Koch curve.

93. Return to the Initiator and Generator. Explain how interpret your expression for the length of Stage n of the Koch curve in terms of the geometry of the Initiator and Generator.

94. What will the length of the Koch curve be? Explain.

95. Each stage of the Koch curve is simply a collection lines, small one-dimensional pieces. Does it seem like the Koch curve will be one-dimensional? Explain.

In the "Introduction" the **Koch snowflake** appeared as an illustration. The stages in its construction are shown in Figure **??**.

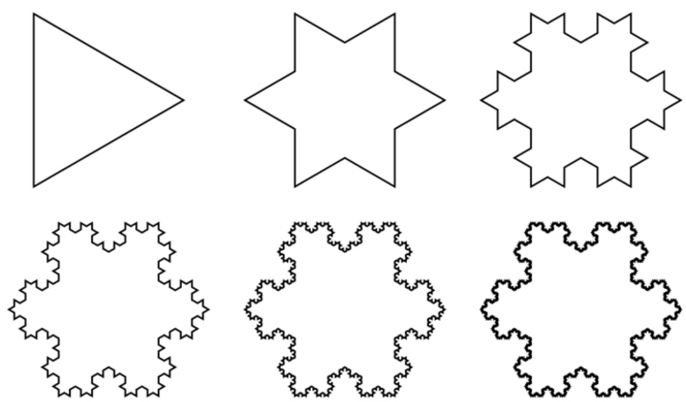


Figure 6.11: Stages in the construction of the Koch snowflake.

96. Explain how the boundary of the Koch curve can be constructed by copies of the Koch curve.

97. What is the minimum number of copies of the Koch curve that can be used to construct the boundary of the Koch snowflake? Explain.,

In "What is Area?" in Discovering the Art of Mathematics - Calculus the area of the Koch snowflake is investigated. Assuming that the sides of the equilateral triangle in Stage 0 are $s = 1$, the area of the Koch snowflake is given by the infinite, *geometric series*

$$A = \frac{\sqrt{3}}{4}\left(1 + \frac{3}{9} + \frac{12}{81} + \frac{48}{729} + \dots\right).$$

98. Explain how each of the terms in parenthesis in expression for the Koch curve area arises.

99. Explain why, despite this series continuing indefinitely, the area of the Koch snowflake must be finite.

The sum of the infinite series, and thus the area of the Koch snowflake, is $A = \frac{\sqrt{3}}{4}\left(\frac{8}{5}\right)$. So, again, we have a situation where an object we have created has a finite area but an infinite perimeter.

6.4.3 Cantor Set

The last fractal you will build is known as the *Cantor set* after **Georg Cantor** (German mathematician; -) whose pioneering work on the infinite[4] was transformative and nicely complements the development of fractals:

> I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not - I do not say divisible - but actually divisible; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

Interestingly, the Cantor set was only rediscovered by Cantor, it had been previously invented by **H.J.S. Smith** (; -) more than 10 years early. Unfortunately, his contribution went unnoticed until much more recently.[5]

The **Cantor set** is the self-similar fractal formed from a initiator that is a line segment of unit length and a generator that is comprised of two line segments of length one-third and which are separated by an empty space one-third of a unit wide - as shown in Figure 6.12. A scale copy of the Cantor set initiator and generator, with nicely sized rule lines, is included in the Appendix.

100. Create Stage 2 of the Cantor set.

101. How many line segments make up Stage 2? How long is each line segment (in terms of the initiator which is 1 unit long)? So what is the overall "length" of Stage 2 of the Cantor set?

102. Create Stage 3 of the Cantor set.

103. How many line segments make up Stage 3? How long is each line segment? So what is the overall "length" of Stage 3 of the Cantor set?

104. Create Stage 4 of the Cantor set.

105. How many line segments make up Stage 4? How long is each line segment? So what is the overall "length" of Stage 4 of the Cantor set?

106. Find an expression for the "length" of the Cantor set in Stage n.

[4] This topic is the central focus of the book Discovering the Art of Mathematics - Infinities.

[5] See "A History of the Cantor Set and Cantor Function" by Julian F. Fleron, *Mathematics Magazine*, vol. 67, no. 2, April 1994, pp. 136-40.

107. Return to the Initiator and Generator. Explain how interpret your expression for the length of Stage n of the Cantor set in terms of the geometry of the Initiator and Generator.

108. What seems to be happening as we continue this process? What will the "length" of the Cantor set will be?

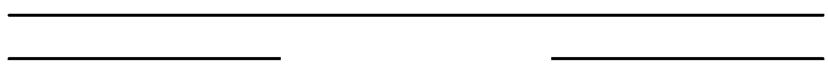


Figure 6.12: Initiator (top) and generator (bottom) for the Cantor set.

The Cantor set does seem to disappear in front of our eyes as we try to generate it. Many have called it "Cantor dust" for this reason. And while it does have *measure* zero it has not disappeared completely. At each stage we only remove the *open* middle third so the endpoints of each copy of the initiator will forever remain. If the initiator was build on a standard number line so its endpoints were $x = 0$ and $x = 1$ then the endpoints $x = \frac{1}{3}$ and $x = \frac{2}{3}$ of the copies of the Initiator that make up the Generator will never be removed; i.e. are points in the Cantor set. And after Stage 2 the points $x = \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}$ will be be endpoints adjacent to middle thirds that have been removed. Hence, they will be in the Cantor set. And, in fact, it is not just points like this that are in the Cantor set. There are exactly as many points in the Cantor set as there are in the whole number line - an amazing discovery of Cantor. In terms of physical size the Cantor set is vanishingly small, in terms of how numerous its elements are it is as large as the line it was first created from![6]

So how can one object be so sparse in some sense and so enormous in another? Perhaps because we are viewing it from the confines of our "limited dimensionality" that A Square spoke of in Flatland.

The Cantor set was born from an Initiator and Generator that were one-dimensional. While it is not one-dimensional, it is far from the zero-dimensional inhabitants of Pointland.

6.5 Fractal Dimensions

> Dimension is not easy to understand. At the turn of the century it was one of the major problems in mathematics to determine what dimension means and which properties it has. And since then the situation has become somewhat worse because mathematicians have come up with some ten different notions of dimension: topological dimension, Hausdoff dimension, fractal dimension, self-similarity dimension, box-counting dimension, capacity dimension, information dimension, Euclidean dimension, and more. They are all related. Some of them, however, make sense in certain situations, but not at all in others, where alternative definitions are more helpful. Sometimes they all make sense and are the same. Sometimes several make sense but do not agreee. The detials can be confusing even for a research mathematician.[7]
>
> **Heinz-Otto Peitgen** (; -)

[6] For more on the Cantor set, see the corresponding chapter in Discovering the Art of Mathematics - Calculus.
[7] From Fractals in the Classroom.

Hartmut Jurgens (; -)

Dietmar Saupe (; -)

In your investigations of scaling perimeters, areas and volumes, each of the results was characterized by:

$$\left(\frac{\text{Length}_1}{\text{Length}_2}\right)^d = \frac{\text{Measure}_1}{\text{Measure}_2} \tag{6.1}$$

where:

- Perimeters, which measure 1-dimensional objects, had $d = 1$,
- Areas and surface areas, which measure 2-dimensional objects, had $d = 2$, and,
- Volumes, which measure 3-dimensional objects, had $d = 3$.

We can use this exact idea in the context of self-similar fractals to give a definition of **self-similar dimension**, or just dimension for short. All we must do is scale our self-similar fractal as we did with squares, cubes, and circles earlier and determine the relationship between the relative linear dimensions and relative measures using 6.1. The self-similar dimension is the unique number d which satisfies this equation.

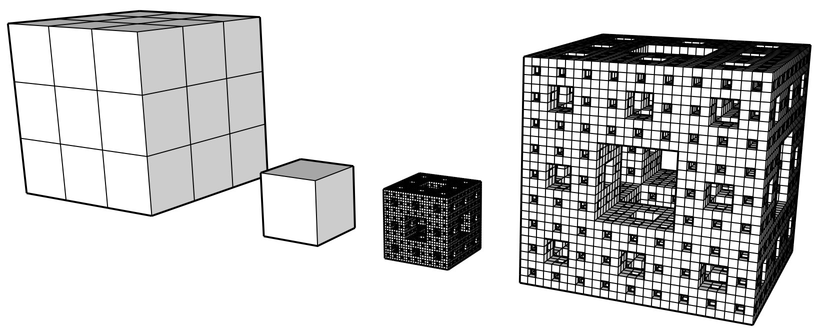


Figure 6.13: Scaled cube and Menger sponge.

Consider the example of the Menger sponge created above. Once the building process is completed, we can scale up the Menger sponge. If the linear scaling factor is 3 then the scaled Menger sponge is 20 copies - or 20 times the measure - of the original Menger sponge. (You showed this in Investigation **52**). As illustrated in Figure 6.13 for the scaled cube and scaled Menger sponge, respectively, the ratios in the equation that defines dimension 6.1 are:

$$\left(\frac{3}{1}\right)^d = \frac{27}{1} \text{ and } \left(\frac{3}{1}\right)^d = \frac{20}{1}.$$

We know $d = 3$ is the unique dimension d which satisfies the equation for the cube. So what is the dimension of the Menger sponge? It is the number d such that $3^d = 20$. $d = 2$ is too small as $3^2 = 9$, so the Menger sponge is not 2-dimensional. But $d = 3$ is too big as $3^3 = 27$. So the dimension of the Menger sponge is *between* 2 and 3. In fact, the dimension of the Menger sponge is $d = \frac{\ln 20}{\ln 3} \approx 2.72683$ as you can check with your calculator.

109. Find an appropriate factor to scale up the Cantor set and compare the Cantor set and its scaled up version by filling appropriate values into equation 6.1.

