CHAPTER 4

Tuning and Intervals

> Sitting on the riverbank, Pan noticed the bed of reeds was swaying in the wind, making a mournful moaning sound, for the wind had broken the tops of some of the reeds. Pulling the reeds up, Pan cut them into pieces and bound them together to create a musical instrument, which he named "Syrinx", in memory of his lost love
>
> **Ovid** (Roman Poet; 43 BC - AD 18/19)

Have you ever watched someone tune a guitar? Or maybe even a piano? The lengths of the strings have to be adjusted by hand to exactly the right sound, by making the strings tighter or looser. Bue how does the tuner know which sound is the right one? This question has been asked throughout history and different cultures at different times have found different answers. Many cultures tune their instruments differently than we do. Listen for instance to the Indian instrument *sarod* in `http://www.youtube.com/watch?v=hobK_8bIDvk`. Also, 2000 years ago, the Greek were using different tuning ideas than we do today. Of course the Greek did not have guitars or pianos at that time, but they were still thinking about tuning for the instruments they had and about the structure of music in general. The **pan flute**, one of the oldest musical instruments in the world, was used by the ancient Greeks and is still being played today. It consists of several pipes of bamboo of increasing lengths. The name is a reference to the Greek god Pan who is shown playing the flute in Figure 1.

FIGURE 1. Pan playing the pan flute.

For the following investigations you need to make your own "pan flute" out of straws. Straws for *bubble tea*[1], work better than regular straws since they have a wider diameter. You need to plug the bottom with a finger to get a clear pitch. Put your lower lip against the opening of the straw and blow across the opening (but not into it). It helps to have some tension in the lips, as if you

[1] "Bubble tea" is the American name for pearl milk tea from Taiwan. You need straws with a larger diameter to drink bubble tea, since the tea contains small balls made of starch.

were making the sounds "p". Also, for shorter straws you need more air pressure than for longer straws.[2]

1. Take a straw and cover the bottom hole while blowing over the top hole. Practice until you can hear a clear note. *Why* do you think we hear a sound?
2. Do you think the sound will be different if the straw is longer or shorter? Explain your thinking.
3. Take a rubber band, hold it tight between two hands and have someone pluck it. Can you hear a clear note?
4. Take a rubber band, stretch it over a container and pluck it. Can you hear a clear note? Why do we hear a sound?
5. Do you think the sound will be different if the rubber band is longer or shorter? Tighter or looser? Explain your thinking.

6. Classroom Discussion: How is sound generated? What exactly is vibrating? What is a *sound wave*? How do different musical instruments like drum, guitar, violin and trumpet generate sound?

For the next investigations we will use the modern piano as a reference tool, so that we can compare our sounds and give them labels. Even with the piano it is quite difficult to hear if two sounds are the same or not. If you have difficulties, turn to someone who has practiced music for a long time for support.

7. Take one straw and cut it such that it has the sound of any white key on a piano (except for the B key, see Figure 2.

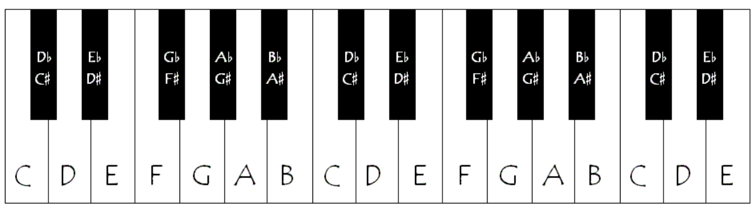

FIGURE 2. piano keys with labels.

We will discover later why the B key doesn't work.) You can go to `http://www.play-piano.org/play_online_piano_piano.html` to use the online piano.

8. Take a second straw and cut it so that it has a length of $\frac{1}{2}$ of the first straw.
9. Take a third straw and cut it so that it has a length of $\frac{2}{3}$ of the first straw. Be precise!
10. Take a fourth straw and cut it so that it has a length of $\frac{3}{4}$ of the first straw. Be precise!
11. Compare the sounds of 2 straws at a time. We call two notes sounding at the same time an **Interval**. We write e.g. $(1, \frac{2}{3})$ for the interval of the first straw and the straw with length $\frac{2}{3}$. Listen carefully: which two straws sound the most alike? You can also sing the notes of the 2 straws and listen to the interval to make your decision.

