MATH 110
Mathematical Explorations
Fall 2017
A Tiling Problem

Combinatorics is an area of mathematics that looks at the number of ways that various quantities or objects may be put together. Combinatorial problems arise in many areas of pure mathematics, notably in algebra, probability theory, topology, and geometry; and it also has many applications in mathematical optimization, computer science, ergodic theory and statistical physics.[1]

We are going to consider a **tiling** problem that comes from the field of combinatorics and connects up with a famous list of numbers.

Problem: Given a whole number n, how many ways can we tile (cover) a $2 \times n$ rectangle using only 2×1 tiles?

For example, there are 3 ways to tile a 2×3 rectangle with 2×1 tiles:

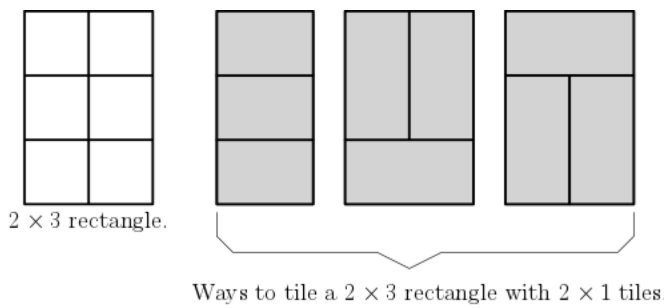

FIGURE 1. The 3 ways to tile a 2×3 rectangle.

Note that the second and third ways are counted as different even though we can rotate the second way 180° to get the third way.

1. Use graph paper to determine the number of ways to tile a 2×5 rectangle with 2×1 rectangles. How do you know that you have them all?
2. Use graph paper to determine the number of ways to tile a $2 \times n$ rectangle with 2×1 rectangles for $n = 1, 2, 3, 4, 6$; then make a table for the number of ways to tile a $2 \times n$ rectangle with 2×1 rectangles for $n = 1, 2, 3, 4, 5, 6$.
3. Find a pattern in your answers to Questions **1** and **2** that allows you to predict the number of ways to tile a 2×7 rectangle based on the number of ways to tile smaller rectangles.
4. Use graph paper to check your answer to Question **3** for $n = 7$. Was your conjecture for $n = 7$ correct? If so, go on to the next question. If not, check your work in determining the number ways to tile a 2×7 rectangle.
5. Based on your answers to Questions **3** and **4** predict the number of ways to tile $2\times n$ rectangles for $n = 8, 9, 10$.
6. Generalize your answer to Question **5** to make a conjecture about how to determine the number ways to tile a $2 \times n$ rectangle, for any given a value for n, from the number of ways to tile some of the smaller rectangles.

[1]From Wikipedia entry on combinatorics.

7. Verify your conjecture in Question **6** for the 2×5 rectangle by showing how to add the appropriate number of 2×1 tiles to the tilings of the smaller rectangles that correspond to those identified in Question **6**.
   Note: You need to add the 2×1 tile(s) to the smaller rectangles in such a way that
   - the width of the rectangle stays at 2,
   - the height of the rectangle changes to 6; and
   - you do not create the same tiling from two different ways of adding the 2×1 tile(s).

   Remember, there are two different orientations of the 2×1 tiles that can be used in covering a $2 \times n$ rectangle.

8. Repeat Question **7** to verify your conjecture in Question **6** for the 2×6 rectangle.
9. Generalize your answers to Questions **7** and **8** to explain why your conjecture in Question **6** is true.
10. Repeat Questions **2 - 9** for tiling $3 \times n$ rectangles with 3×1 rectangles.
11. What are some of the similarities and differences between the ways we can determine the number of tilings for $2 \times n$ rectangles and $3 \times n$ rectangles by 2×1 and 3×1 tiles respectively.

Extra Credit: Extend Question **10** to explore the number of ways of tiling $k \times n$ rectangles with $k \times 1$ tiles for $k = 4, 5$ and 6. Can you identify a pattern that allows you to find a way for determining the number of ways of tiling $k \times n$ rectangles with $k \times 1$ tiles for any k? Prove it.