110. Using your calculator, find an approximate value for d that solves this version of equation 6.1.

The number you have just found is the dimension of the Cantor set. Again, it is *not* a whole number. The Cantor set is so small that it lives in a dimension below the linear, first dimension.

At this point some people object that this is all a clever ploy to describe complex mathematical objects. Dimensions that are whole numbers don't really exist. (See Further Investigation **F9** to consider other potential objections.)

In fact, they exist. Barnsley's warning above was echoed many years before when Mandelbrot first began to champion the notion of fractals in his groundbreaking book The Fractal Geometry of Nature:

> Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

111. Pictured in Figure 6.14 is bolt of lightning. Explain how the lightning bolt exhibits signs of self similarity.

112. Similarly, explain how clouds exhibit signs of self-similarity.

113. Similarly, explain how the fern leaf exhibits signs of self-similarity.

114. The fern leaf in Figure 6.14 was generated electronically on a computer by Barnsley. If it had a realistic background, would you think it was a forgery or would you think it was real?

115. What about mountains, rivers, and tree bark? Are they self-similar? Explain.

116. Find a dozen other examples of real-world objects that exhibit self-similarity.

A wonderful statement of the role of fractals in the universe is from **H.W. Smith** (Artist and mathemtician; -) who founded Art Matrix in the late 1970's to help share the beauty of fractals with others:

> The physical universe is basically an iterated system, so actually it is surprising we have made the progress we have, using only simple evaluation. The equations have been around forever. The physical universe has been USING them almost forever. The equations have as part of their very nature things like fixed points, period cycles, chaotic cycles, basins of attraction, etc., so you can be sure all these things are manifested in the physical universe INCLUDING FRACTALNESS.
> To say therefore that fractals have nothing to do with anything and have not explained or proven useful in our understanding of the universe is more a statement about the people who are working with fractals rather than a statement about the pertinence

of fractals to the world at large. Fractals are so pertinent to the universe no one can see it yet. Long time ago, they thought math did not pertain either. The "why" was God. The "why" might still be God, but if it is, then clearly God is a mathematician of significant merit, and no doubt a fractal enthusiast.

Images of fractals abound on the Internet. We highly recommend Viewpoints: Mathematical Perspective and Fractal Geometry by Annalis Crannell and Marc Franz which could serve as a text for a course very much like the one you are currently enrolled in. Books considered the classics in the area include The Fractal Geometry of Nature by Mandelbrot, The Beauty of Fractals: Images of Complex Dynamical Systems by **Heinz-Otto Peitgen** (; -) and **P.H. Richter** (; -), and Fractals Everywhere by Barnsley. We also recommend Chaos Under Control: The Art and Science of Complexity by **Michael Frame** (; -) and **David Peak** (; -).

So fractals are part of nature. When did humans catch on? In terms of the history European/American mathematics one of the oldest fractals is the *Cantor set* which you will investigate below. Funny thing, while the set is named after **Georg Cantor** (German mathematician; -), widely considered the father of the infinite in mathematics, it was discovered a decade earlier by the lesser known **H.J.S. Smith** (; -).[8] As noted, discoveries and investigations by von Koch, Sierpinski, Fatou and Julia followed. Also notable was the work by **Lewis Fry Richardson** (American meteorologist, physicist, mathematician and teacher; -) who measured coastlines on smaller and smaller scales. Looking back on his work now we would say that he discovered that many coastlines are fractals *and* he even found the fractal dimensions of these coastlines significantly before Mandelbrot and others brought things into a systematic organization.[9]

We should note that like many histories, that related here is narrow and one-sided. In his book African Fractals: Modern Computing and Indigenous Design **Ron Eglash** (; -) provides significant evidence that fractal-like design has been used in Africa for decades. Since fractals occur in nature, it is likely that many other indigenous cultures used them as well. Which makes one wonder why the European/Americans took so long to catch on. Aren't we supposed to be more sophisticated?

6.6 A Few More Fractals

To construct the Cantor set begin by drawing a line segment of unit length.

117. Divide this segment into thirds. You are to remove the middle third, leaving the endpoints $\frac{1}{3}$ and $\frac{2}{3}$. Draw a picture of what remains after this first step.

118. Divide each of the line segments that remain from Investigation **117** into thirds. Delete each of the middle thirds, again leaving the endpoints. Draw a picture of what remains after this second step.

119. Divide each of the line segments that remain from Investigation **118** into thirds. Delete each of the middle thirds, again leaving the endpoints. Draw a picture of what remains after this third step.

[8] See "A history of the Cantor set and Cantor function" by Julian F. Fleron in ???

[9] See the discussion in the chapter "How long is the coast of Britian" in Mandelbrot's Fractals: Form, Chance and Dimension for more details.

120. Draw a picture of what remains after the fourth step in this construction.

The **Cantor set** is the set of points that are never removed from the unit interval when this process is continued indefinitely.

121. Find a number of points that are in the Cantor set. How many such points are there?

122. One can show that the point $\frac{1}{4}$ is not an endpoint of any of the removed intervals. Nonetheless, it is a member of the Cantor set. Does this surprise you? Why?

123. What is the linear scaling factor for the Cantor set?

124. Determine the self-similarity dimension of the Cantor set.

Do Menger Sponge.

6.7 Conclusion

How to end? Do the bread, balled paper and other do-it-yourself fractals examples. Video feedback is good too. Get student to think about the fact that these things really should be intermediaries in the dimensional ladder.

> You should really think of my pictures [of fractals] as a metaphor for all living things.
>
> **John Hubbard** (French mathematician; -)

[10]

[10] From the videotape "The Beauty and Complexity of the Mandelbrot Set."

6.8 Further Investigation

6.8.1 Zipf's Law and the Pareto Principle

Just before his death, **George Kingsley Zipf** (American Linguist and Statistician; 1902 - 1950) self published the book Human Behavior and the Principle of Least Effort. Benoit Mandelbrot says:

> I know very few books...in which so many flashes of genius, projected in so many directions, are lost in so thick a gangue of wild notions and extravgence. on the one hand it includes a chapter dealing with the shape of sexual organs and another in which the Anschuluss of Austria into [Nazi] Germany is justifiable by means of a mathematical formula. On the other hand... [it has] been of considerable historical importance... and has not yet been exhausted.

The centerpiece of its importance is what has become known as *Zipf's law.*

Do it for language. Do the city thing later? City thing is easier to understand.

Rank order things. The population is then inversely proportional to the rank. Do it first as inversely proportional.

Give references to Pareto. Both his importance in economics and how his ideas foreshadow what we are thinking about below.

F1. Determine an algebraic formula for p as a function of r.

F2. This is well and good. But for our purposes we would like to envision this a bit differently. Now do the scaling thing. If you double the population, what can you expect to happen for the rank? Does it matter where you start? Make sure that the numbers come out well. Maybe this is where we need to have a hypothetical example. Certainly things at least need to work out.

F3. Have them do Zipf's theorem for several countries, using at least a dozen of the biggest cities. How well does it seem to work?

F4. Now show them the log-log data and have them comment on how nicely this fits a line.

> When a straight line, the mathematical evidence of a law, failed ao appear on the paper, I suggested different kinds of logarithmic paper. If you cannot simplify the curve on one kind of paper, simplify the paper.[11]
>
> **Theodore von Karman** (; -)

F5. Now let the cat out of the bag and talk to them about the real percentage error in these things.

F6. Get them to reflect on believing everything that you read.

F7. Analyzing Zipf's theorem leads directly to finite sums of the harmonic series. Have them develop one or two of these so they see the pattern.

[11] Quoted in Fractals: Form, Chance and Dimension by Benoit Mandelbrot, p. 273.

F8. Then make notes about the non-convergence of the harmonic series. Because it diverges so slowly, this is not an issue in any particular instance. However, it does raise some important theoretical considerations. Among them, that the exponent need not be $s = 1$, but rather $s > 1$. All of a sudden we are in the realm of *p-series* and then *Riemann hypothesis* and the *Riemann Zeta function*!!

F9. Give excerpts from Krantz article. Is it available online? Have them comment on it. Maybe compare it to other scientific fads? String theory?

6.8.2 Kleiber's Law

With cubes to make things work out more nicely. It is cool that the major point is that the exact constants that make up the shape do not really matter.

Have them derive the surface area of a cube as a function of the volume. Have them see that it is scale invariant.

As a function of mass, metabolic rate is scale invariant. BUT, the exponent is 3/4, not 2/3. The biological, chemical, and physical issues interact with the geometry.

Discuss how remarkable it is that Kleiber's law holds so universally.

Get them to do an example to show how things scale up and down - mouse and elephant.

6.8.3 More on Scaling Perimeters, Areas and Volumes

In the section Section 6.2 you explored the scaling of perimeters areas and volumes for squares, cubes, circles and spheres. For squares and cubes, the side length was a natural linear dimension to track how the objects were scaled. For circles and spheres the radius is natural. However, neither of these is required to be the linear dimension that gives rise to the scaling factor, nor are the results you obtained particular to these common shapes - these results are universal.

F10. Choose a two-dimensional, planar object whose perimeter and area you can calculate. Also choose a linear dimension that you will track as the object is scaled. Calculate the perimeter and area of your object.

F11. Scale the object by doubling the linear dimension. What are the perimeter and area now?

F12. Scale the object by tripling the linear dimension. What are the perimeter and area now?

F13. Repeat this process for a different shape.

F14. Use these examples to complete the following important theorems, explaining intuitively why they are reasonable results:

Theorem 1. *Let R be a two-dimensional, planar region whose boundary has perimeter P. If R is scaled by a factor of m then the perimeter of the scaled region is* $\boxed{} \times P$.

and

Theorem 2. *Let R be a two-dimensional, planar region whose area is A. If R is scaled by a factor of m then the area of the scaled region is* $\boxed{} \times A$.

F15. Choose a three-dimensional, solid object with two-dimensional boundary whose surface area and volume you can calculate. Also choose a linear dimension that you will track as the object is scaled. Calculate the surface area and volume of your object.