12. Classroom Discussion: Share your intervals with the class. Decide together which fraction gives the "most alike" interval.

We call the interval that sounds the most alike an ***Octave***. Human brains seem to be hard-wired to perceive these sounds as alike or even the same. The thalamus is a part in the brain of mammals that is built in layers of neurons that correspond to octaves. See Figure 3. Additionally research shows that rhesus monkeys have "same" responses to melodies that are one or two octaves apart but "different" responses to other melody shifts.

[2]Tubes with diameter $\frac{1}{10}$ of their length are easiest to play!

FIGURE 3. Thalamus in the Human Brain.

This explains why we can find octaves in cultures all over the world even though their music may sound very different. Even though all cultures share octaves, there are many ways to divide the octave into smaller intervals. We call those choices ***scales***. In modern western culture, the major and minor scale are the most prominent scales. For example the C major scale corresponds to the white keys on a piano. Notice that on a piano you have to go up or down 8 white keys to travel an octave (starting on a white key and counting this first key as one of the 8).

You can go to `http://www.play-piano.org/play_online_piano_piano.html` to play the C-major scale. Take the intervals $(1, \frac{2}{3})$ and $(1, \frac{3}{4})$ and see if you can find the corresponding intervals on a piano.

13. Take your pair of straws for the interval $(1, \frac{1}{2})$. How many white keys are between the notes if you count the beginning and the end note as well?
14. Take your pair of straws for the interval $(1, \frac{2}{3})$. How many white keys are between the two straw-sounds if you count the beginning and the end key as well?
15. Why do we call the interval $(1, \frac{2}{3})$ a **fifth**[3]? Explain!
16. Why do we call the interval $(1, \frac{3}{4})$ a **fourth**? Explain!

You have probably heard of the mathematician and philosopher **Pythagoras of Samos** (Greek Philosopher and Mathematician; 570 BC - 495 BC), but did you know about the secret society called the ***Pythagoreans***? The Pythagoreans believed that *everything* in the world could be explained using mathematics, including music. There is not much evidence about the life of Pythagoras and his disciples, see Further Investigation 3. However, they are credited with some important discoveries in mathematics. The Pythagoreans believed that all music could be explained using mathematics. They used, for instance, the musical fifths to get to all other notes in their scales as the next Investigations illustrate. The tuning they used is called ***Pythagorean Tuning***.

17. Take the interval $(1, \frac{2}{3})$. Now take a third straw and cut it such that the length is $\frac{2}{3}$ of the previous $\frac{2}{3}$ straw. How many times is your new, very short straw compared to the longest straw? Write your answer as a fraction and explain your reasoning.
18. What is the label of your new straw on the piano? Is it in the same octave as the first two straws? Can you see how to use the fraction to determine whether your new note is in the first octave or not? From now on we will call this octave (between our first two straws) our ***main octave***.
19. Compare the two fractions $\frac{1}{1}$ and $\frac{1}{2}$, whose sounds lie an octave apart. Which fraction operation do we have to do to get from one to the other? Explain how to go up and down octaves using fractions.

[3] We have to distinguish between the musical fifth (which is a specific interval between two notes), and a mathematical fifth (which is the fraction $\frac{1}{5}$.)

20. By looking at *any* fraction, how can you tell whether the corresponding note will be in the main octave or not? Explain your reasoning.
21. Take the fraction from Investigation 17. How can we use it to get a new fraction corresponding to the same note in the *main* octave?
22. You just found a fraction representation of a note in your main octave that corresponds to a fifth above a fifth. Continue the pattern by taking the next fifth and so forth. If you can't hear the sound of your straw anymore, see if you can find the mathematical pattern to continue this quest in theory. You should find a list of fractions.
23. Draw a number line from $\frac{1}{2}$ to 1 and label the first 5 fractions you found.
24. Look at a piano keyboard. How many steps are there in a fifth if you include the black keys?
25. We said earlier that a fifth corresponds to five white keys on the piano keyboard if you don't start from a B. Use Investigation 24 to argue why did we had to exclude the B.
26. Using investigation 24, how many fifths do we have to go up on a piano keyboard before we return to the same note (some octaves higher)?
27. Now we will use the fraction $\frac{2}{3}$ to go up by fifths. Find the fraction representation of the note in the main octave that corresponds to 12 fifths above your original note. Explain your strategies.
28. How far is the fraction from investigation 27 from 1? Did you expect this answer? Explain.
29. Does the chain of fifths ever end? Use fractions to explain your answer.
30. Use the chain of fifths to explain problems that arise with Pythagorean tuning.