F16. Scale the object by doubling the linear dimension. What are the surface area and volume now?

F17. Scale the object by tripling the linear dimension. What are the surface area and volume now?

F18. Repeat this process for a different shape.

F19. Use these examples to complete the following important theorems, explaining intuitively why they are reasonable results:

Ask them if they can generalize this for volumes:

Theorem 3. *Let R be a three-dimensional solid whose surface area is A. If R is scaled by a factor of m then the area of the scaled region is* $\boxed{} \times A$.

Theorem 4. *Let R be a three-dimensional solid whose volume is V. If R is scaled by a factor of m then the volume of the scaled region is* $\boxed{} \times V$.

6.8.4 Menger Sponge Surface Area

Above you discovered that the Menger Sponge has zero volume and it was noted that it has infinite surface area - a strange behavior that is typical for fractals.

You can determine the surface area as you did other measures of other fractals, by tracking it at each stage.

F20. At each stage the surface area is formed by congruent squares. Find the area of each of these squares as a function of the stage number, n.

F21. The initiator - a cube - has 6 square faces. In the initiator, how many squares make up the part of the surface that lie on original faces of the cube?

F22. In Stage 2, how many squares make up the part of the surface that lie on original faces of the cube?

F23. Continue this pattern to find a formula for the number of squares in Stage n that make up the part of the surface that lie on original faces of the cube as a function of the stage number, n.

F24. Show that the portion of the surface area of the Menger Sponge that lies along the original faces of the cube is zero. Does this compare to other fractals considered above?

The fact that the surface area along the original square faces is zero should not be surprising, it is the holes in a sponge - mathematical, biological, or synthetic - that give it its enormous surface area.

F25. In the generator, how many square faces does each hole add to the surface? How many holes are there in the generator?

F26. How much surface area is added by the holes created in forming the generator?

F27. In Stage 1, how many square faces does each of the smaller holes added in this stage add to the surface? How many such holes are there?

F28. How much surface area is added by the holes created in forming Stage 1?

F29. In Stage 2, how many square faces does each of the smaller holes added in this stage add to the surface? How many such holes are there?

F30. How much surface area is added by the holes created in forming Stage 2?

F31. Find a pattern that allows you to determine how much surface area is added by the holes created in forming Stage n.

F32. Explain why this proves that the surface area of the Menger Sponge is infinite.

6.9 Teacher Notes

> I think that it is time that teachers of geometry become a little more ambitious... It seems to me regrettalbe that students are not given the opportunity, while still at school, of learning a good deal more about the real subject matter out of which modern geometrical systems are built. It is probably easier, and certainly vastly more instructive than a, great deal of what they are actually taught... I have never not yet encountered a student who finds difficulty with such ideas [projective geometry, the nature of axiom systems, and perspective]... [We must] widen the horizon of knowledge, reconising, as regards the niceites of logic, sequence, and exposition, that the elementary geometry of schools is a fundamentally and inevitably illogical subject.
>
> **G.H. Hardy** (English mathematician; -)

[12]

Somewhere in here there needs to be a note that a function is a power function if and only if it scales. This is important to note. We have chosen to approach the topic via the perspective of scaling for several reasons:

- It is least algebra intensive, which is appropriate for this audience.
- It mimics the language in which many in the social sciences, natural sciences, and even humanities talk about growth. Relative growth is critical here. There is no presupposition that exact, closed term formulas are possible or even appropriate. Relative growth and scaling are more natural tools.
- We believe that the development of the definition of self-similarity dimension is then much more natural.

[12] From "What is Geometry?" Hardy's Presidential Address to The Mathematical Association in 1925. Reprinted in The Changing Shape of Geometry edited by Bhris Pritchard, Cambridge University Press, 2003.

6.10 JF Notes

I have some notes written up in Word about the introduction and conclusion. There is also the issue of Dimension paralleling the history of the development of number. Don't forget to look at these later.

6.10.1 Things still to get in there.

Area and perimeter of the Koch snowflake. Very good self-similarity argument in Sandefur, p. 113.

Frame and Peak thing on Chernobyl.

Jackson Pollack fractal dimension thing.

Picture of Newton's method for $z^4 = 1$.

MUST get them to get to the computer and zoom in on these things. Actually, this should go with the stuff about self-similarity.

Certainly the Mandelbrot set must be in there somewhere.

Stephen Smale: The Mathematician Who Broke the Dimension Barrier. What does this refer to? Check this out. Might be good to have something about Smale in here. The links to the now-solved Poincare conjecture. Etc.

Scaling up/down and the link to Penrose tilings. MUST be mentioned.

Put Buzz's picture of the microorganism that looks like a cube here.

Another way to describe how things are scaled is by a magnification factor. Zoom on a camera, Reduce/Enlarge on a photocopy machine, or even the magnification factor on a magnifying glass, microscope, binoculars, or telescope. How would we have to scale, or magnify, the photo for it to be life-sized?

Cut from first draft:

The complexity of these marvelously beautiful objects, several of which you will construct below, is captured by their dimension. Although their infinitely repeating complexity might suggest higher, or infinite, dimensions for these objects, their dimensions capture the infinite in another, finite way. Namely, their dimensions are represented by numbers other than whole numbers! Their definitions are given by infinite decimals which are of finite size. Numbers such as 0.6309... and 1.5850... They are objects that are neither 0-dimensional, 1-dimensional, nor 2-dimensional. They are objects that "live" between our familiar dimensions.

The history of number saw first the natural numbers 1, 2, 3, ... and then, over the millenia, increasingly complex and surprising expansion of our understanding of number. Our study of dimension is beginning to look similar. We have managed to move beyond the 0th, 1st, 2nd, and 3rd dimensions - to the 4th, 5th, ..., and even to infinite dimensions. Now we have made the step to decimal dimensions.

Perhaps this sounds fabulously complicated. But in fact, much of what we need to fill out the claims above is simply a reinterpretation of our notion of the scaling exponent from the previous lesson. Let us start with the unit cube. Instead of thinking of the cube in relation to some larger cube, like we did in the Delian problem, let us think of the unit cube on its own. In analogy with problem ?? we can decompose this cube into eight smaller cubes, as shown in the figure below.

Notice that the length of any portion of one of these smaller cubes (edge, diagonal, etc.) is exactly half a long as the corresponding length in the original cube. We say that the scaling factor between the larger and smaller cubes is s = 1/2. The unit cube is composed of 8 smaller cubes, and cubes have dimension 3. How are the numbers 1/2, 3 and 8 related? By the formula sd = 1/n, where d is the dimension, n is the number of pieces, and s is the scaling factor. For certainly we have sd = (1/2)3 = (1/2)(1/2)(1/2) = 1/8 =1/n.

Following Mandelbrot, we'll call objects whose self-similarity dimension is not a natural number a fractal. You'll meet several of these objects below. Like the Cantor set which has a similarity with 2 components and scaling factor 1/3, meaning that its dimension d must satisfy (1/3)d = 1/2. The number d that satisfies this equation is the irrational number d = (log 2)/(log 3) = 0.6309... We have an entire new species of object, those with dimensions that are not whole numbers! The Pythagoreans would be truly shocked.

Introduce the term **scale invariance**.

6.11 Scaling

Euclid and circles:

$$\frac{A_{C_1}}{A_{C_2}} = \left(\frac{r_1}{r_2}\right)$$

We will call the quantity on the left the area scale. We will call the quantity on the right the radii scale.

Notice that the radii scale is a linear scale - it is a length that is to be measured.

Links back to scaling section.

We have seen a number of examples from different fields. From this point forward we will be considering only geometric objects. So let us see how our results specialize.

In each of the investigations ???? the result has the form:

$$\left(\frac{\text{Length}_0}{\text{Length}_1}\right)^d = \frac{\text{Measure}_0}{\text{Measure}_1} \tag{6.2}$$

Here the ratio in parenthesis on the left is the **linear scaling factor**. It is the degree in which we have magnified the linear scale. The ratio on the right is the measured result of this scaling - be it perimeter, area, volume or any other measure.

Again, this language is more than two centuries old. The *only* shift in thinking that is needed to arrive at fractals is to reconceptualize the ratio when considering *self-similar* geometric objects.

125. Divide a square each of whose sides is $\text{Length}_0 = 1$ into four smaller, congruent squares. What is the length, Length_1, of each of these smaller squares? What are the areas, Measure_0 and Measure_1, of the larger and smaller squares?

126. What is the linear scaling factor in this situation? How is the area scaled?

127. What exponent d will make the equation in 6.2 hold?

128. Repeat Investigation **125** when the unit square is divided into nine smaller, congruent squares.

129. What is the linear scaling factor? How is the area scaled?

130. What exponent is needed now?

131. Can you repeat Investigation **125** with different linear scaling factors? If so, explain by describing how the area has been scaled and what happens to the exponent d. If not, explain why not.

What about cubes?

132. Show how a cube, each of whose sides is $\text{Length}_0 = 1$ can be divided into eight smaller, congruent cubes. What is the linear scaling factor? How has the volume been scaled?

133. What exponent d will make the equation in ?? hold?

134. Repeat Investigation **132** with linear scaling factor $\frac{1}{3}$. How has the volume been scaled?

135. What exponent d will make the equation in ?? hold?

136. Can you repeat Investigation **132** with different linear scaling factors? If so, explain by describing how the volume has been scaled and what happens to the exponent d. If not, explain why not.

137. Can the process in Investigation **125** and Investigation **132** be carried out for lines? If so, describe how and what the results are. If not, explain why not.

138. In these examples, how much the length, area, and volume have been scaled can be determined without actually calculating any of the lengths, areas, or volumes because of self-similarity. Explain how.

A square is, of course, two dimensional and a cube three dimensional. You've known this for years. In the previous chapter we observed how certain patterns allowed us to codify what we meant by topological dimension. We saw this extended our notion of dimension, allowing us to investigate higher dimensions that the third in a rigorous way.

The situation is similar here. (Please excuse the pun.) The following can be taken as the formal definition of dimension. Lines will have dimension 1, squares dimension 2, and cubes dimension 3, just as they should.

The **self-similarity dimension** of a self-similar object is the unique number d which satisfies

$$\left(\frac{\text{Length}_0}{\text{Length}_1}\right)^d = \frac{n_0}{n_1} \tag{6.3}$$

where $\frac{\text{Length}_0}{\text{Length}_1}$ is the linear scaling factor and n_1 is the number of scaled copies it takes to make the original object which consists of n_0 components.

$\frac{n_0}{n_1}$ comes from your observations in Investigation **138**. In each of the examples above $n_0 = 1$. This is generally the case.

Figure 6.14: Lightning bolt, clouds, and a fern leaf.

Figure 6.15: Diagram for Investigation ??.

Chapter 7

So, What is Geometry? [Working Draft]

7.1 Introduction

At the outset of this book we urged you to think about entire new worlds simply by opening your minds. We hope that your exploration through dimension has been mind opening. We hope that you have broken free of some of the "prejudices of your dimensions." We hope that you are more comfortable in recognizing all sorts of geometry in the three-dimensional world around you. And we certainly hope that your exploration of the higher dimension and non-integer dimensions was compelling.

If these things have happened, or if other things of value have happened, then this is success. Wonderful!

Should we just end here? You may if you choose. But our hope is that you will continue on a bit more - looking forward to other areas of geometry to explore. We wanted to open your eyes to areas of geometry that are often left out of the typical school geometry. As we worked to decide what areas of geometry to include we had a very hard task. For we could have written a several books about wonderful, accessible areas of geometry that most people are wholly unaware of. It was tough to pick.

As we progressed through the book we have tried to include links to some of these areas so you can find out more. In some cases others have already written wonderful books that we have mentioned: Mathematical Perspective and Fractal Drawing by Crannell and Franz; Groups and Symmetry by Farmer; Symmetry, Shape, and Space by Kinsey and Moore; etc. In other cases there is not as much, yet.

But there is a big unanswered question given all of this: what is geometry? If so many people are unaware of these areas of geometry which we claim are fundamental parts of geometry then what is going on? Why is this geometry so different than what we saw through elementary, middle, and high school? We think this is a fundamental question to address.

This then is the main topic of this final chapter.

To be able to answer this question there are some areas that are critical to mention - areas that we have yet to mention yet. Additionally, because most experiences with school geometry are so limited, some historical perspective is important as well.

So, in here we will guide you through an exploration of some key areas of geometry and some revolutions in the subject that will help us answer the question, "So, what is geometry anyway?"

7.2 Ancient Geometry

The etymology of the word geometry is from the combination of two Greek words: *ge* for "earth" and *metréō* for "to measure." Ancient geometry, as far as we can tell from the historical and archeological record, began as a practical art. **Herodotus** (Greek Historian; -) reports what he learned on a visit to the Nile River valley circa 460 B.C.:

> They said also that this king [Sesostris] divided the land among all Egyptians so as to give each one a quadrangle of equal size and to draw from each his revenues, by imposing a tax to be levide yearly. But every one from whose part the river tore away anything, had to go to him and notify what had happened. He then sent the overseers, who had to measure out by how much the land had become smaller, in order that the owner might pay on what was left, in proportion to the entire tax proposed. In this way, it appears to me, geometry originated.

As the surveyers often used ropes as their measuring tools, they are known as ***rope stretchers***.

The geometric abilities that developed from surveying, architecture, and navigation were quite sophisticated, as we shall see.

As a starting point we would like you to experience geometric questions, problems, and strategies that mimic those of the rope stretchers. I.e. we want you to think about how we measure things in our world.

In Investigation **19** of the opening chapter you built a scale model of a significant three-dimensional structure. As you worked on your model you certainly found many, many different quantities to be measured to find the dimensions of components of your building. Below you are asked to (re-)discover methods for making both direct and indirect measurements. If you are like most of our students, we expect that you will rediscover methods that were known to the ancient geometers and are known as *shadow reckoning.*

The principle idea in shadow reckoning is one of proportion or similarity. The power of this idea appears to have been universally recognized by ancient cutures. As **Frank Swetz** (Mathematical Historian; -) tells us:

> Many ancient societies relied on shadow observations for agricultural and religious purposes. Shadow lengths helped determine Summer and Winter solstices thus fixing a year in time upon which planting seasons could be scheduled. Egyptian and Hindu priests fixed religious rituals according to the sun's position in the sky as determined by shadow lengths. In later Islamic societies, three of the five prescribed times for daily prayer are based on shadow lengths. Existing evidence indicates that the Babylonians, Egyptians, and Chinese developed rather precise celestial observation techniques using merely a vertical staff or pole and noting shadow positions.[1]

[1] From "Trigonometry comes out of the shadows" by Frank J. Swetz in Learning From the Masters, Mathematical Association of America, 1995, p. 57

1. Figure 7.1 shows a schematic diagram of a light source located 21' 5 15/16" above the ground. The light source is located 17' 3" from the furthest point on the base of a vertical pole. The length of the shadow cast by the pole is 26' 4 7/8" long. Describe a context in which such a situation might arise. How tall is the pole?

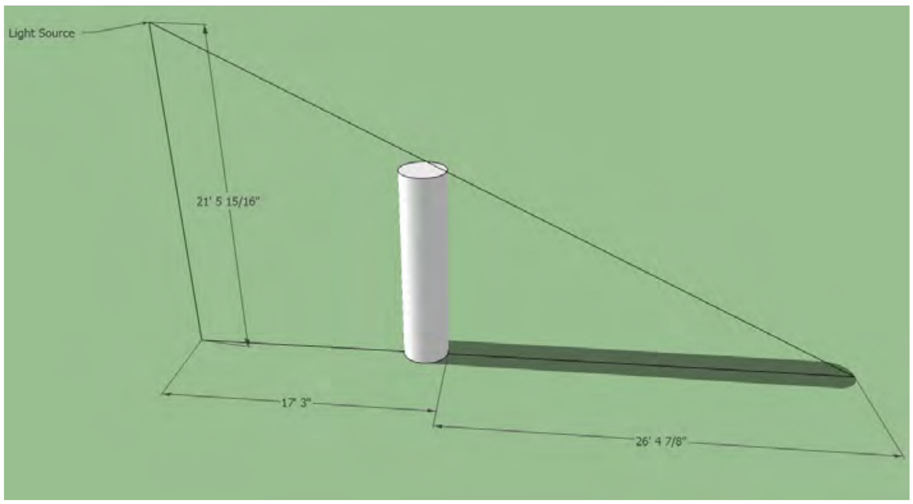


Figure 7.1: Diagram for Investigation **1**.

Figure 7.2: Diagram for Investigation **2**.

2. A schematic for two crossed rulers held up against the backdrop of a building is pictured in Figure 7.2. The width of the door, which can be easily found to be 12 feet, measures 2 3/8 inches on the ruler. The circular stained glass window measures 4 1/4 inches on the ruler. What is the actual diameter of this circular window?

3. How are the approaches in Investigation **2** and Investigation **??** related? I.e. it seems nice to have ready access to digital cameras, but is their use that much different than what was available to the ancients?

4. The images in Figure 7.3 are of the Carillon Tower at Stanley Park in Westfield, MA. The figure in Figure 7.4 is an illustration of the way that the tower can be sighted in a mirror via its reflection. An actual mirror sighting of this type is pictured in Figure 7.5. When sighted in this way, the distance from the camera lens to the tape on the mirror was 64 inches horizontally and 40 inches vertically. The horizontal distance from the tape on the mirror to the center of the tower's dome was 118 feet. How tall is the Carillon Tower?[2]

Figure 7.3: The Carillon Tower at Stanley Park in Westfield, MA; front view and view from base.

5. Figure 7.6 show's Google SketchUp's character "Bruce" standing next to a vertical pole of unknown height. Sunlight casts their shadows on the ground. The lengths of the shadows are 9' 6" and 30' 8". If Bruce is 5' 10", how tall is the pole?

6. **Plutarch** (Greek Historian; -) relates the reverence with which the King of Egypt held the mathematical work of **Thales of Miletus** (Greek Philosopher and Mathematician; circa 625 B.C. - 547 B.C.) using shadow reckoning:

> Although he admired you for other things, yet he particularly liked the manner by which you measured the height of the pyramid without any trouble or instrument.[3]

Use one of the approaches above to explain how Plutarch could have determined the height of one of the pyramids with no other instrument other than a pole and ruler.

[2]It is interesting to note that the Park's WWWebsite lists the height of the tower as 98 feet. This is badly inaccurate, as you will see from your results.
[3]From Burton, The History of Mathematics, p. 94.

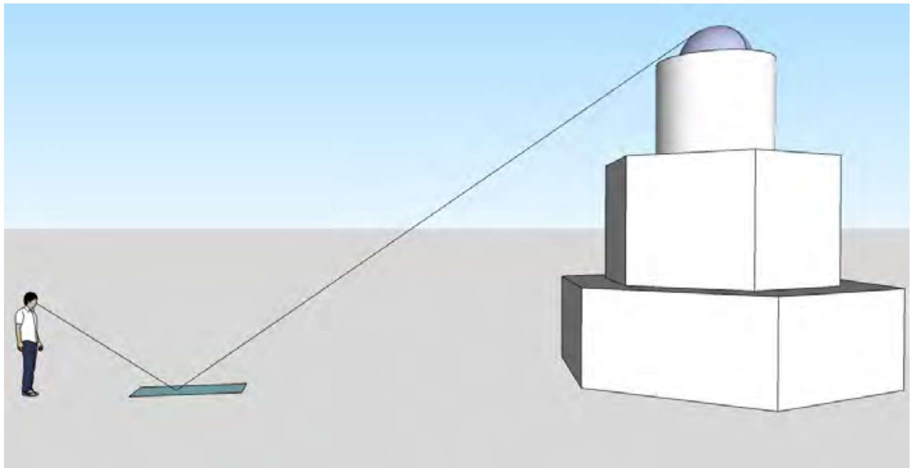


Figure 7.4: The experimental setup for *mirror reckoning*.