31. Classroom Discussion: Does the chain of fifths end or not? Compare your result of the fraction computation with the result on the piano keyboard. How perfect is Pythagorean tuning?

It is common to measure the "height" of a note, also called ***pitch***, with frequencies. The frequency measures how fast the sound wave vibrates. In a long straw (big number) the air vibrates more slowly (small number) and in a short straw (small number) the air vibrates faster (big number), which means the length of the straws is anti-proportional to the speed of vibration. For simplicity we will assume that the fractions for frequency are just the reciprocals of the fractions for length, i.e.

$$\text{frequency} = \frac{1}{\text{length}}.$$

For example a straw of length $\frac{1}{2}$ sounds with a frequency of $\frac{2}{1}$.

The unit of frequency is hertz (Hz), named after **Heinrich Hertz** (German Physicist; 1857 - 1894). 1 Hz means that an event repeats once per second.

We want to redo the above investigations thinking about frequency instead of length.

32. Write the intervals $(1, \frac{1}{2})$, $(1, \frac{2}{3})$, and $(1, \frac{3}{4})$ using frequencies instead of length.
33. By comparing the two frequencies that make our main octave, which fraction operation do we use to go up and down octaves? Explain.
34. Compute the ascending fifths as above using frequencies instead of length. Explain your strategies.
35. Draw a number line from 1 to 2. Label your first 5 frequency fractions.
36. Since the process of taking more and more fifths results in notes that sound out of tune, the Pythagoreans used the fraction $\frac{3}{4}$ to help them. Recall the key on the piano corresponding to the forth, i.e. to the fraction $\frac{3}{4}$. How many fifths do we use to go up on the keyboard in order to get to the same note as the forth (ignoring octaves)?
37. Why is it more accurate to work with the forth instead of the fifths in investigation 36?
38. Label the frequency that corresponds to the fraction $\frac{3}{4}$ on your number line.

Your main straw could have been any length in the above investigations and hence correspond to any note from a white key (excluding B, of course). For the next section we will assume that it corresponds to the note C. The mathematics works out the same if you use another note as your starting point, but it makes it easier to read if we agree on a base note.

We want to discover how the Pythagorean fifths will give us the entire C-major scale!

39. Fill in the first row in table 1. If your main straw would correspond to the note C, how do the other frequency fractions we found relate to the keys on the piano? You can use the fractions you computed in the above investigations. Just match them with the C-major scale instead of the scale from your straws.

TABLE 1. Frequency Table

Note	C	D	E	F	G	A	B	C
Frequency Fraction	$\frac{1}{1}$							$\frac{2}{1}$
Ratios between Frequency Fractions								

40. Classroom Discussion:

Compare the first row in table 1. Now look at the ratios[4] between adjacent fractions on your number line. Fill in row 2 in table 1. What patterns do you notice?

You just discovered the so called ***Pythagorean Tuning*** based on C. Unfortunately there are some problems with this tuning method... you will discover some of these in the next Investigations:

41. Compute the frequency of the last fifth FC using ratios.

42. Compare the frequencies of the fifth CG to the frequency of the fifth FC. What do you notice?

43. Why do you think the last fifth FC is called the ***wolf interval***?

44. Your piano is tuned in Pythagorean tuning based on C. Imagine you have a melody staring with the fifth CG. Do you think the song would sound different if you started playing it on the piano key F? Explain.

So it seems that for some melodies the piano will sound in tune while for other melodies or other starting points of you melody it might sound out of tune. Musician would say: "If I played a song that uses a different ***key*** it would sound out of tune!". This *key* is not the same as a key on a keyboard. It is an abstract term roughly describing a set of notes that a piece of music is most likely to use. You can for instance say that a song is being played in the key of "C major".

That is not what we wanted! It gets even weirder:

45. Compare the ratios for a half step and a whole step in Pythagorean tuning (table 1). What do you notice? Are two half steps really a whole step? Remember to use ratios and differences in your argument.