Perhaps a more impressive indirect measurement is the measurement of the circumference of the earth that was completed by **Eratosthenes of Cyrene** (Greek Mathemtaician and Geographer; circa 276 B.C. - 195 B.C.) some 2,250 years ago.

One of the greatest urban legends of all time is the myth that **Christopher Columbus** (Italian Explorer; 1451 - 1506) discovered the earth was round by attempting to reach the East by sailing West. In fact, all extant evidence suggests that by several millennia *before* the birth of Christ the world's major cultures *all* believed the earth was spherical. The Christopher Columbus legend stems largely from the widely popular work The Life and Voyages of Christopher Columbus which was written by **Washington Irving** (American Author; 1783 - 1859) in 1828.

7. Classroom Discussion: Arguments attributed to the ancients' view that the earth was spherical were, not surprisingly, geometric. Describe how sailors' views of ships on a distant horizon could help them conclude that the earth is round. Alternatively, describe how views seen during a lunar eclipse could help astronomers conclude that the earth is round. Can you think of other evidence that the ancients may have used to conclude that the earth is round?

Eratosthenes described his result in his book On the Measurement of the Earth which was lost but is described by many subsequent authors, leaving little question of his approach. The approach is shown schematically in Figure 7.7. Eratosthenes noticed that at noon on the Summer Solstice in Syene the sun was directly overhead; presumably something he found by looking down a well and seeing no shadow. At exactly the same time a vertical pole at Alexandria cast a shadow measuring $7\frac{1}{5}$ degrees.

8. Explain why each of the angles labelled $7\frac{1}{5}$ must be congruent.

9. Eratosthenes measured the distance between the well and the staff to be about 490 miles.

Figure 7.5: Citing the Carillon Tower using mirror reckoning. Note the piece of tape at the top the image of the dome.

[4] Using this measurement and the angle from the previous investigation, determine the circumference of the earth that Eratosthenes would have obtained.

10. How close is Eratosthenes value to the actual circumference of the earth? How remarkable is it that the circumference was known this exactly so long ago?

11. Figure Figure **??** show three right triangles with base angles of 18 degrees. The height of the first triangle, correct to two decimal places, is 0.31. Determine the missing lengths of the other two triangles.

12. Given the height of *any* right triangle with base angle 18 degrees, is it possible to determine the length of the hypotenuse? Explain.

13. Conversely, given the hypotenuse of *any* right triangle with base angle 18 degrees, is it possible to determine the height? Explain.

In fact, the positive observations in Investigations **12-13** are what help form the basis of trigonometry. In this case, it allows us to define the *sine* of an angle. All that is required to define the sine for other angles is knowledge of the ratio of the opposite side to the hypotenuse. Rough estimates for these values can be found experimentally, as was done in ancient cultures. They can

[4]Of course, this measurement is difficult and there remains debate about how precisely this distance was measured. See, e.g. Burton's History of Mathematics, pp. 205-6.

Figure 7.6: Set-up for Investigation **5**; Google SketchUp's Bruce and a pole of unknown height.

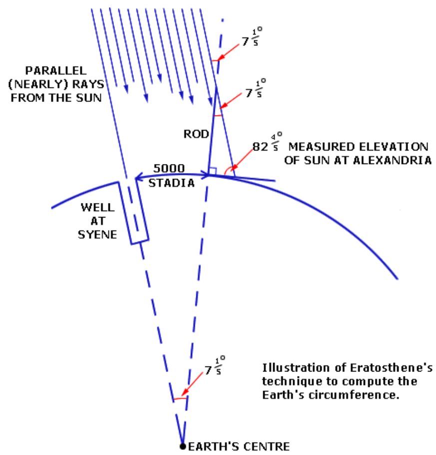


Figure 7.7: Schematic of the set-up for Eratosthenes measure of the Earth's circumference.

be found theoretically, as was done later, and compiled into tables. With the advent of computers and calculators, numerical algorithms can compute these values to a very high degree of accuracy.

The knowledge of these ancient rope stretchers brought us here:

> Shadow reckoning dominated calendrical computation and time keeping for over two thousand years. They shaped the sciences of land survey, cartography, and navigation. Concern with shadow ratios led to the eventual derivation of the basic trigonometric functions that we know today... Certainly, shadow reckoning constituted an important phase of early mathematical activity.[5]

14. Classroom Discussion: Are there mathematical principles that each of the investigations above share? What are they and how does their application differ in the problems?

[5]From "Trigonometry comes out of the shadows" by Frank J. Swetz in Learning From the Masters, Mathematical Association of America, 1995, p. 67

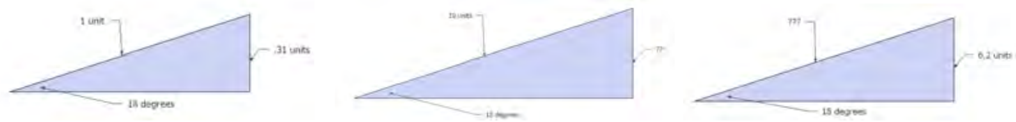


Figure 7.8: Three triangles with base angles of 18 and 90 degrees.

7.3 Euclid and the Platonic Ideal

Having developed the practical side sufficiently to perform the tasks they needed, geometry continued to be widely practiced. Great structures were build (Great Pyramids, give many other examples ???) A significant division opened that remains a fundamental part of the organization of mathematics and many of the sciences. In addition to the practitioners who applied mathematical ideas to the world around them, there developed a significant group which concerned themselves with the more theoretical aspects of mathematics.

Careful. Is this really a good way to characterize this? Certainly this is a good place for a plug for A People's History of Science. Euclid and the Pythagoreans - who we look back at with reverence because it makes a nice, tidy story, borrowed much of what they learned from practitioners whose names are long lost.

> Mathematics owes its existence and a great deal of its development to surveyors, merchants, clerk-accountants, and mechanics of many millennia.[6]
>
> **Clifford D. Conner** (; -)

Nowhere is this illustrated better than the Platonic ideal in geometry.

Pythagoreans

Ruler and compass constructions. This is an Art form. What are the limits to this art form? Links to the reasoning book where the Three Greek problems are considered.

(The equations of lines and circles are y=mx+b and $(x-a)^2+(y-b)^2=r^2$. I.e. we have only linear and quadratic equations. We can get square roots (have them do them!), but where cube roots? Fourth roots from square roots of square roots. But fifth roots? Then state the theorem about constructible numbers. Send them to George Martin and Hadlock and new symmetry book if they want more details.) While a dichotomy between pure and applied remains to this day, each is informed by the otherWhat are the activities for this section?

Discussion about the rigor of Euclid.

The Platonic ideal where lines, triangles, and circles rule. We live in 3D and draw in 2D. There is nothing else.

This is not true. Lines are not everything. This is coming in many different ways.

What Euclid did. How Euclid is portrayed in contemporary mathematics:

Separate Euclid's Elements as a single book at a single point in time from his inspiration to follow logic?

> The early study of Euclid made me a hater of geometry.
>
> **Arthur Cayley** (; 1821 - 1895)

[6]P. 3 of A People's History of Science: Miners, Midwives, and "Low Mechanicks"

> Euclid's work ought to have been any educationist's nightmare. The work presumes to begin from a beginning; that is, it presupposes a certain level of readiness, but it makes no other prerequisites. Yet it never offers any "motivations," it has no illuminating "asides," it does not attempt to make anything "intuitive," and it avoids applications to a fault. It is so "humorless" in its mathematical purism that, although it is a book about "Elements," it nevertheless does not unbend long enough in its single-mindedness to make the remark, however incidentally, that if a rectangle has a base of 3 inches and a height of 4 inches then it has an area of 12 square inches. Euclid's work never mentions the name of a person; it never makes a statement about, or even an (intended) allusion to, genetic developments of mathematics... In short, it is almost impossible to refute an assertion that the Elements is the work of an insufferable pedant and martinet.
>
> **S Bochner** (; -)

> Only impractical dreamers spent two thousand years wondering about proving Euclid's parallel postulate, and if they hadn't done so, there would be no spaceships exploring the galaxy today.
>
> **Marvin Jay Greenberg** (; -)

> It has been customary when Euclid, considered as a textbook, is attacked for his verbosity or his obscurity or his pedantry, to defend him on the ground that his logical excellence is transcendent, and affords an invaluable training to the youthful powers of reasoning. This claim, however, vanishes on a close inspection. His definitions do not always define, his axioms are not always indemonstrable, his demonstrations require many axioms of which he is quite unconscious. A valid proof retains its demonstrative force when no figure is drawn, but very many of Euclids earlier proofs fail before this test... The value of his work as a masterpiece of logic has been very grossly exaggerated.
>
> **Bertrand Russell** (; -)

What are the problems that we are going to have students work on? What is the interactive part?

Need a few axiomatics. 180 degrees in a triangle is good. Can get them doing several different proofs and then show they Euclid's.

In 306 the stuff that I did at the outset was good. What can be constructed? Should have them do some of these things with a compass and a straightedge. Really get them doing hands on things. Can also do some of the construction things that relate to the Three Greek Problems. These are nice things to work with.

Should use the term ***synthetic geometry*** in discussing Euclid. But should also consider the rise of Analytic geometry - it really is a big deal. Can do this with the links to dynamic geometry software which they should certainly see. Part of this should certainly be the triangle centers stuff. This is nice because it fist in well historically too. The Euler line is after Descartes. Is this something that was done analytically? I imagine so. It would be nice to have this historical thread in here. The GeoGebra script that we can make for the triangles - does it show all of the algebraic information as an applet? If so, this would be great seeing these parallel representations working together. It really does illustrate the connection in an important way.

15. **Classroom Discussion:** Having spent many years in mathematics classes, having gone through the material on logic, reasoning and proof in the Student Toolbox, and having worked

again a bit with Euclidean geometry, discuss the following: Is Euclid's Elements a model for logical reasoning? Has the structure of Euclid's Elements and its influence on mathematics classes you have been in provided beneficial in your mathematics education? Should we give Euclid's work different prominence depending on whether we are considering the history and/or philsophy of mathematics and thought versus when we are considering questions of teaching, learning, and pedagogy? Do you have an opinion the role the style, structure, and approach Euclid's Elements should have in contemporary mathematics classrooms? (Note: You may want to revisit this last question once you have proceeded further through the investigations that make up this book.)

7.4 Analytic Geometry

Not till Descartes, 1985 years after the death of Plato, published his analytic geometry, did geometry escape from its Platonic straightjacket.[7]

E.T. Bell (; -)

Have this section be a continuation of the chapter on Euclid only now dressed in the guise of algebra. We can see how powerful this is in many ways.

Use the triangle centers investigations via GeoGebra scripts. This is a very powerful venue for conjecturing. This is the real nature of mathematics - how mathematics works. See how this shift empowered those like Euler to find the Euler line which was there all along, waiting to be discovered.

Links to Morgan's theorem - another perfect story to illustrate the potential of all to make contributions to mathematics. Also gives a nice link to the importance of different ways of viewing things - the computer as a tool to explore. Maybe include Papert quotes here?

7.5 Non-Euclidean Geometry

Most people are unaware that around a century and a half ago a revolution took place in the field of geometry that was as scientifically profound as the Copernican revolution in astronomy and, in its impact, as philosophically important as the Darwinian theory of evolution.

Marvin Jay Greenberg (; -)

[8]

What is a line? Lines on a cylinder with ribbon.

Maps and map projections. Our earth - where we live - is only locally Euclidean!!

7.6 Klein's Erlanger Program

The universe is an enormous direct product of representations of symmetry groups.

Steven Weinberg (American, Nobel Prize Winning Physicist; 1933 -)

[7] From Men of Mathematics, p. 32.

[8] From Euclidean and Non-Euclidean Geometries: Development and History.

Figure 7.9: Images of “The Bean” in Chicago, Illinois’ Millennium Park.

Is this too much for this? Are there other big hooks that we can put in here? Poincare conjecture is really a topological thing. Are there other big open problems that typify what modern geometry is about?

Symmetry.

Links back to perspective drawing which is a subset of projective geometry.

Although this particular program was not ultimately how mathematicians view modern geometry, the flavor of its unique approach was a revolutionary new idea whose spirit lives on in our contemporary conception of geometry.

Farmer references.

Kinsey and Moore - Chapters 5 and 8.

Not sure how to do this without repeating what has been done in these nice books.

Google SketchUp does some of this stuff nicely.

In the heaven of the great god Indra is said to be a vast and shimmering net, finer than a spider’s web, straching to the outermost reaches of space. Strung at each intersection of its diaphanous threads is a reflecting pearl. Since the net is infinite in extent, the pears are infinite in number. in the flistening surface of each pearl are reflected all the other pearls, even those in the furthest corners of the heavens. IN each reflection, again are reflected all the infinitely many other pearls, so that by this process, reflections of reflections continue without end. Fronticpiece from Indra's Pearls: The Vision of Felix Klein.

7.7 The Shape of the Universe

Jeff Weeks stuff. Tic Tac Toe on a cylinder? History of this stuff? This seems a bit out of place because the Erlanger program is the overarching flavor of contemporary geometry. Maybe this can go right after Non-Euclidean geometry because it talks about our earth's geometry? Nice lead in. But then things would be a bit out of historical order. Maybe the last section could be Contemporary Views of Geometry and the Erlanger program could be one of the topics.

7.8 So What is Geometry?

Notice self-similar title. Have several discussion questions. Formulate this in a broad, modern way. See article by Atiyah among other things.

Further Investigations

Modern measuring tools and their relationship to geometry. GPS, parabolic reflectors, triangulation, orienteering,)

7.9 JF Notes

Look up stuff in A People's History of Science!

Appendix

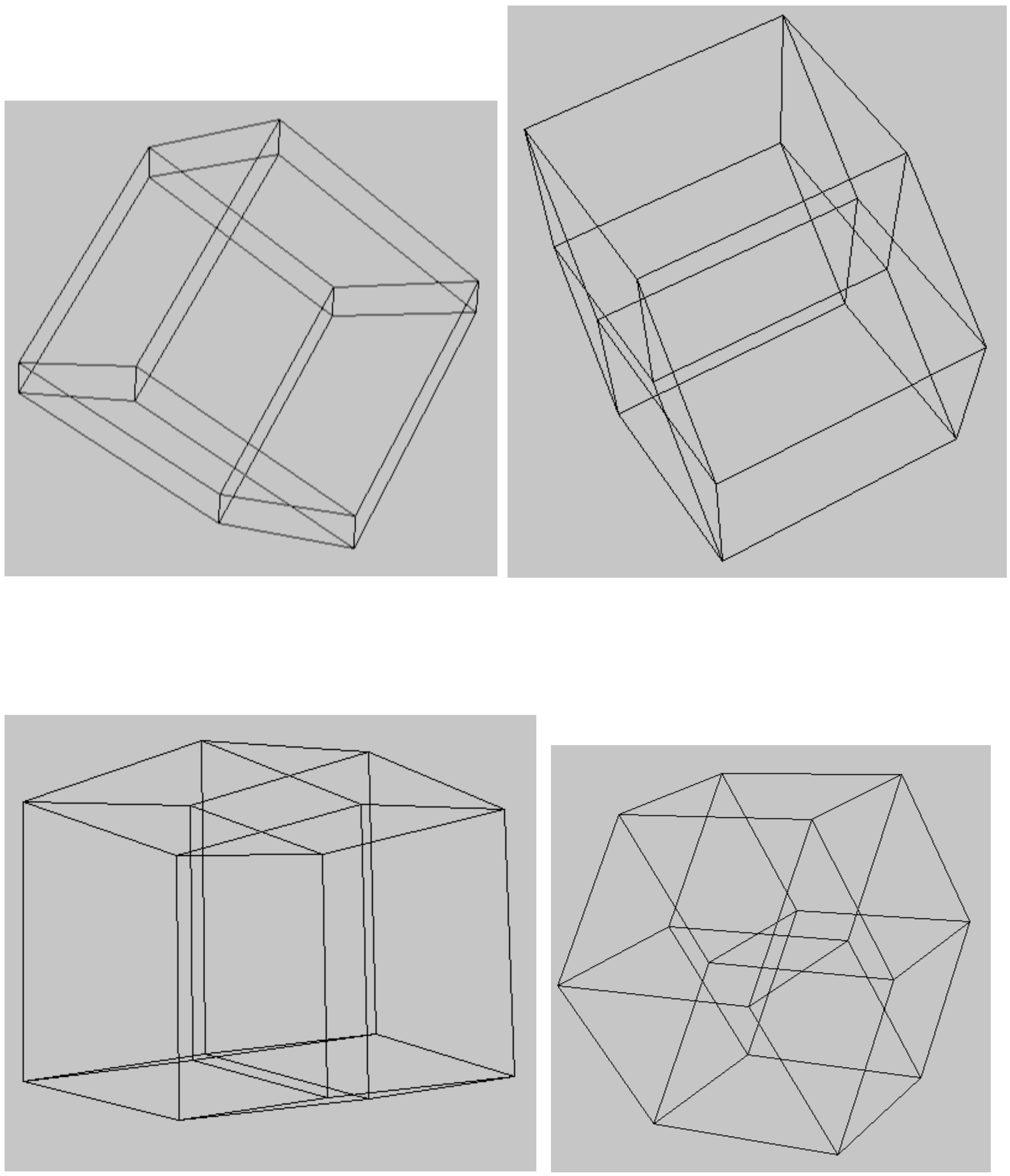


Figure 7.10: Images for Investigation **12** in Chapter 5.