46. Why is Pythagorean tuning a very natural way of tuning, even though problems arise?

Since the Pythagorean tuning is not the same for all *keys*, other ways of tuning were developed over time. In the 18th century ***well tempering*** was used, in which compromises were made such that every *key* would sound good but slightly different. One advantage of each *key* sounding different is that the mood of a piece of music can be expressed by the choice of *key*.

Since the middle of the 19th century ***equal temperament*** is most commonly used. This tuning requires a new mathematical idea which you will discover in the next Investigations. We know that the frequency interval $(1, 2)$ gives us an octave. It is customary in Western Music to have 12 steps in an octave. Therefore we need to find a way to split the interval between 1 and 2 into 12 "equal" steps. Since we are dealing with ratios here, we need all the steps to have the same ratio. Look back at table 1 to see 7 steps (ratios of frequency fractions) that are not all equal.

[4] To find the ratio between two fractions you need to divide one fraction by the other - you compute a fraction of fractions. We will divide the larger fraction by the smaller to make it easier to compare.

47. Split the interval between 1 and 2 into 2 "equal" steps such that the *ratios* are the same. This means we are looking for a fraction, say x, between 1 and 2, such that the ratio of 2 and x is the same as the ratio of x and 1. What is x? Describe your strategy.

48. Compare your solution with the following problem: Split the interval between 1 and 2 such that *differences* are the same. This means we have to find a number, say y, between 1 and 2 such that the difference between y and 2 is the same as the difference between y and 1. What is y? Did you get the same answer as in the last investigation?

49. Classroom Discussion: Compare the two solutions above to get "equal size" steps in the interval $[1, 2]$. Compare your strategies. What does "equal size" mean? Compare your results for the fractions in Investigation 47. Discuss if $\sqrt{2}$ can be written as a fraction or not. Can you prove your conjecture?

50. Split the interval between 1 and 2 into 3 steps with equal ratios. Describe your strategy.
51. Split the interval between 1 and 2 into 4 steps with equal ratios. Describe your strategy.
52. Split the interval between 1 and 2 into 5 steps with equal ratios. Describe your strategy.
53. Split the interval between 1 and 2 into 12 steps with equal ratios. Describe your strategy.
54. Summarize how to find the frequencies for the *equal temperament tuning*.
55. What are some advantages and some disadvantages of *equal temperament tuning*?

You really understand Pythagorean tuning and equal temperament tuning now, and you have traveled through many centuries of music and mathematics history. Hidden in the above mathematics is some history about numbers:

The Pythagoreans believed that *every* number could be written as a fraction. Mathematicians call these numbers **Rational Numbers**. According to legend **Hippasus of Metapontum** (Greek Philosopher; 500 BC -) was put to death by Pythagoras because he had revealed the secret of the existence of ***irrational numbers***: numbers that can not be written as fractions. It might seem easy to grasp for us now, but every time mathematicians expand their ideas of numbers it is like a small revolution. And there are more than just irrational numbers! There are for instance *complex numbers* and *imaginary numbers* and *surreal numbers*. For the latter you can read the book Discovering the Art of Mathematics: The Infinite.

56. Do you find it surprising that the Hippasus was put to death?
57. Name one irrational number. Do you know more?

1. Further Investigations

The way Greek mathematicians first encountered irrational numbers was not in music, but in geometry. You will solve their problem in the next Investigation.

F1. In a square with side length equal to 1, what is the length of the diagonal?

F2. Find a proof of the fact that $\sqrt{2}$ is an irrational number. You can look at books or go online. Explain the proof to someone else without looking at your notes to see if you fully understood it.

F3. Read "The Ashtray: Hippasus of Metapontum (Part 3)" by ERROL MORRIS published in the New York Times Opinionator. What do we *actually* know about Hippasus?

F4. Understand how to draw graphs of waves with different frequencies, see Figure 4. How does this relate to waves of air in the straws?

Check out Ruben's Tube videos on `youtube.com`. How does this connect to graphs of sound waves? See Figure 5.

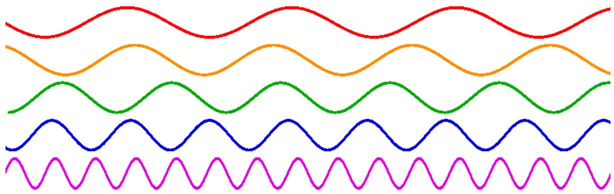

Figure 4. Graphs of waves with different frequencies.

Figure 5. A Ruben's Tube Experiment.