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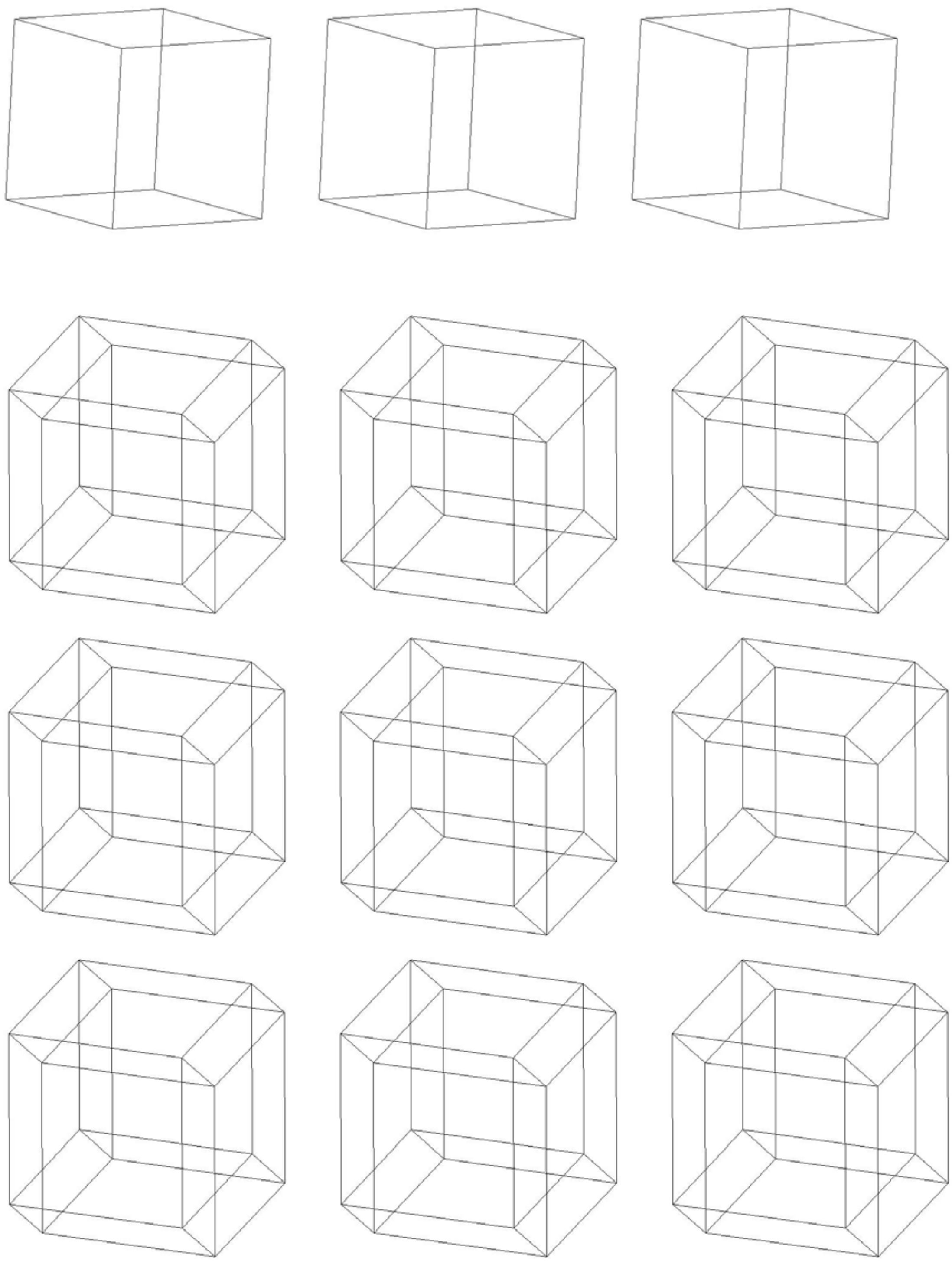

Figure 7.11: Images for Investigation **22** in Chapter 5.

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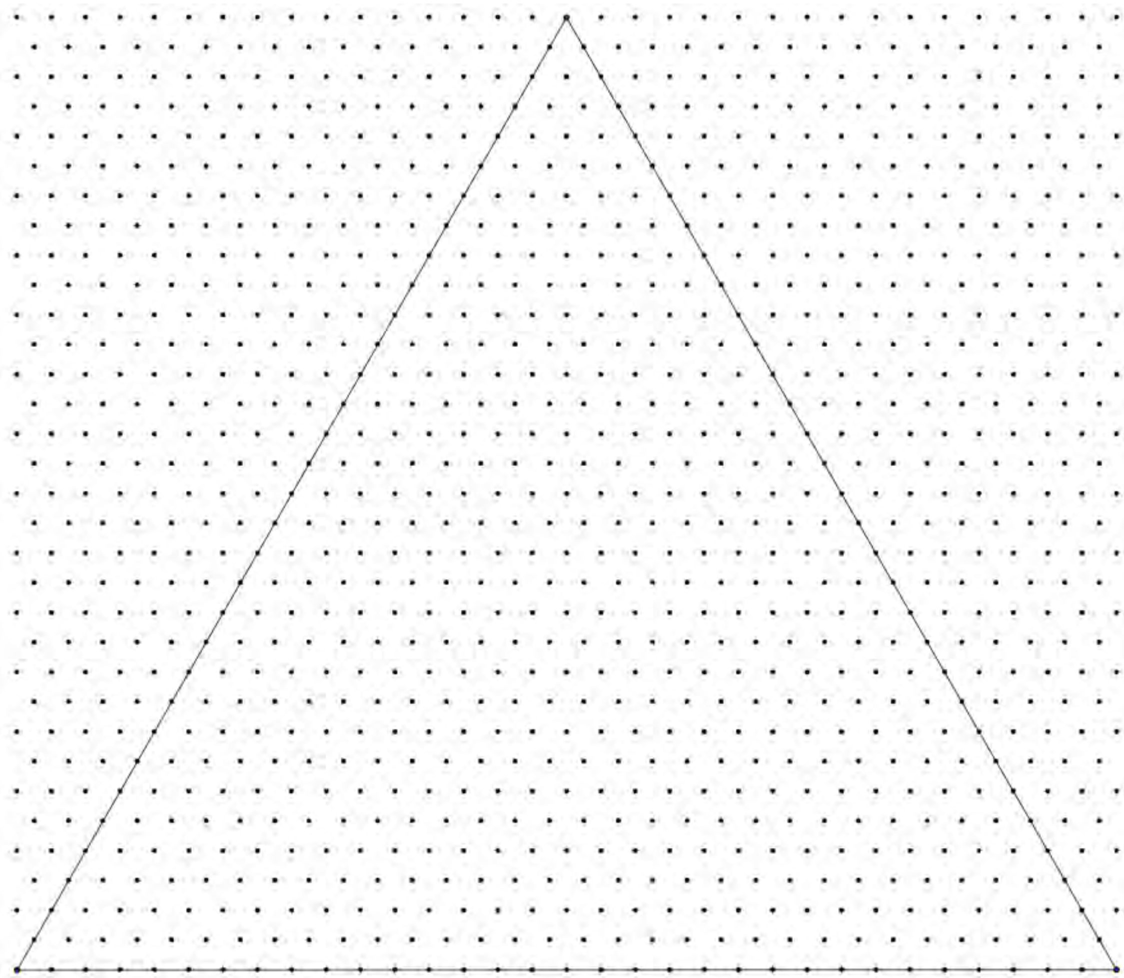

Figure 7.12: Image for Investigation **64** - Investigation **67** in Chapter 6.

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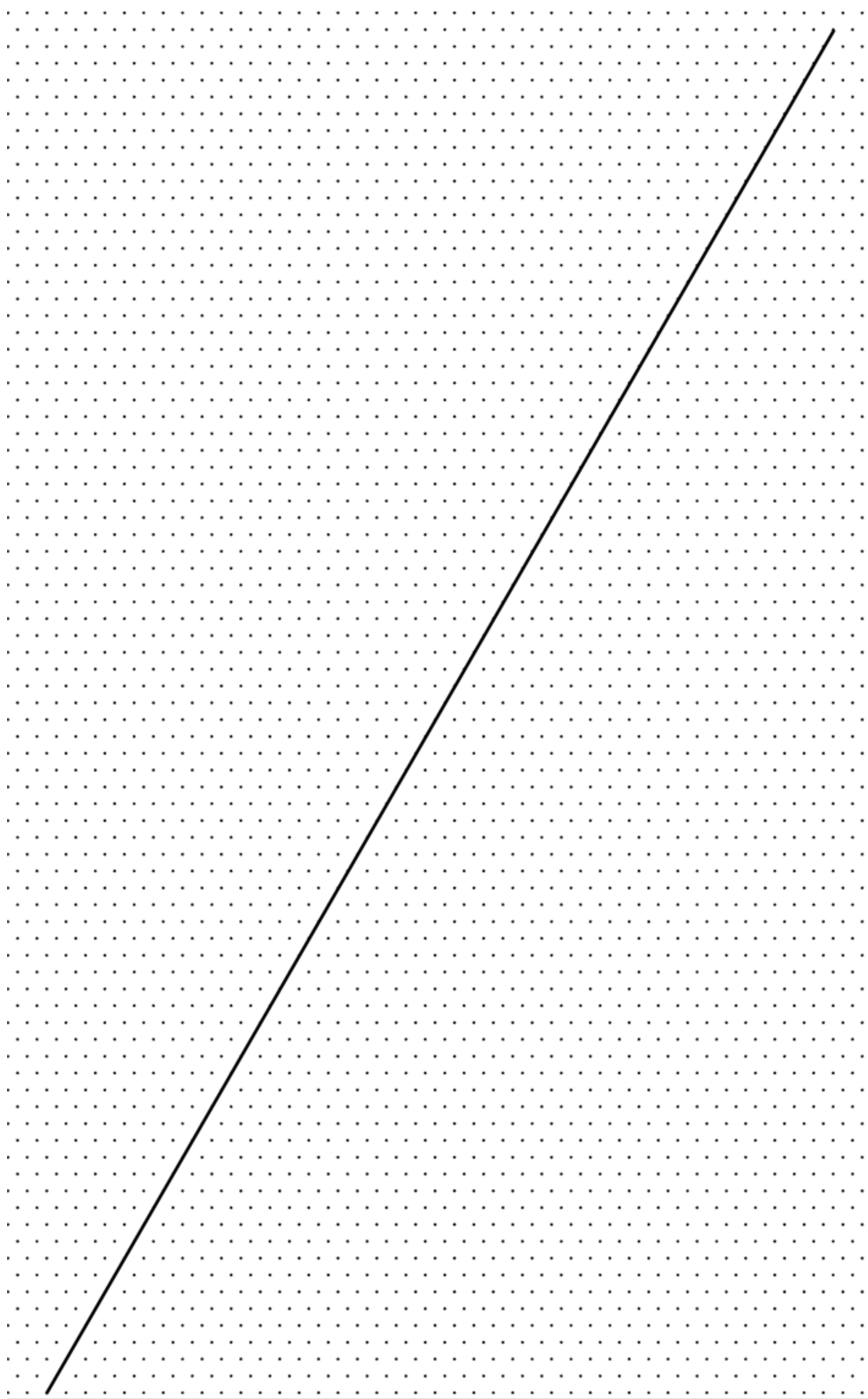

Figure 7.13: Koch curve Initiator for investigations in Chapter 6.

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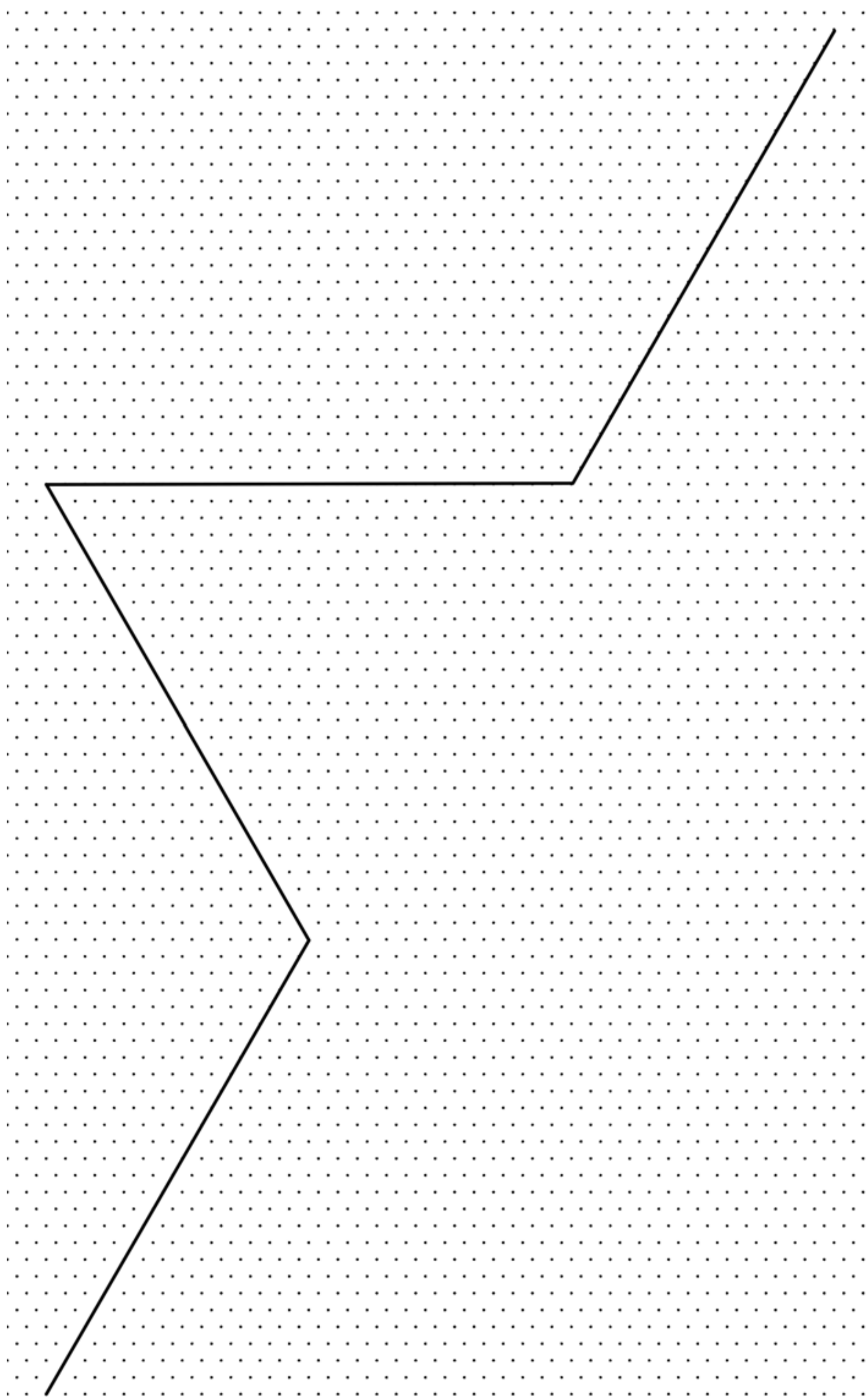

Figure 7.14: Koch curve Generator for investigations in Chapter 6.

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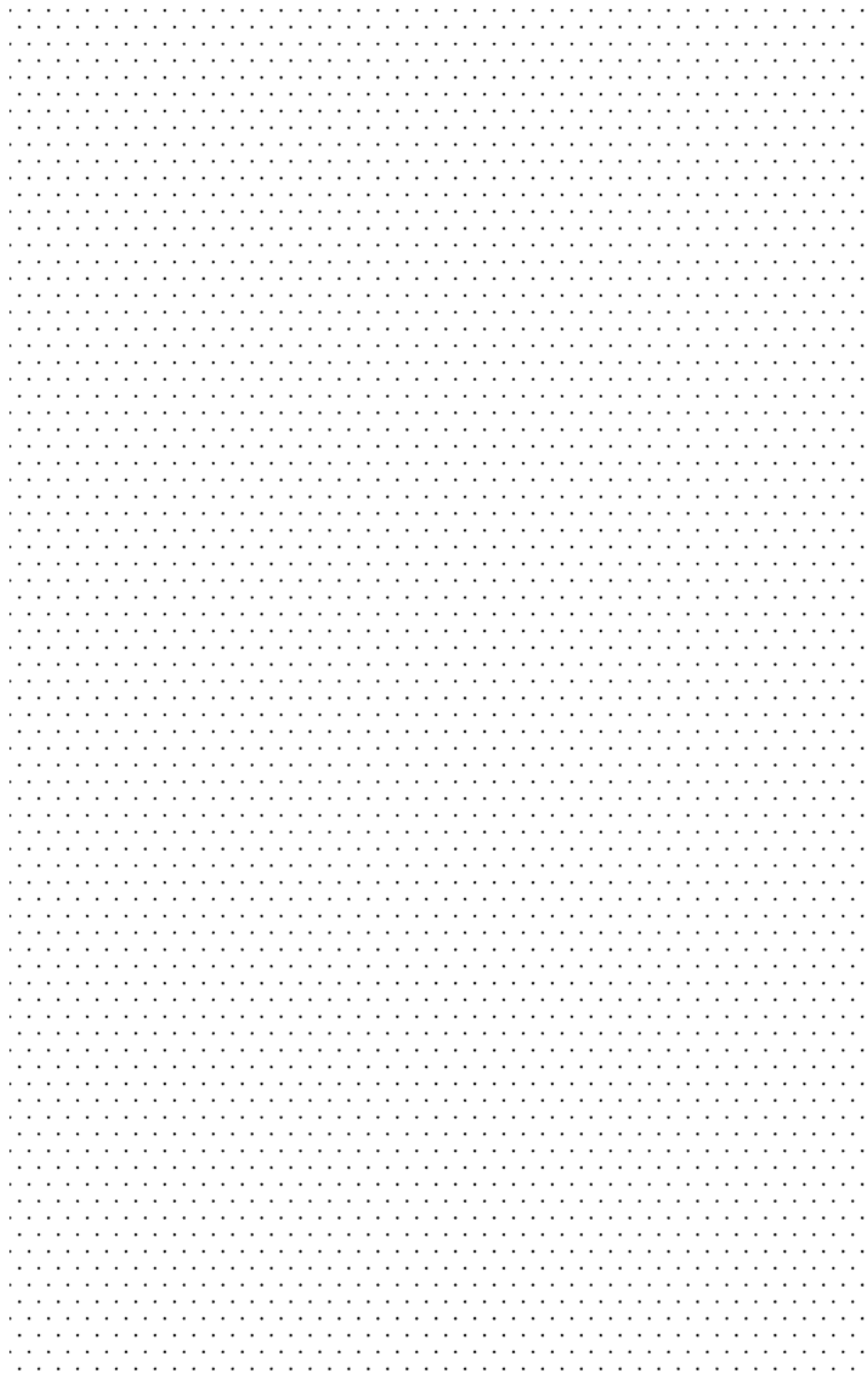

Figure 7.15: Koch curve dot paper for investigations in Chapter 6.

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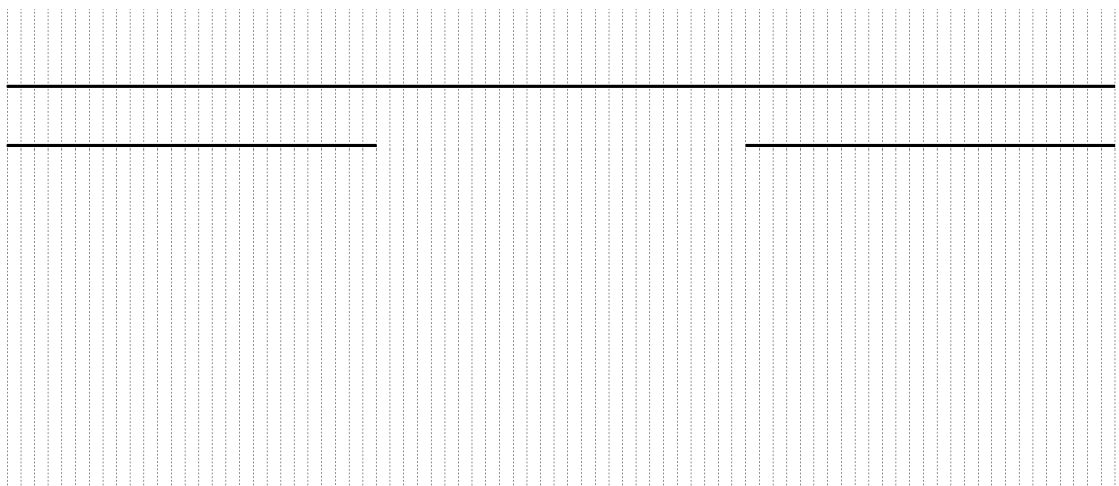

Figure 7.16: Cantor set Initiator and Generator investigations in Chapter 6.

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